Field theory for recurrent mobility

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A thesis submitted in fulfillment of the requirements for the degree of Master in Physics of Complex Systems in the

IFISC-CSIC

September 19, 2017
In the field of geography of mobility exist two main modeling frameworks: radiation and gravity. In the last years, there has been an open discussion about the which of these models produce better results for each type of mobility: from commuting to migration. Here we introduce a new perspective translating the problem into a field theory formalism. One of the two models, gravity, fulfils the conditions of having a null rotational (it can be derived from a potential) and the Gauss theorem holds. Our results show that empirical commuting data support these features. Furthermore, the analysis of the potential provides a method to find centers of mobility.
Acknowledgments

To José, for his advice, guidance and help during this project; to Antònia for providing me the necessary data; and to Riccardo and Aleix for their tips and help with the code,

to the IFISC, for his special help and for having allowed me to study this master,

to my master fellows Ana, Víctor, Joan L., Luca, Alejuandro, Oriol, Gianmarco, Joan P., Patrick, Miguel P. R. and Edu for their company during this year (I could write lines and lines talking about you, but I guess that this work is about science and not about friendship),

to my parents Marta and Jordi, and my brothers Martona, Xavi, Tuky, María, Jordi, Núria, Jaume and Rafa, for their support during all these years and for making me the person that I am today,

to my adoptive parents during this year, Mercedes and Javier, and my adoptive siblings Bosco, Álvaro, Marta, Gonzalo and Inés, for having received me like another one of their beautiful family,

and last but not least, specially thanks to María, for having supported me during all these years and for continuing faking interest when I talk about physics. Without you I could never have walked this road.

Gràcies! Gracis! Gracias! Thanks!
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Chapter 1

Introduction

As we know from our daily lives, people do not move randomly in space but tend to exhibit high levels of regularity. Thus, it is interesting to study this phenomenon using the language of mathematics. In 1885, Ravenstein published his seminal paper [1] in migration laws, and since then the people mobility has been an active branch of research in social science. However, during the awakening of this new science all the accessible data was static, in the sense that the researchers only had access to data coming from census, which is a data source that is updated in time scales of decades. But nowadays, thanks to the information technologies (ICT), we have access to dynamic data on how people move at every moment. Therefore, using this new data we can study in a deeper way the human mobility.

During the last years a lot of work have been done in this branch of the science and it has been shown that has a lot of applications in different aspects: disease spreading [2]–[4], transport planning [5], control of migrations [6], community detection in cities [7], [8] and more. One important research question is the characterization of periodically recurring travels between one’s place of residence and place of work, also known as commuting. When facing of commuting from a modeling perspective two main approaches appear: gravity and radiation models.

It is widely accepted, in the gravity approach, that the commuting is hindered by geographical distance and on the other hand is benefited by the human density in the destination. This point of view can be mapped to the Newton’s law or more complex versions of it. These models are known as “gravity models” and have a long tradition in quantitative
geography [9]–[12]. On the other hand, there are another kind of models where the distance does not play such a major role as a travel cost, but the only important quantity is the number of opportunities at a certain distance. These models are known as “intervening models” and were proposed by Stouffer [13], and suggest that the number of persons traveling a given distance is proportional to the number of opportunities at that distance and inversely proportional to the number of intervening opportunities. The main type of intervening model is the so-called “radiation model”, proposed by Simini et al. in [14] and extended in [15]. The radiation model is inspired by a simple model of bodies that can radiate and absorb particles (in our case the bodies are the cities or municipalities and the particles are persons). The gravity model has been used to model flows of population [16], accessibility to health services [17] and phone communications [18]. On the other hand, the radiation model has been used in the modeling of migrations [14], but it has been showed that it fails when reproducing the intra-city movements [19]–[21].

The gravity and the radiation models have been compared during the last decades [22] showing that usually the two approaches have comparable performances. However, a work by Lenormand et al. [23] showed that in the most cases the gravity law with an exponential decay factor is the one with the best performances in almost all the cases of commuting.

In this work we introduce a new approach to the topic, using as inspiration the electric field and approach it from the perspective of vector field and Maxwell-like equations. We will interpret the commuting of people in the city as vectors and we will derive the corresponding field theory equations -mimicing Maxwell equations- for commuting. Once the Maxwell-like equations have been derived we will adress the problem of conurbations\(^1\) and city centers. Using the equations of the fields we will derive a potential that will allows us to make a formal definition of a city and apply it to the case of conurbations in order to detect the different cities inside of them.

\(^1\)A conurbation is a region comprising a number of urban areas that have merged to form one continuous urban or industrially developed area. In most cases, a conurbation is a polycentric urbanised area.
Chapter 2

Methods

In this section we will describe how we have performed the computations. First we will present the data and then we will present the mathematical formulation of the models under study.

2.1 Data

In this work we have focused in two urban areas, London and Manchester-Liverpool (more areas have been studied and the results will be presented briefly in the Appendix). The area of London contains a population of 17,140,493 (2014) and the Manchester-Liverpool conurbation has a population of 5,680,000 (2015). In the case of London we have 5,731,896 tweets of 343,138 users and in Manchester-Liverpool we have 5,506,862 tweets of 258,985 users from a time period going from 2014 to 2016. Despite the fact the number of users is much less than the population of the cities it has been shown [24] that it is enough to study the mobility in cities. The datasets used in this work consist of geolocalized tweets in different urban areas around the world. The data that we have used have the structure of the table 2.1

<table>
<thead>
<tr>
<th>ID</th>
<th>LON</th>
<th>LAT</th>
<th>Year</th>
<th>Month</th>
<th>Day</th>
<th>Hour</th>
<th>Min</th>
<th>Sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>3607772952</td>
<td>8.4554525</td>
<td>49.9876383</td>
<td>2015</td>
<td>12</td>
<td>07</td>
<td>23</td>
<td>45</td>
<td>20</td>
</tr>
</tbody>
</table>
Using a grid with equal area cells (500m x 500m) we have assigned to each tweet a cell. Then, we specify a certain weight for each tweet $w_t$ emitted by the user $i$ as $w_t = \frac{1}{t_i}$, where $t_i$ is the number of tweets of the user $i$. Using this definition of weight we avoid that users that had emitted a huge amount of tweets count more than those which have emitted less. Finally, we assign to each cell the value of the sum of all the weights in that cell. In this way we obtain a heat map of the population density, which look like Figure 2.1a.

Furthermore, we assign to each user a home cell (the cell where more tweets has been emitted between 8PM and 8AM) and a work cell (the cell where more tweets has been emitted between 8AM and 8PM). To have sufficient statistic we have considered only those users with more than 10 tweets. If we plot the population density for those users which work and live in different cells we obtain the Figures 2.1b and 2.1c, where it can be observed that in the work case the densities are more concentrated, and in the home map the densities are more distributed around the peak, which means that the people tends to move toward the center when commuting.

![Figure 2.1](image)

**Figure 2.1:** Heat map of the population distribution (A) for 343,138 Twitter users in London and the work place (B) and home (C) for 95,690 users (those who does not have home and work place in the same cell home and work place and have at least 10 tweets).
Once we have assigned to each user a home and a work cell we can compute the commuting flows for each cell. The commuting flow $\vec{T}_{ij}$ is defined as

$$\vec{T}_{ij} = |T_{ij}| \hat{n}_{ij}$$

(2.1)

where, $|T_{ij}|$ is the number of people commuting from $i$ to $j$ and $\hat{n}_{ij}$ is an unitary vector that points from $i$ to $j$. For each cell we can define an effective vector $\vec{T}_{i}^{\ast}$, which is the result of the sum of all the vectors starting at $i$, this is

$$\vec{T}_{i}^{\ast} = \sum_{j} \vec{T}_{ij}$$

(2.2)

With this definition, those cells which have a prefered direction of commuting will have a big vector pointing on that direction, but those cells where the people commute without preferences will have a vector with modulus almost zero. In Figure 2.2 we have the result of $\vec{T}_{i}^{\ast}$ at each cell of Manchester and Liverpool. We can see how the vectors point outwards the city centers (in principle the vectors must point towards the city center, but we have plot them outwards in order to make a plot easier to understand).

**Figure 2.2:** Vector map of London commuting. The bigger the arrows the most people that commutes in that direction.

We can also measure the distribution of population of the city. In Figure 2.3 we have plotted the population inside a circle $m$ versus the radius $R$ of the circle, and as we can see
it fits very well to the function $m(R) = m^*(1 - \exp(-\beta r))$, where $m^*$ is the total population of the city and $\beta$ a parameter that controls the size of the city. On the other hand we can also plot the distribution of population per cell, and in Figure 2.4 we can see how the population per cell follow a long-tailed distribution, which means that we can find cells with a lot of population and other cells with almost no population.

**Figure 2.3:** Population inside a circle of radius $R$ centered at the city center. We have fitted the data to the function $m(R) = \kappa(1 - \exp(-\beta r))$, where $\kappa$ is the total population of the London urban area.

**Figure 2.4:** Probability distribution of the population per cell. As it can be observed, the distribution is fat-tailed.
Finally, another interesting measure is the distribution of the commuting distances. In Figure 2.5 we can see that the urban areas with conurbations and urban areas with only one city have different distributions of commuting distances. As expected in conurbations the commuting distances are bigger than in normal cities, because there will be some people that commute from one city of the conurbation to another inside the same conurbation.

![Figure 2.5: Distribution of the commuting distances. The distribution changes in function of the urban area under study, in fact for one single city (A) the distribution seems to follow an exponential decay, while in conurbations (B) looks like the decaying is fat-tailed. Notice than in (A) the plot is log-linear and in (B) is log-log.](image)

### 2.2 Models

In this section we will present the two main models that have been used in the modelling of commuting: gravity and radiation.

#### 2.2.1 Gravity models

In the simplest form of the gravity law, the flow between two positions $i$ and $j$ is proportional to the product of the origin population $m_i$ and destiny population $m_j$, and inversely proportional to the travel cost between the two positions $f$. In the gravity models the travel cost between the positions is a function of the distance between the positions, so $f = f(d_{ij})$. 

---

2.2. Models
Therefore the flow from \(i\) to \(j\) is given by
\[
\vec{T}^g_{ij} \propto \frac{m_i^\alpha m_j^\beta}{f(d_{ij})} \hat{n}
\] (2.3)

where \(f(d_{ij})\) is usually chosen as an exponential \(f_e(d_{ij}) = \exp(d_{ij}/d_0)\) or as a power law \(f_p(d_{ij}) = d_{ij}^\delta\) and \(\hat{n}\) is an unitary vector that points from \(i\) to \(j\). In our case we will focus in the exponential case, because as shown in [23] is better in reproducing results than the power law decaying function. We will also take for simplicity \(\alpha = \beta = 1\), so our expression for the flow from \(i\) to \(j\) is
\[
\vec{T}^g_{ij} = \kappa m_i m_j e^{-d_{ij}/d_0} \hat{n}
\] (2.4)

where \(\kappa\) is a constant such that \(T_i = \kappa m_i\), where \(T_i = \sum_j |\vec{T}_{ij}|\) is the number of commuters starting at the cell \(i\). Therefore, at cell \(i\) the commuter vector field defined at 2.2 is
\[
\vec{T}^g(i) = \kappa m_i \sum_j m_j e^{-d_{ij}/d_0} \hat{n}
\] (2.5)

Notice that this equations can be easily generalized to the continuum changing \(i\) by \(\vec{r}\) and \(j\) by \(\vec{r}'\) and the vector fields read as
\[
\vec{T}^g(\vec{r}) = \kappa m(\vec{r}) \int m(\vec{r}') e^{-d(\vec{r},\vec{r}')/d_0} \hat{n} d\vec{r}'
\] (2.6)

### 2.2.2 Radiation models

As we have said before this model is inspired in radiating and absorbing particles, let’s see now how it works:

1. We associate to every particle \(X\) emitted from the location \(i\) an absorption threshold \(z_X^{(i)}\). The higher the threshold the less likely to be absorbed. The threshold \(z_X^{(i)}\) is defined as the maximum number obtained after \(m_i\) extractions of the distribution \(p(z)\), where
2.2. Models

$m_i$ is the population of the site $i$. We will show late that the choice of $p(z)$ do not affect to the final results.

2. The locations surrounding $i$ have a certain probability to absorb the particle $X$: $z_X^{(j)}$ is the absorbance of the location $j$ for the particle $X$. The absorbance $z_X^{(j)}$ is defined as the maximum number obtained after $m_j$ extractions of the distribution $p(z)$, where $m_j$ is the population of the site $j$.

3. The particle $X$ is absorbed by the closest location whose absorbance is greater than its absorption treshold.

4. Repeat this process for all the emitted particles of all the sites.

In Figure 2.6 we can see a simple example of how works the radiation model.

![Figure 2.6](image)

**Figure 2.6:** an individual applies for jobs in all counties and collects potential employment offers. The number of job opportunities in each county is $m_j/n_{jobs}$, chosen to be proportional to the resident population $m_j$. Each offer’s attractiveness (benefit) is represented by a random variable with distribution $p(z)$, the numbers placed in each county representing the best offer among the $m_j/n_{jobs}$ trials in that area. Each county is marked in green (red) if its best offer is better (lower) than the best offer in the home county (here $z = 10$). An individual accepts the closest job that offers better benefits than his home county. In the shown configuration the individual will commute to the closest county whose benefit $z = 13$ exceeds the home county benefit $z = 10$. Figure and caption from [14].

The vector field given by this model is

$$\vec{T}_{ij} = \kappa m_i \frac{m_i m_j}{(m_i + s_{ij})(m_i + m_j + s_{ij})} \hat{n}$$

(2.7)
where $m_i$ is the population of the starting place, $m_j$ the population of the destination and $s_{ij}$ is the total population inside a circle of radius $R = d_{ij}$ minus the population in $i$ and $j$.

The analytical derivation of this equation can be found in Appendix B.
Chapter 3

Results & Discussion

In this section we will present our theoretical and empirical results. First we will present a comparison of the two models predictions when reproducing the data and then we will present our method for city detection.

3.1 Fluxes

In order to study and compare the two methods we have studied the fluxes of the vector field through different circumferences. For the gravity model we will present some theoretical and analytical results and we will compare them with the observed data. For the case of radiation model we will present only numerical results and we will also compare them with the data.

3.1.1 Gravitational Fluxes

In all the equations related with the gravity model appears the constant $d_0$, so it is very important to obtain its value in each city. One way to obtain the of $d_0$ we can interpret that the cell $i$ with population $m_i$ creates a vector field at $j$ of the form

$$\vec{W}^g_i(j) = \kappa m_i e^{-d_{ij}/d_0} \hat{n}$$  \hspace{1cm} (3.1)
such that the commuter vector of \( j \) to \( i \) is given by \( \vec{T}_{ij}^g = m_j \vec{W}_{ij}^g(j) \), in analogy with the electric fields, where the force between particles is given by \( \vec{F} = q \vec{E} \). Therefore, computing the vectorial flux of the vector map \( \vec{W}_i^g \) created by one particle through a circumference \( C \) of radius \( R \) and fitting the theoretical prediction with the data we can obtain the value of the constant \( d_0 \). The flux of a vector map is defined as

\[
\Phi_{C,i}^g = \oint_C \vec{W}_i^g d\vec{n}
\]

(3.2)

where \( d\vec{n} \) is a normal vector perpendicular to \( C \) in each point of the perimeter. By means of the divergence theorem

\[
\Phi_{C,i}^g = \int_A \nabla \cdot \vec{W}_i^g dS
\]

(3.3)

where the integration is done over the area \( A \). The divergence of the vector map created by a cell centered at \( (x_i, y_i) \) is

\[
\nabla \cdot \vec{W}_i^g = \kappa m_i \exp\left(-\frac{\sqrt{(x-x_i)^2+(y-y_i)^2}}{d_0}\right) \left(\frac{1}{\sqrt{(x-x_i)^2+(y-y_i)^2}} - \frac{1}{d_0}\right)
\]

(3.4)

therefore, the integral that gives us the flux of the vector field through the circumference is

\[
\int_A \kappa m_i \exp\left(-\frac{\sqrt{(x-x_i)^2+(y-y_i)^2}}{d_0}\right) \left(\frac{1}{\sqrt{(x-x_i)^2+(y-y_i)^2}} - \frac{1}{d_0}\right) dS
\]

(3.5)

In general this integral is difficult to compute, but if the cell is at \( (0, 0) \) the problem reduces to one that can be solved analytically. With the cell centered at \( (0, 0) \) the divergence becomes

\[
\nabla \cdot \vec{W}_i^g = \kappa m_i \exp\left(-\frac{\sqrt{x^2+y^2}}{d_0}\right) \left(\frac{1}{\sqrt{x^2+y^2}} - \frac{1}{d_0}\right)
\]

(3.6)
and the flux is now

\[ \int_A \kappa m_i \exp \left( -\frac{\sqrt{x^2 + y^2}}{d_0} \right) \left( \frac{1}{\sqrt{x^2 + y^2}} - \frac{1}{d_0} \right) dS \]  

(3.7)

Since the area in which we are performing the integration is a circle and we have terms of the form \(x^2 + y^2\) we can use polar coordinates, obtaining finally the result of the flux

\[ \Phi_{C,i}^g = \kappa m_i \int_0^{2\pi} \int_0^R \exp \left( -\frac{r}{d_0} \right) \left( \frac{1}{r} - \frac{1}{d_0} \right) r dr d\theta = 2\pi \kappa m_i R e^{-R/d_0} \]  

(3.8)

This equation can be understood as the commuting version for the first Maxwell equation in his integral form, that in the case of electrodynamics reads as \(\Phi_E = \oint_A E dS = 4\pi Q\). In Figure 3.1a we have plotted the divergence of the vector field and in 3.1b the flux in function of the circle radius. It is interesting to notice that the maximum flux is achieved at \(r^* = d_0\), and this is due to the fact that \(\vec{\nabla} \vec{W}_i(g) > 0\) if \(|\vec{r}| < d_0\) and \(\vec{\nabla} \vec{W}_i(g) < 0\) if \(|\vec{r}| > d_0\). The value of the distance \(d_0\) gives us the decaying factor of the function when \(R > d_0\). This means that the flux is almost zero when \(R\) is much greater than \(d_0\) (for example if \(R = 10d_0\) we have that \(\Phi_{l,C}^g(10d_0)/\Phi_{l,C}^g(d_0) \approx 10^{-3}\)).

![Figure 3.1](image)

**Figure 3.1:** (A) divergence of the vector field created by a cell of population \(m_i = 1\) at \((0,0)\). The function changes the sign at \(R = d_0\). An important property of this function is that \(\int_0^{d_0} \vec{\nabla} \vec{W}_i(g) dR = -\int_{d_0}^{\infty} \vec{\nabla} \vec{W}_i(g) dR\), this is that the blue and the red area are the same. In (B) we have the value of the flux in function of the circle radius, and as expected we have a maximum at \(R = d_0\).
Chapter 3. Results & Discussion

For the case of a cell at an arbitrary \((x_i, y_i)\) we first perform the shift coordinates system origin from \((0, 0)\) to \((x_i, y_i)\), and we will use \((\tilde{x}, \tilde{y})\) for the new coordinates. If we use now polar coordinates we need to notice that the circumference through which we want to compute the flux is centered at \((-\tilde{x}_i, -\tilde{y}_i)\), and now the polar coordinates of that circle are

\[
\tilde{r} = r_i \cos(\theta - \phi_i) + \sqrt{R^2 - r_i^2 \sin^2(\theta - \phi_i)},
\]

where \(r_i = \sqrt{x_i^2 + y_i^2}\) and \(\phi_i = \arctan \frac{y_i}{x_i}\). The change of coordinates is depicted in Figure 3.2. The integral is then

\[
\Phi_{g, i}^C = \kappa m_i \int_0^{2\pi} \int_0^r r_i \cos(\theta - \phi_i) + \sqrt{R^2 - r_i^2 \sin^2(\theta - \phi_i)} \exp \left( - \frac{r}{d_0} \right) \left( \frac{1}{r} - \frac{1}{d_0} \right) rdrd\theta (3.9)
\]

and integrating over \(r\) we obtain

\[
\Phi_{g, i}^C = \kappa m_i \int_0^{2\pi} \left( r_i \cos(\theta - \phi_i) + \sqrt{R^2 - r_i^2 \sin^2(\theta - \phi_i)} \right) \times \exp \left( - (r_i \cos(\theta - \phi_i) + \sqrt{R^2 - r_i^2 \sin^2(\theta - \phi_i)})/d_0 \right) d\theta (3.10)
\]
and due to symmetry reasons the value of the flux can not depend on the value of $\phi_i$, and then we can eliminate it in the integral

$$
\Phi_{C,i}^g = \kappa m_i \int_0^{2\pi} \left( r_i \cos \theta + \sqrt{R^2 - r_i^2 \sin^2 \theta} \right) \exp \left( - \left( r_i \cos \theta + \sqrt{R^2 - r_i^2 \sin^2 \theta} \right) / d_0 \right) d\theta
$$

(3.11)

However this equation is not solvable analitically, and if we want to compute it we will need to used numerical methods. The results for the above equation for different values of $r_i$ have been plotted in Figure 3.3, and as we can see the form of the curves are qualitatively equal and all have a tail that decays exponentially, but with the peak displaced and with a lower maximum.

![Figure 3.3: Numerical computation of the flux of $\vec{W}^g$ created by cells at different $r_i/d_0$ through circumferences of radius $R/d_0$ centered at $(0, 0)$. Inset plot: log-linear plot that shows the exponential decaying of the tails.](image)

Note that all these equations depend on the constant $\kappa$ and $d_0$ that must be obtained from the real data. In order to obtain these values we can use equation 3.8 and fit it to the observed data. In order to compute the empirical flux $\Phi_{C,i}$ of the field $\vec{W}_i$ we can use the fact that we know the empirical value of $\vec{T}_{ij}$, and therefore we can obtain the empirical value of the vector field as $\vec{W}_i = \frac{T_{ij}}{m_j}$. Thence, centering a system of coordinates at cell $i$ with
population $m_i$, and computing the flux of the vector field $\vec{W}_i$ (the vector field created by $i$) through a circumference of radius $R$ and repeating this process for different cells we can check if the empirical curve is in accordance with the theoretical. In fact, if we divide the flux of $\vec{W}_i$ by $m_i$ we must obtain the universal curve $\Phi = 2\pi \kappa R e^{-R/d_0}$. The results for the case of London and Manchester-Liverpool are plotted in 3.4, and we can see how our model predict the fluxes created by individual cells. The fits gives us the values of $d_0$ for different cities.

Finally, we can run another test for the gravity model, computing the total flux of people through a circumference of radius $R$ centered at the city center. We have compute the observed and the expected flux of people commuting for London and plotted them together in Figure 3.5, and as we can see our model can predict very well the observed fluxes of people, with a correlation coefficient $R^2 = 0.88$.

### 3.1.2 Radiation Model Fluxes

In the case of the radiation model we can not use the same decomposition as in the gravity model, such that $\vec{T}_{ij} = m_j \vec{W}_i(j)$ because $\vec{T}_{ij}$ is not linear in $m_j$. However, we are able to compute define the vector field $\vec{W}_i(j)$ for some simple cases. One case is the one given by a
3.1. Fluxes

![Graphs showing experimental and predicted fluxes](image)

**Figure 3.5:** Measured flux of people commuting through a circumference centered at the city center and the predicted one using the gravity model for the case of London.

uniform distribution of population in the plane, this is \( \rho(r, \theta) = \rho \). In this case we have

\[
\begin{align*}
  s_{ij} &= \int_0^{2\pi} \int_0^{r_{ij}} \rho(r, \theta) r dr d\theta - \rho(r_i, \theta_i) - \rho(r_j, \theta_j) = \rho \pi r_{ij}^2 - 2\rho \\
  \Rightarrow \quad T_{ij}^- &= \kappa \frac{\rho^3}{(\rho \pi r_{ij}^2 - \rho) \rho \pi r_{ij}^2} \hat{n} = \kappa \frac{\rho}{\pi^2 r_{ij}^2 - \pi r_{ij}^2} \hat{n}
\end{align*}
\]

(3.12)

therefore

\[
\vec{T}_{ij}^- = \kappa \frac{\rho^3}{(\rho \pi r_{ij}^2 - \rho) \rho \pi r_{ij}^2} \hat{n} = \kappa \frac{\rho}{\pi^2 r_{ij}^2 - \pi r_{ij}^2} \hat{n}
\]

(3.13)

which is the case of a gravity law with a power-law decaying function \( f(d_{ij}) \propto d_{ij}^{-\gamma} \) with \( \gamma = 4 \).

However, it is important to notice that the case of a uniform distribution of the population is not fulfilled in reality, as has been shown in Figure 2.4. Also, the term \( s_{ij} \) it is not easily calculable in general, even for simple population distributions, therefore all the predictions of the radiation model will be numerical. Since the model have no free parameters to be adjusted we can directly implement it and compare with the data. The observed results of the commuting fluxes through circumferences of different radius \( R \) centered at the city center for London are plotted in 3.6 together with the predicted ones. As we can see, the predictions are worse than in the case of the gravity model, with a correlation of \( R^2 = 0.39 \).
Figure 3.6: Measured flux of people commuting through a circumference centered at the city center and the predicted one using the radiation model for the case of London.

3.2 Potential theory

In Section 3.1.1 we have shown that the vector field under study fulfill some kind of first Maxwell law. In this section we will see how the same vector field also follow another Maxwell law. Let’s start computing the curl of this vector field. In 2D the curl of a vector field $\vec{F}(x,y) = (U(x,y), V(x,y))$ is defined as

$$\nabla \times \vec{F} = (0, 0, U_y - V_x)$$  \hspace{1cm} (3.14)

and it turns out that the curl of the vector field $\vec{W}_g$ is

$$\nabla \times \vec{W}_g = (0, 0, 0)$$  \hspace{1cm} (3.15)

This equation could be interpreted as the commuting version of the third Maxwell law, also known as the Maxwell-Faraday law, which has the form $\nabla \times \vec{E} = \partial_t \vec{B}$. Since the curl of the vector field vanishes the vector field is conservative, and therefore we can write the vector field as the divergence of a scalar potential $V_i^g$

$$\vec{W}_i^g = -\nabla V_i^g$$  \hspace{1cm} (3.16)
Solving the above equation one finds out that the potential created by the cell \((x_i, y_i)\) with population \(m_i\) is

\[
V^g_i(x, y) = -\kappa m_i d_0 \exp \left( -\frac{\sqrt{(x - x_i)^2 + (y - y_i)^2}}{d_0} \right)
\]  

(3.17)

Now, with the values of \(d_0\) obtained with the fitting we can plot the potential created by all the cells of the city using

\[
V^g_{total}(x, y) = -\kappa \sum_i m_i d_0 \exp \left( -\frac{\sqrt{(x - x_i)^2 + (y - y_i)^2}}{d_0} \right)
\]  

(3.18)

The form of the potential for Manchester-Liverpool and some equipotential lines are plotted in Figure 3.7.

![Figure 3.7](image)

**Figure 3.7:** (A) Potential created by the urban area of Manchester-Liverpool and (B) some equipotential lines of the potential. The potential exhibits some clear wells that will act as attracting points for the commuters.

### 3.2.1 City detection

Now that we have shown that the gravity model can reproduce the commuting fluxes on a city, we can use it in order to which zones of the urban areas under study behave as a city. Our definition of a city is those points that minimize the potential defined in Equation 3.18. This definition is based in the fact that those points are the ones towards which the people
feels attracted, as it occurs in classical mechanics. Therefore, to detect cities on a urban area we just need to compute the potential for that area and find the points that minimize that potential.

Applying this method to the case of London we have found that this urban area has only one center of attraction, as we have expected. It is interesting to notice that the minimum of the potential does not have to coincide with the point where the concentration of population is maximum. The result after applying this method for the case of London is plotted in Figure 3.8a.

In the case of Manchester-Liverpool we have found that many city centers appear. Most of the detected cities are the ones expected, such as Manchester, Liverpool, Warrington, Sheffield and Leeds among others. However it is interesting to notice that there are a lot of cities that are not detected by the method, in fact more than 30 cities are in the area under study and we have only detected 13. This is showing that all this cities does not behave as attracting centers for the commuting, and according to our definition this cities are not real cities. In fact, we can say that these cities are integrated into a big urban area that behaves as one big city. The results are plotted in Figure 3.8b.

![Figure 3.8: Results of the method for city detection applied to London (A) and Manchester-Liverpool (B).](image-url)
Chapter 4

Conclusions & Perspective

In this work we have studied the commuting problem applying and comparing the gravity and radiation model. First, we have shown that the gravity model is better than the radiation when predicting the commuting, and we have showed this by computing the fluxes of people through different closed curves. After showing this we have derived the field equations for the gravity model and showed that they can be used to predict fluxes of people through closed curves, as it happens with the usual Maxwell equations. After this, we have used the fact that the vector field for the gravity model comes from a potential to compute the potential created by the city. Finally, we have defined the city centers as those points of the area under study that minimize the potential created by the population. We have also shown that in the case of London only one center is detected, as we expect, but in the case of Manchester-Liverpool we have only detected some of the cities, which means that a lot of cities does not behave as attractors for commuting. Therefore, we have defined a method that can be applied to conurbations in order to observe how many cities are present in these urban areas.

As further work, we propose that this model can be applied to detect the boundaries of the cities using the form of the potential and its equipotential curves. Also, this model can be applied to a priori mono-center cities, in order to study is they are really mono-center, or if they have more than one center of attraction. Another step that can be done in the future is to develop a method to predict the value of $d_0$ without the need of empirical data, because sometimes we have areas of interest in which we do not have data of the mobility.
Appendix A

Other regions: Paris, Holland, Colonia

In this section we present other urban areas that we have studied during this work: Paris, Hollanda and Colonia. Paris is a single city, while Holland and Colonia have some conurbations.

In the case of Paris we have only detected one center, as in the case of London. In the cases of Holland and Colonia we have detected more centers, but as in the case of Manchester-Liverpool, we have detected less cities than the expected. All the results are plotted together in Figure A.1
Appendix A. Other regions: Paris, Holland, Colonia

**Figure A.1:** (A-C) Fit of the model to the real data, (D-F) potential created by the population, (G-I) results of the method for city detection. (A, D, G) Paris, (B, E, H) Holland, (C, F, I) Colonia.
Appendix B

Radiation model derivation

The probability of emission/absorption between any two locations can be computed analytically, and therefore obtain predictions of the flows between those locations.

The probability that a particle emitted at \(i\) with population \(m_i\) is absorbed at \(j\) with population \(m_j\), given that \(s_{ij}\) is the total population (except \(i\) and \(j\)) inside the circle of radius \(r_{ij}\) centered at \(i\) (\(r_{ij}\) is the distance between the sites \(i\) and \(j\)) is

\[
P(1|m_i, m_i, s_{ij}) = \int_0^\infty dz P_{m_i}(z) P_{s_{ij}}(<z) P_{m_j}(>z) \tag{B.1}
\]

where \(P_{m_i}(z)\) is the probability that the maximum value extracted from \(p(z)\) after \(m_i\) trials is \(z\).

Similarly, \(P_{s_{ij}} = p(<z)^{s_{ij}}\) is the probability that \(s_{ij}\) numbers extracted from \(p(z)\) distribution are all less that \(z\), and \(P_{m_j} = 1 - p(<z)^{m_j}\) is the probability that among \(m_j\) extractions at least one is greater than \(z\). In view of all of this we have

\[
P_{m_i}(<z) = \int_0^z dt P_{m_i}(t) \implies P_{m_i}(z) = \frac{dP_{m_i}(<z)}{dz} = m_i p(<z)^{m_i-1} \frac{dp(<z)}{dz} \tag{B.2}
\]

and using the definition of \(P_{m_i}(<z)\) we obtain \(P_{m_i}(<z) = p(<z)^{m_i}\) and then

\[
P_{m_i}(z) = m_i p(<z)^{m_i-1} \frac{dp(<z)}{dz} \tag{B.3}
\]

Hence, \(P(1|m_i, m_i, s_{ij})\) represent the probability that one particle is emitted at \(i\) and it is
not absorbed by the closest locations \( s_{ij} \) and it is finally absorbed by \( j \). The integral can be evaluated analytically as

\[
P(1|m_i, m_i, s_{ij}) = m_i \int_0^\infty dz dp(<z) \left( p(<z)^{m_i+s_{ij}-1} - p(<z)^{m_i+m_j+s_{ij}-1} \right) \frac{dp(<z)}{dz} \frac{m_i m_j}{(m_i + s_{ij})(m_i + m_j + s_{ij})}
\]

where we can see that the probability \( P(1|m_i, m_i, s_{ij}) \) is independent of the choice of the distribution \( p(z) \).

Now, the probability \( P(T_{i1}, T_{i2}, ..., T_{iL}) \) for a sequence of absorptions \( (T_{i1}, T_{i2}, ..., T_{iL}) \), where \( T_{im} \) represent the absorption of particle emitted at \( i \) by \( j \), is given by a multinomial distribution

\[
P(T_{i1}, T_{i2}, ..., T_{iL}) = \prod_{j\neq i} \frac{T_{ij}!}{T_{ij}^j p_{ij}^{T_{ij}}} ; \quad \sum_{j\neq i} T_{ij} = T_i
\]

where \( T_i \) is the total number of particles emitted at \( i \), and \( p_{ij} = \frac{m_i m_j}{(m_i + s_{ij})(m_i + m_j + s_{ij})} \). Here we can use the fact that \( T_i \) is proportional to the population of the location \( i \), and therefore \( T_i = m_i \kappa \).

To obtain the probability that \( T_{ij} \) particles emitted from \( i \) are absorbed at \( j \) can be computed by marginalizing the above distribution

\[
P(T_{ij}|m_i, m_i, s_{ij}) = \sum_{T_{ik}, k \neq i, j} P(T_{i1}, T_{i2}, ..., T_{iL}) = \frac{T_i!}{T_{ij}!(T_i - T_{ij})!} p_{ij}^{T_{ij}} (1 - p_{ij})^{T_i - T_{ij}}
\]

One can notice that the above distribution is a binomial with an expected value \( \langle T_{ij}^r \rangle \) given by

\[
\langle T_{ij}^r \rangle = \kappa m_i \frac{m_i m_j}{(m_i + s_{ij})(m_i + m_j + s_{ij})}
\]
and with variance

\[ \sigma^2[T_{ij}] = \kappa m_i \frac{m_i m_j}{(m_i + s_{ij})(m_i + m_j + s_{ij})} \left(1 - \frac{m_i m_j}{(m_i + s_{ij})(m_i + m_j + s_{ij})}\right) \]  \hspace{1cm} (B.8)
Bibliography


