Fuzzifying topology induced by a strong fuzzy metric

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Abstract

A construction of a fuzzifying topology induced by a strong fuzzy metric is presented. Properties of this fuzzifying topology, in particular, its convergence structure are studied. Our special interest is in the study of the relations between products of fuzzy metrics and the products of the induced fuzzifying topologies.

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1. Introduction

After fuzzy metric was defined by I. Kramosil and J. Michalek [18] and later redefined in a slightly different way by A. George and P. Veeramani [7], many researchers became interested in the topological structure of a fuzzy metric space. In particular, properties of topologies induced by fuzzy metrics were studied by A. George and P. Veeramani, V. Gregori, S. Romaguera, A. Sapena, D. Mihet, S. Morilas et al.. see e.g. [7], [8], [13], [12], [9], [10], [23]. In most papers, the topology induced by a fuzzy metric is actually an *ordinary*, that is a *crisp* topology on the underlying set. However recently some authors showed interest in a *fuzzy-type topological structures* induced by fuzzy (pseudo-)metrics, see [35], [20][1]. It is also the principal goal of the

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[1]We are grateful to an anonymous referee for paying our attention to these works

present paper to study this problem.

To state our idea more precisely, we recall the three basic approaches to the concept of a fuzzy topology[2]. The first approach was initiated by Zadeh's student C.L. Chang [3] and soon it was conceptually generalized by J.A. Goguen [6]. It realizes an L-fuzzy topology on a set X (where L is a complete lattice, or more generally a cl-monoid) as a certain crisp subset T of the L-powerset of a set X, that is $T \subseteq L^X$. We refer to this view on a fuzzy topology as *a crisp-fuzzy approach*. The second approach, first presented by U. Höhle [14] and then independently rediscovered by M.S. Ying [31], realizes an L-fuzzy topology $\mathcal{T}$ on a set X as an L-fuzzy subset of the powerset of X, that is as a mapping $\mathcal{T} : 2^X \to L$, satisfying certain conditions. Following M.S. Ying, such structures are usually called fuzzifying topologies. We view this approach as *a fuzzy-crisp* one. Finally, the last one of the three approaches interprets a fuzzy topology as a fuzzy subset $\mathcal{T}$ of the fuzzy powerset of a set X, that is as a mapping $\mathcal{T} : L^X \to L$. It was first presented (independently) by T. Kubiak [19] and by the second named author of this paper [28, 29]. We call it by *a fuzzy-fuzzy* approach.

Developing the general fuzzy viewpoint on the topological structure of a fuzzy metric space, here we (as well as the authors of the both above mentioned papers [35], [20]) start with the fuzzy-crisp approach. It is just the main goal of the present paper to develop the foundations of this approach, that is to work out the concept of a fuzzfying topology induced by a strong fuzzy metric and to present our results obtained in this field so far.

The structure of the paper is as follows. The next section, Preliminaries, is divided into four, rather independent subsections, in which we expose basic definitions and results concerning fuzzy metrics that are needed in the sequel, give an introduction into the theory of fuzzifying topologies, present a construction of a fuzzifying topology from an ordered family of topologies, and apply this construction to describe the product of fuzzifying topologies. In the third section, we describe the construction of a fuzzifying topology induced by a strong fuzzy metric, consider some properties of this construction and describe the convergence structure of a fuzzifying topology induced by a strong fuzzy metric. In Section 4 we define and study products, separately finite and countable, of fuzzy metrics and show that the fuzzifying topology

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[2]Here we restrict ourselves to the fixed basis fuzzy topologies as it is specified by S.E. Rodabaugh [24]

induced by products of strong fuzzy metrics coincides with the products of the fuzzifying topologies induced by these fuzzy metrics. In the last Section 5, Conclusions, we state some problems which we could not solve in the process of writing this paper and discuss several prospectives for the future work. Besides we make here some remarks concerning the relations of our approach and the approaches proposed in [35] and [20] to the study of fuzzifying topology induced by a fuzzy metric.

2. Preliminaries

2.1. Fuzzy metrics

Basing on the concept of a statistical metric introduced by K. Menger [22], see also [27], I.Kramosil and J.Michalek [18] introduced the notion of a fuzzy metric. Later A. George and P. Veeramani [7] slightly modified the original concept of a fuzzy metric. At present in most cases research involving fuzzy metrics is done in the context of George-Veeramani definition. This approach is accepted also in our paper.

Let X be a non-empty set, $* : [0,1] \times [0,1] \to [0,1]$ be a continuous t-norm (see e.g. [22], [27])) and $\mathbb{R}^+ = (0, +\infty)$.

Definition 2.1. [7] *A fuzzy metric on the set X is a pair $(M, *)$, or simply M where $M : X \times X \times \mathbb{R}^+ \to [0,1]$ (that is M is a fuzzy subset of $X \times X \times \mathbb{R}^+$), satisfying the following conditions for all $x, y, z \in X$, $s, t \in \mathbb{R}^+$:*

(1GV) $M(x, y, t) > 0$;
(2GV) $M(x, y, t) = 1$ *if and only if* $x = y$;
(3GV) $M(x, y, t) = M(y, x, t)$;
(4GV) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;
(5GV) $M(x, y, -) : \mathbb{R}^+ \to [0,1]$ *is continuous.*

If $(M, *)$ is a fuzzy metric on X, then the triple $(X, M, *)$ is called a fuzzy metric space.
If axiom (2GV) is replaced by a weaker axiom

(2′GV) *if $x = y$, then $M(x, y, t) = 1$*

we get definitions of a fuzzy pseudo-metric, and the corresponding fuzzy pseudo-metric space.

Note that axiom (4GV) combined with axiom (2′GV) implies that the fuzzy metric $M(x, y, t)$ is non-decreasing in the third argument.

Definition 2.2. [12] *A fuzzy (pseudo-)metric M on X is called strong if, in addition to the properties* (1GV) - (5GV), *the following stronger version of axiom* (4GV) *is satisfied*

(4^{s}GV) $M(x, z, t) \geq M(x, y, t) * M(y, z, t)$ *for all $x, y, z \in X$ and for all $t > 0$.*

Definition 2.3. [13] *A fuzzy metric M on X is said to be stationary, if M does not depend on t, i.e. if for each $x, y \in X$, the function $M_{x,y}(t) = M(x, y, t)$ is constant. In this case we write $M(x, y)$ instead of $M(x, y, t)$.*

The next concept implicitly appears in [12]:

Definition 2.4. *Given two fuzzy metric spaces $(X, M, *_M)$ and $(Y, N, *_N)$, a mapping $f : X \to Y$ is called continuous if for every $\varepsilon \in (0, 1)$, every $x \in X$ and every $t \in \mathbb{R}^+$ there exist $\delta \in (0, 1)$ and $s \in \mathbb{R}^+$ such that $N(f(x), f(y), t) > 1 - \varepsilon$ whenever $M(x, y, s) > 1 - \delta$. In symbols:*

$$\forall \varepsilon \in (0, 1), \forall x \in X, \forall t \in \mathbb{R}^+ \; \exists \, \delta \in (0, 1), \exists s \in \mathbb{R}^+ \; such \; that$$

$$M(x, y, s) > 1 - \delta \Longrightarrow N(f(x), f(y), t) > 1 - \varepsilon$$

Fuzzy metric spaces as objects and continuous mappings between them as morphisms form a category which we denote **FuzMS**.

In a fuzzy (pseudo-)metric space $(X, M, *)$ a (crisp) topology can be introduced on X as follows, see e.g. [7], [8]. Given a point $x \in X$, a number $\varepsilon \in (0, 1)$ and $t \in \mathbb{R}^+$, we define the ball at the level t with the center x and the radius ε as the set $B^M(x, \varepsilon, t) = \{y \in X \mid M(x, y, t) > 1 - \varepsilon\}$. Obviously

$$t \leq s \Longrightarrow B^M(x, \varepsilon, t) \subseteq B^M(x, \varepsilon, s) \text{ and } \varepsilon \leq \delta \Longrightarrow B^M(x, \varepsilon, t) \subseteq B^M(x, \delta, t).$$

As shown in [7], [8], the family $\{B^M(x, \varepsilon, t) \mid x \in X, t \in \mathbb{R}^+, \varepsilon \in (0, 1)\}$ is a base for a topology on X; we denote this topology by T^M. One can easily verify the following well-known proposition:

Proposition 2.5. [7] *Given two fuzzy metric spaces $(X, M, *_M)$ and $(Y, N, *_N)$ a mapping $f : (X, M, *_M) \to (Y, N, *_N)$ is continuous if and only if the mapping of the induced topological spaces $f : (X, T^M) \to (Y, T^N)$ is continuous.*

Hence, by assigning to a fuzzy metric space $(X, M, *)$ the induced topological space (X, T^M) and assigning to a continuous mapping $f : (X, M, *_M) \to (Y, N, *_N)$ the mapping $f : (X, T^M) \to (Y, T^N)$, we get a functor

$$\Phi : \textbf{FuzMS} \longrightarrow \textbf{TOP}$$

where **TOP** is the category of topological spaces and their continuous mappings.

The following example gives a standard construction of a fuzzy metric from a usual metric on the same set:

Example 2.6. Let (X, d) be a metric space. Let M_d be the fuzzy set defined on $X \times X \times \mathbb{R}^+$ by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

Then $(M_d, \cdot)$, where the symbol $\cdot$ stands for the product t-norm, is a fuzzy metric on X called the *standard fuzzy metric* generated by d. The topologies induced by the metric d and the fuzzy metric M_d coincide: $T_d = T^{M_d}$. Further, it is well known that $(X, M_d, \cdot)$ is strong [26].

2.2. Fuzzifying topologies

The concept of a fuzzifying topology (under the name of *a fuzzy topology*) was introduced in 1980 by U. Höhle [14], as a certain probabilistic modification of the concept of topology. Later, in 1991, the same concept was independently introduced by M.S. Ying [31], under the name of a *fuzzifying topology*. M.S. Ying rediscovered this concept by making a deep logical analysis of topological axioms and different properties of topological spaces. Later the theory of fuzzifying topologies got a profound development in the works by different authors, see e.g. [32], [33], [34], [16], et. al..

Definition 2.7. [14], [31] *Given a set X, a mapping $\mathcal{T} : 2^X \to [0, 1]$ is called a fuzzifying topology on X if it satisfies the following axioms:*

1. $\mathcal{T}(\emptyset) = \mathcal{T}(X) = 1$;
2. $\mathcal{T}(A \cap B) \geq \mathcal{T}(A) \wedge \mathcal{T}(B) \ \forall A, B \in 2^X$;
3. $\mathcal{T}\left(\bigcup_i A_i\right) \geq \bigwedge_i \mathcal{T}(A_i) \ \forall \{A_i : i \in I\} \subseteq 2^X$.

The pair $(X, \mathcal{T})$ is called a fuzzifying topological space.

Remark 2.8. The intuitive meaning of the value $\mathcal{T}(A)$ is the degree to which a set $A \subseteq X$ is open. In particular, an ordinary topology T on a set X can be realized as a fuzzifying topology $\mathcal{T} : 2^X \to \{0, 1\} \subset [0, 1]$ by assigning $\mathcal{T}(A) = 1$ if and only if $A \in T$, and $\mathcal{T}(A) = 0$ otherwise.

Remark 2.9. In an obvious way, the definition of a fuzzifying topology was extended to the concept of an L-fuzzifying topology where L is a complete infinitely distributive lattice: one has just to replace the unit interval $[0, 1]$ in the definition of a fuzzifying topology by a complete infinitely distributive lattice L. However we restrict here to the case $L = [0, 1]$ since fuzzifying topologies obtained from fuzzy metrics always take their values in the interval $[0, 1]$.

Definition 2.10. [14], [31]. *Given two fuzzifying topological spaces $(X, \mathcal{T}^X)$ and $(Y, \mathcal{T}^Y)$, a mapping $f : (X, \mathcal{T}^X) \to (Y, \mathcal{T}^Y)$ is called continuous if*

$$\mathcal{T}^X \left(f^{-1}(B) \right) \geq \mathcal{T}^Y(B) \quad \forall B \subseteq Y.$$

Since the composition $g \circ f : (X, \mathcal{T}^X) \to (Z, \mathcal{T}^Z)$ of two continuous mappings $f : (X, \mathcal{T}^X) \to (Y, \mathcal{T}^Y)$ and $g : (Y, \mathcal{T}^Y) \to (Z, \mathcal{T}^Z)$ is obviously continuous and since the identity mapping id : $(X, \mathcal{T}^X) \to (Y, \mathcal{T}^Y)$ is continuous, we come to the category **FzfTop** of fuzzifying topological spaces as objects and their continuous mappings as morphisms.

2.3. Construction of a fuzzifying topology from a family of crisp topologies

Let a fuzzifying topology $\mathcal{T} : 2^X \to [0, 1]$ on a set X be given and $\alpha \in [0, 1]$. One can easily notice (see also e.g. [29]) that

$$\mathcal{T}_\alpha = \{A \in 2^X : \mathcal{T}(A) \geq \alpha\}$$

is an ordinary topology on the set X; we refer to this topology as the α-level of the fuzzifying topology $\mathcal{T}$. Obviously $\alpha \leq \beta$ implies that $\mathcal{T}_\alpha \supseteq \mathcal{T}_\beta$. Thus a fuzzifying topology $\mathcal{T} : 2^X \to [0, 1]$ can be decomposed into a decreasing family of α-level topologies $\{\mathcal{T}_\alpha : \alpha \in [0, 1]\}$. Moreover, it is easy to see that the decomposition $\{\mathcal{T}_\alpha : \alpha \in [0, 1]\}$ is lower semi continuous, that is

$$\mathcal{T}_\alpha = \bigcap_{\beta < \alpha} \mathcal{T}_\beta, \text{ where } \mathcal{T}_0 = 2^X \text{ as the intersection of the empty family.}$$

In the following we describe a construction allowing to restore a fuzzifying topology from a decreasing family of ordinary topologies:

Let a dense set $K \subseteq [0, 1]$ and family of topologies $\{T_\alpha : \alpha \in K\}$ on a set X be given such that

$$\alpha < \beta, \alpha, \beta \in K \implies T_\alpha \supseteq T_\beta.$$

6

We define a mapping $\mathcal{T} : 2^X \to [0, 1]$ by setting

$$\mathcal{T}(A) = \sup\{\alpha \in K : \ A \in T_\alpha\} \ \ \text{where} \ \sup \emptyset = 0.$$

The mapping $\mathcal{T} : 2^X \to [0, 1]$ thus obtained is a fuzzifying topology. Indeed,

1. $\mathcal{T}(\emptyset) = \mathcal{T}(X) = 1$, since $\emptyset, X \in T_\alpha$ for every α.
2. To show that $\mathcal{T}(A \bigcap B) \geq \mathcal{T}(A) \wedge \mathcal{T}(B)$ for any $A, B \subseteq X$, let $\mathcal{T}(A) = \alpha$, $\mathcal{T}(B) = \beta$ and assume that $\alpha \leq \beta$. In case $\alpha = 0$ the statement is obvious. Otherwise, $\alpha = \sup\{\lambda : \lambda < \alpha, \lambda \in K\}$, and $A, B \in T_\lambda$ for every $\lambda < \alpha, \lambda \in K$. Hence $A \cap B \in T_\lambda$ whenever $\lambda < \alpha, \lambda \in K$, and therefore $\mathcal{T}(A \cap B) \geq \alpha = \mathcal{T}(A) \wedge \mathcal{T}(B)$ by the definition of $\mathcal{T}$.
3. To show that $\mathcal{T}(\bigcup_i A_i) \geq \bigwedge_i \mathcal{T}(A_i)$ for every family $\{A_i : i \in I\} \subseteq 2^X$, let $\bigwedge_i \mathcal{T}(A_i) = \alpha$. In case $\alpha = 0$ the statement is obvious. Otherwise, $\alpha = \sup\{\lambda : \lambda < \alpha, \lambda \in K\}$, and hence $\mathcal{T}(A_i) \geq \lambda$ for every $i \in I$ and every $\lambda < \alpha, \lambda \in K$. Therefore for every $\lambda < \alpha, \lambda \in K$ the family T_λ contains all $A_i, i \in I$. However, this means that $\bigcup_i A_i \in T_\lambda$ for every $\lambda < \alpha, \lambda \in K$, and hence, by the definition of $\mathcal{T}$ we have $\mathcal{T}(\bigcup_i A_i) \geq \alpha = \bigwedge_i \mathcal{T}(A_i)$.

Note that this construction can be generalized for the case when L is a completely distributive lattice and $K \subseteq L$ is its dense subset. However this generalization will involve additional techniques and we do not reproduce it here since it is not needed for our merits.

One can easily prove the following statement, see e.g. [29], [30]:

Proposition 2.11. *Let $(X, \mathcal{T}^X)$, $(Y, \mathcal{T}^Y)$ be fuzzifying topological spaces and let $K \subseteq [0, 1]$ be a dense subset. A mapping $f : (X, \mathcal{T}^X) \to (Y, \mathcal{T}^Y)$ is continuous if and only if all mappings $f : (X, \mathcal{T}_\alpha^X) \to (Y, \mathcal{T}_\alpha^Y)$, $\alpha \in K$ are continuous. ($\mathcal{T}_\alpha^X$ and $\mathcal{T}_\alpha^Y$ are the α-level topologies of the fuzzifying topologies $\mathcal{T}^X$ and $\mathcal{T}^Y$ respectively.)*

Remark 2.12. Assume that $K \subseteq [0, 1]$ is dense and let the family $\{T_\alpha : \alpha \in K\}$ in the previous construction be lower semi continuous, that is $T_\alpha = \bigcap\{T_\lambda : \lambda \in K, \lambda < \alpha\}$ for every $\alpha \in K$. Further, let $\mathcal{T}$ be the fuzzifying topology constructed as above. One can easily notice (see e.g. [29]) that $\mathcal{T}_\alpha = T_\alpha$ for every $\alpha \in K$ where $\mathcal{T}_\alpha = \{A \in L^X : \mathcal{T}(A) \geq \alpha\}$ is the α-level of $\mathcal{T}$.

2.4. Product of fuzzifying topologies

For our merits, we need only finite and countable products of fuzzifying topologies.Nevertheless we describe here the general construction of the product since its description actually does not depend on the number of factors.

Given a family of fuzzifying topological spaces $\{(X^i, \mathcal{T}^i) : i \in I\}$, we define the fuzzifying topology on the product $\prod_i X^i$ as the initial fuzzifying topology for the family of projections $p_i : \prod_i X^i \to (X^i, \mathcal{T}^i)$, see e.g. [29], [30]. Explicitly it can be described as follows:

For every $i \in I$ we define the fuzzifying topology $\tilde{\mathcal{T}}^i$ on the product $\prod_i X^i$ as the preimage of the fuzzifying topology $\mathcal{T}^i$ under projection $p^i : \prod_i X^i \to (X^i, \mathcal{T}^i)$, that is

$$\tilde{\mathcal{T}}^i(B) = \begin{cases} \mathcal{T}^i(A) & \text{if } B = p_i^{-1}(A) \text{ for some } A \subseteq X^i \\ 0 & \text{otherwise.} \end{cases}$$

Thus we obtain a family of fuzzifying topologies $\{\tilde{\mathcal{T}}^i : i \in I\}$ on the set $\prod_i X^i$. Now we define the fuzzifying topology $\mathcal{T}$ on the set $\prod_i X^i$ by setting $\mathcal{T} = \sup_i \tilde{\mathcal{T}}^i$.

To obtain the level-wise description of the product fuzzifying topology of the family $\{(X^i, \mathcal{T}^i) : i \in I\}$ we act as follows. Given $\alpha \in K \subseteq [0,1]$, where K is a dense subset of $[0,1]$, for each $i \in I$ we consider the (usual) topology $\mathcal{T}^i_\alpha = \{A : A \subseteq X^i, \mathcal{T}^i(A) \geq \alpha\}$ on X^i. Let T_α be the product of these topologies, that is T_α is the topology on $X = \prod_{i \in I} X^i$ defined by $\bigcup_{i \in I} \mathcal{T}^i_\alpha$ as a subbase.

Since the decomposition of a fuzzifying topology into a family of α-level topologies is lower semi continuous, one can easily notice that its description does not depend on the choice of the dense subset $K \subseteq [0,1]$ and verify the validity of the following theorem:

Theorem 2.13. *The α-level topologies $\mathcal{T}_\alpha$, $\alpha \in K$ of the fuzzifying topology $\mathcal{T}$ on the set $\prod_i X^i$ coincide with the topology T_α of the product $\prod_{i \in I}(X^i, \mathcal{T}^i_\alpha)$ where $\mathcal{T}^i_\alpha$ are α-levels of fuzzifying topologies $\mathcal{T}^i$.*

Corollary 2.14. *The product fuzzifying topology $\mathcal{T}$ on the product $\prod_i X^i$ can be reconstructed from the family of ordinary topologies $\{T_\alpha : \alpha \in K\}$ on $\prod_i X^i$ by setting $\mathcal{T}(A) = \bigvee\{\alpha : A \in T_\alpha,\}$ where T_α is the product of $\mathcal{T}^i_\alpha$ and $\mathcal{T}^i_\alpha$ are the α-levels of the fuzzifying topologies T^i.*

3. Fuzzifying topology induced by a strong fuzzy metric

3.1. Construction of a fuzzifying topology on a strong fuzzy metric space

Let $(X, M, *)$ be a strong fuzzy metric space. In order to construct a fuzzifying topology induced by this metric, we first have to define a relation between "distance-type" properties of the fuzzy metric for a fixed parameter $t \in \mathbb{R}^+$ and the α-levels of the "future" fuzzifying topology. For this merit we take a strictly increasing continuous bijection $\varphi : \mathbb{R}^+ \to (0, 1)$. Although here one can take any continuous strictly increasing bijection φ, in order to make situation fixed, we will use here the mapping $\varphi : \mathbb{R}^+ \to (0, 1)$ defined by $\varphi(t) = \frac{t}{t+1}$ where $t \in \mathbb{R}^+$. Obviously its inverse is given as the mapping $\psi : (0, 1) \to (0, \infty)$, where $\psi(\alpha) = \varphi^{-1}(\alpha) = \frac{\alpha}{1-\alpha}$ for each $\alpha \in (0, 1)$.

We fix $\alpha \in (0, 1)$ and consider the family

$$B_\alpha^M = \{B^M(x, r, t) : x \in X, r \in (0, 1)\}, \text{ where } t = \varphi^{-1}(\alpha).$$

B_α^M is a base of a topology T_α^M on the set X. Indeed, it is easy to verify (see e.g. [12]) that $M_t : X \times X \to [0, 1]$ defined by $M_t(x, y) = M(x, y, t)$ for $x, y \in X$, is a stationary fuzzy metric on X whose topology has as a base the family $\{B^{M_t}(x, r) : x \in X, r \in (0, 1)\}$. This topology is characterized in the next theorem:

Theorem 3.1. *Let $\alpha \in (0, 1)$ and let $U \in 2^X$. Then $U \in T_\alpha^M$ if and only if for each $x \in U$ there exists $r \in (0, 1)$ such that $B^M(x, r, t) \subseteq U$, where $t = \varphi^{-1}(\alpha)$.*

Proof Fix $\alpha \in (0, 1)$ and let $U \in 2^X$. Suppose that $U \in T_\alpha^M$ and $x \in U$. Then there exist $y \in X$ and $\varepsilon \in (0, 1)$ such that $x \in B^M(y, \varepsilon, t) \subseteq U$ where $t = \varphi^{-1}(\alpha)$. So, one can find $\delta \in (0, \varepsilon)$ such that $M(x, y, t) > 1 - \delta$. Since the t-norm $*$ is continuous and $1 - \delta > 1 - \varepsilon$, there exists $r \in (0, 1)$ such that $(1 - r) * (1 - \delta) > 1 - \varepsilon$. We show that $B^M(x, r, t) \subseteq B^M(y, \varepsilon, t)$. Indeed, if $z \in B^M(x, r, t)$ then $M(x, z, t) > 1 - r$, and since the fuzzy metric M is strong, it follows that $M(y, z, t) \geq M(y, x, t) * M(x, z, t) > (1 - \delta) * (1 - r)$. Thus, $z \in B^M(y, \varepsilon, t)$.

Conversely, if for each $x \in U$ we can find $r_x \in (0, 1)$ such that $B^M(x, r_x, t) \subseteq U$, where $t = \varphi^{-1}(\alpha)$, then $U = \bigcup_{x \in U} B^M(x, r_x, t)$ and so $U \in T_\alpha^M$. $\square$

From this theorem and taking into account that the inclusion $B^M(x, r, s) \subseteq B^M(x, r, t)$ holds for each $x \in X$, for each $r \in (0, 1)$, and for every $t > 0$ whenever $0 < s < t$, we obtain the next corollary:

Corollary 3.2. *If $U \in T_\alpha^M$, then $U \in T_\beta^M$ whenever $\beta < \alpha$, and hence the family $\{T_\alpha^M : \alpha \in (0,1)\}$ is decreasing.*

Now, referring to Subsection 2.3, we get the following theorem from the previous corollary:

Theorem 3.3. *Let $(X, M, *)$ be a strong fuzzy metric space. By setting $\mathcal{T}^M(A) = \bigvee \{\alpha : A \in T_\alpha^M\}$ for every $A \in 2^X$ we get a fuzzifying topology $\mathcal{T}^M : 2^X \to [0,1]$.*

Remark 3.4. Observe that the continuous t-norm $*$ plays an important role in the definition of a fuzzy metric space, and particularly, in the definition of a strong fuzzy metric space. (Indeed, the standard fuzzy metric M_d is fuzzy metric for the minimum t-norm, but in case d is not an ultrametric, fuzzy metric M_d is not strong for this t-norm. Notice also that M_d is a strong fuzzy metric for each d in case of the product t-norm.) Nevertheless, the topology induced by a (strong) fuzzy metric is independent on the t-norm used in its definition, and, in particular, the fuzzifying topology considered in the last theorem does not dependent on the t-norm that defines the strong fuzzy metric.

In the sequel we refer to the fuzzifying topology $\mathcal{T}^M$ constructed in the previous theorem as the fuzzifying topology induced by the strong fuzzy metric M.

Remark 3.5. Unfortunately the family of topologies $\{T_\alpha^M : \alpha \in K\}$ used in the previous construction generally is not lower semi continuous. Although, as it is clear from the construction, the inclusion $T_\alpha \subseteq \bigcap_{\beta < \alpha} T_\beta$ is obviously true, the opposite inclusion generally does not hold. Indeed, in Example 3.8 of our paper, we will show that the set E belongs to T_α^M for each $\alpha \in (0, \frac{1}{2})$, and hence $E \in \bigcap_{\alpha < \frac{1}{2}} T_\alpha^M$ although $E \notin T_{\frac{1}{2}}^M$.

Definition 3.6. [10] *A fuzzy metric space $(X, M, *)$ is called principal (and corresponding M is principal) if $\{B^M(x, r, t) : r \in (0,1)\}$ is a local base at $x \in X$ for each $x \in X$ and each $t \in \mathbb{R}^+$.*

Remark 3.7. Applying the definition of the principal fuzzy metric and Corollary 3.2, one can easily verify that if $(X, M, *)$ is a strong principal fuzzy metric space, then $T_\alpha^M = T^M$ for all $\alpha \in [0,1]$. Hence for every $A \in 2^X$ we have that
$$\mathcal{T}^M(A) = \begin{cases} 1, & \text{if } A \in T^M \\ 0, & \text{otherwise} \end{cases}$$

Thus $\mathcal{T}^M$ is a two-valued fuzzifying topology, that it is actually an ordinary topology induced by the fuzzy metric M.

As the next three examples show, in case of non-principal fuzzy metrics the situation may be different: there may exist subsets $A \subseteq X$ whose openness measure in the induced fuzzifying topology is a number strictly between 0 and 1, that is $\mathcal{T}^M(A) \in (0,1)$.

Example 3.8. Let $X = (0,1]$, $G = X \cap \mathbb{Q}$, and $H = X \setminus G$. Following [10] we define the fuzzy metric $(M, \cdot)$ on the set X as follows:

$$M(x,y,t) = \begin{cases} \frac{\min\{x,y\}}{\max\{x,y\}} \cdot t, & \text{if } x \in G, y \in H \text{ or if } x \in H, y \in G, \text{ and } t \in (0,1), \\ \frac{\min\{x,y\}}{\max\{x,y\}}, & \text{otherwise.} \end{cases}$$

As before, let the bijection $\varphi : \mathbb{R}^+ \to (0,1)$ be defined by $\varphi(t) = \frac{t}{t+1}$ for each $t \in \mathbb{R}^+$. We show that, given $x \in G$ and $\varepsilon < \min\{x, 1-x\}$, it holds $\mathcal{T}^M(E) = \frac{1}{2}$, where $E = (x - \varepsilon, x + \varepsilon) \cap G$.

Fix $\alpha \in \left(0, \frac{1}{2}\right)$. Since $t = \varphi^{-1}(\alpha) = \frac{\alpha}{1-\alpha}$, one can verify that B_α^M is given by the next expression:

$$B_\alpha^M = \left\{ \left\{ \left(\left(\frac{(1-r)z}{t}, \frac{zt}{1-r} \right) \cap H \right) \cup \left(\left((1-r)z, \frac{z}{1-r} \right) \cap G \right) : r \in (1-t, 1) \right\} \cup \right.$$

$$\cup \left\{ \left((1-r)z, \frac{z}{1-r} \right) \cap G : r \in (0, 1-t) \right\} : z \in G \right\} \bigcup$$

$$\bigcup \left\{ \left\{ \left(\left(\frac{(1-r)z}{t}, \frac{zt}{1-r} \right) \cap G \right) \cup \left(\left((1-r)z, \frac{z}{1-r} \right) \cap H \right) : r \in (1-t, 1) \right\} \cup \right.$$

$$\left. \cup \left\{ \left((1-r)z, \frac{z}{1-r} \right) \cap H : r \in (0, 1-t) \right\} : z \in H \right\}.$$

Let $y \in E$ and take $r_y < \min\{\frac{y+\varepsilon-x}{y}, \frac{y-x-\varepsilon}{x+\varepsilon}, 1-t\}$; it is easy to verify that $U_y = (y(1-r_y), \frac{y}{1-r_y}) \in B_\alpha^M$ and $U_y \subseteq E$. Therefore, $E = \bigcup\{U_y : y \in E\}$ and hence $E \in T_\alpha^M$.

Since $\alpha \in (0, \frac{1}{2})$ is arbitrary, we have that $E \in T_\alpha^M$ for each $\alpha \in (0, \frac{1}{2})$ and hence $\mathcal{T}^M(E) = \bigvee\{\alpha \in (0,1) : E \in T_\alpha^M\} \geq \frac{1}{2}$.

Now, take $\alpha \in (\frac{1}{2}, 1)$. Then, $\varphi^{-1}(\alpha) \geq 1$, and hence one can verify that

$$B_\alpha^M = \left\{ \left((1-r)z, \frac{z}{1-r} \right) : z \in X, r \in (0,1) \right\}.$$

Therefore, $E \notin T_{\frac{1}{2}}^M$ and this allows us to conclude that $\mathcal{T}^M(E) = \frac{1}{2}$.

In a similar way, one can show that $\mathcal{T}^M(F) = 1$ for the set $F = (x - \varepsilon, x + \varepsilon)$.

Example 3.9. Consider the non-principal strong fuzzy metric space, [10], $(X, M, \cdot)$ where $X = \mathbb{R}^+$ and the fuzzy metric M is given by

$$M(x, y, t) = \begin{cases} \frac{\min\{x,y\}}{\max\{x,y\}} \cdot t, & t \in (0, 1], \\ \frac{\min\{x,y\}}{\max\{x,y\}}, & t \in (1, \infty). \end{cases}$$

Further, let $\varphi(t) = \frac{t}{t+1}$ for each $t \in \mathbb{R}^+$. We will see that for every $x \in \mathbb{R}^+$, it holds $\mathcal{T}^M(\{x\}) = \frac{1}{2}$.

Fix $\alpha \in (0, \frac{1}{2})$. Since $t = \varphi^{-1}(\alpha) = \frac{\alpha}{1-\alpha}$, then $t \in (0, 1)$ and so one can verify that B_α^M is given by the next expression

$$B_\alpha^M = \{x : x \in X\} \cup \left\{ \left(\frac{x(1-r)}{t}, \frac{xt}{1-r} \right) : r \in (1-t, 1) \right\},$$

and hence $\mathcal{T}^M(\{x\}) \geq \frac{1}{2}$.

On the other hand, for each $\alpha \in [\frac{1}{2}, 1)$ the corresponding $t = \varphi^{-1}(\alpha)$ is in the set $[1, \infty)$. However, this means that

$$B_\alpha^M = \left\{ \left(x(1-r), \frac{x}{1-r} \right) : r \in (0, 1) \right\}.$$

Therefore, $\{x\} \notin B_\alpha^M$ for each $\alpha \in [\frac{1}{2}, 1)$ and hence $\mathcal{T}^M(\{x\}) = \frac{1}{2}$.

Example 3.10. Let $X = \mathbb{R}^+$. We define the fuzzy set M on $X \times X \times \mathbb{R}^+$ as follows:

$$M(x, y, t) = \begin{cases} \frac{t}{\max\{x,y\}}, & 0 < t \leq \min\{x, y\}, x \neq y, \\ \\ \frac{\min\{x,y\}}{\max\{x,y\}}, & \text{otherwise.} \end{cases}$$

It is easy to verify that $(X, M, \cdot)$ is a non-principal strong fuzzy metric space. Consider $\varphi(t) = \frac{t}{t+1}$ for each $t \in \mathbb{R}^+$. We will show that $\mathcal{T}^M(\{x\}) = \frac{x}{1+x}$ for every $x \in \mathbb{R}^+$.

Let $\alpha \in (0, 1)$ and consider $t = \varphi^{-1}(\alpha) = \frac{\alpha}{1-\alpha}$. Then,

$$B_\alpha^M = \left\{ \left(x(1-r), \frac{x}{1-r} \right) : x \in (0, t), r \in (0, 1) \right\} \cup$$

12

$$\cup \left\{ \left[t, \frac{t}{1-r} \right) : x \in [t, \infty), r \in \left(\frac{x-t}{x}, 1 \right) \right\} \cup \left\{ x : x \in [t, \infty), r \in \left(0, \frac{x-t}{x} \right] \right\}.$$

In case $\alpha \leq \frac{x}{1+x}$, we have $t = \varphi^{-1}(\alpha) \leq x$. Therefore, $B_\alpha^M(x, r, t) = \{x\}$ for each $r \in \left(0, \frac{x-t}{x} \right]$. This means that $\{x\} \in T_\alpha^M$ and hence $\mathcal{T}^M(\{x\}) \geq \frac{x}{1+x}$.

Take now $\alpha > \frac{x}{1+x}$, then $t > \varphi^{-1}(\alpha) = x$. Hence, for each $r \in (0, 1)$ we have that $B_\alpha^M(x, r, t) = \left(x(1-r), \frac{x}{1-r} \right)$. This means that for each $A \in \mathcal{T}_\alpha^M$, such that $x \in A$, there exists $r \in (0, 1)$ such that $\left(x(1-r), \frac{t}{1-r} \right) \subseteq A$ and hence $\{x\} \notin T_\alpha^M$. Therefore, $\mathcal{T}^M(\{x\}) = \frac{x}{1+x}$.

Thus in this case the spectrum of values of the fuzzifying topology $\mathcal{T}^M$ is the whole unit interval $[0, 1]$.

3.2. Convergence structure of a fuzzifying topology induced by a strong fuzzy metric

As before, throughout this section, $(X, M, *)$ will be a strong fuzzy metric space and the bijection $\varphi : \mathbb{R}^+ \to (0, 1)$ is defined in Subsection 3.1.

Let $\{x_n\}$ be a sequence in X, $x_0 \in X$ and $\alpha \in (0, 1)$. We define:

$$Con(\{x_n\}, \alpha)(x_0) = \begin{cases} 1 & \text{if for each } U \in \mathcal{T}_\alpha^M, \text{ with } x_0 \in U \text{ there exists} \\ & n_0 \in \mathbb{N} \text{ such that } x_n \in U \text{ for all } n \geq n_0. \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.11. *We say that $\{x_n\}$ is fuzzy convergent to x_0 if there exists $\alpha \in (0, 1]$ such that $Con(\{x_n\}, \alpha)(x_0) = 1$.*

Definition 3.12. *We say that a fuzzy convergent sequence $\{x_n\}$ converging to x_0 is λ-fuzzy convergent if $\lambda = \bigwedge \{\alpha \in (0, 1] : Con(\{x_n\}, \alpha)(x_0) = 1\}$.*

Definition 3.13. *[23] A sequence $\{x_n\}$ in X is called p-convergent to x_0 if $\lim_n M(x_n, x_0, t_0) = 1$ for some $t_0 \in \mathbb{R}^+$. Equivalently, if there exists $t_0 \in \mathbb{R}^+$ such that for each $\varepsilon \in (0, 1)$ one can find $n_0 \in \mathbb{N}$ such that $x_n \in B^M(x_0, \varepsilon, t_0)$ for all $n \geq n_0$.*

Remark 3.14. *[10] If $\lim_n M(x_n, x_0, t_0) = 1$ then $\lim_k M(x_{n_k}, x_0, t_0) = 1$ for each subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$.*

Theorem 3.15. *A sequence $\{x_n\}$ is fuzzy convergent to $x_0 \in X$ if and only if it is p-convergent to x_0.*

Proof   Assume that a sequence $\{x_n\}$ fuzzy converges to $x_0 \in X$ and find $\alpha \in (0,1)$ such that $Con(\{x_n\}, \alpha)(x_0) = 1$. Then, for each $U \in 2^X$, such that $x \in U$ and $\mathcal{T}^M(U) \geq \alpha$, there exists $n_0 \in \mathbb{N}$ with the property that $x_n \in U$ for all $n \geq n_0$.

Consider $t = \varphi^{-1}(\alpha)$ and take $\varepsilon \in (0,1)$. Then, $x_0 \in B^M(x_0, \varepsilon, t)$ and $\mathcal{T}^M(B^M(x_0, \varepsilon, t)) \geq \alpha$. Thus, we can find $n_0 \in \mathbb{N}$ such that $x_n \in B^M(x_0, \varepsilon, t)$ for all $n \geq n_0$. However this means that $\{x_n\}$ is p-convergent to x_0.

Conversely, suppose that a sequence $\{x_n\}$ is p-convergent to x_0, while $\{x_n\}$ is not fuzzy convergent to x_0. Then, we can find $t_0 > 0$ such that $\lim_n(x_n, x_0, t_0) = 1$, and $Con(\{x_n\}, \alpha)(x_0) = 0$ for all $\alpha \in (0,1)$.

Choose $\lambda \in (0,1)$ for which $\varphi^{-1}(\lambda) > t_0$. Since $\{x_n\}$ is not fuzzy convergent to x_0, we can find $U_\lambda \in 2^X$, with $x_0 \in U_\lambda$ and $\mathcal{T}^M(U_\lambda) \geq \lambda$ and such that for each $n \in \mathbb{N}$ there exists $m_n > n$ for which $x_{m_n} \notin U_\lambda$. On the other hand, since $x_0 \in U_\lambda$ and $\mathcal{T}^M(U_\lambda) \geq \lambda$, we conclude that $U_\lambda \in \mathcal{T}_\alpha^M$ for each $\alpha \in (0, \lambda)$. Take now $\alpha_0 \in (0, \lambda)$ such that $t_0 = \varphi^{-1}(\alpha_0)$ (this choice is possible since φ is a strictly increasing bijection). Therefore we can choose $r \in (0,1)$ such that $B^M(x_0, r, t_0) \subseteq U_\lambda$. To complete the proof we will inductively construct a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ as follows.

For the first step find n_1 such that $x_{n_1} \notin U_\lambda$ and hence $x_{n_1} \notin B^M(x_0, r, t_0)$. Now find $n_2 > n_1$ such that $x_{n_2} \notin U_\lambda$ and so $x_{n_2} \notin B^M(x_0, r, t_0)$. Continuing by induction on k, we construct a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \notin B^M(x_0, r, t_0)$ for all $k \in \mathbb{N}$. Therefore, $\lim_k M(x_{n_k}, x_0, t_0) \leq 1 - r$ and hence $\lim_n M(x_n, x_0, t_0) \leq 1 - r < 1$. The obtained contradiction completes the proof. $\qquad\square$

3.3. Continuity of mappings of fuzzifying spaces versus continuity of mappings of strong fuzzy metric spaces.

We first recall that a mapping $f : (X, \mathcal{T}^X) \to (Y, \mathcal{T}^Y)$ of fuzzifying topological spaces is called continuous (Definition 2.10) if $\mathcal{T}^X(f^{-1}(V)) \geq \mathcal{T}^Y(V)$ for each $V \in 2^Y$.

On the other hand a mapping $f : (X, M, *_M) \to (Y, N, *_N)$ of fuzzy metric spaces is called continuous (Definition 2.4) if for every $\varepsilon \in (0,1)$, every $x \in X$ and every $t \in \mathbb{R}^+$ there exist $\delta \in (0,1)$ and $s \in \mathbb{R}^+$ such that $N(f(x), f(y), t) > 1 - \varepsilon$ whenever $M(x, y, s) > 1 - \delta$.

Unfortunately, as different from the situation in case of fuzzy metrics and induced topologies, (see Proposition 2.5), the concept of continuity of fuzzy metric spaces is not coherent with the concept of continuity of the induced fuzzifying topological spaces. Therefore in order to describe the

relations between continuity of mappings between fuzzy metric spaces and the continuity of the corresponding mappings between the fuzzifying topological spaces whose topology is induced by strong fuzzy metrics we need to consider the following stronger version of continuity for mappings of fuzzy metric spaces introduced in [9]:

Definition 3.16. *A mapping $f : (X, M, *_M) \rightarrow (Y, N, *_N)$ will be called strongly continuous at a point $x \in X$ if given $\varepsilon \in (0,1)$ and $t \in \mathbb{R}^+$ there exists $\delta \in (0,1)$ such that $M(x, y, t) > 1 - \delta$ implies $N(f(x), f(y), t) > 1 - \varepsilon$. We will say that $f : (X, M, *_M) \rightarrow (Y, N, *_N)$ is strongly continuous (on X) if it is strongly continuous at each point $x \in X$.*

Remark 3.17. In the paper [9] this property of a mapping $f : (X, M, *_M) \rightarrow (Y, N, *_N)$ was called t-continuity. We think it reasonable to rename this property as *strong continuity*, first because it is well related with the concept of a strong fuzzy metric which is fundamental for this paper, and second, because the prefix t in front the adjective "continuous" seems to be misleading in this context.

Theorem 3.18. *A mapping $f : (X, \mathcal{T}^M) \rightarrow (Y, \mathcal{T}^N)$ of fuzzifying topological spaces $(X, \mathcal{T}^M)$, $(Y, \mathcal{T}^N)$ induced by fuzzy metrics $M : X \times X \times \mathbb{R}^+ \rightarrow [0,1]$ and $N : Y \times Y \times \mathbb{R}^+ \rightarrow [0,1]$ is continuous if and only if the mapping $f : (X, M, *_M) \rightarrow (Y, N, *_N)$ is strongly continuous.*

    Proof Suppose that a mapping $f : (X, \mathcal{T}^M) \rightarrow (Y, \mathcal{T}^N)$ is continuous and fix $x \in X$. Let $\varepsilon \in (0,1)$ and $t \in \mathbb{R}^+$. Since $B^N(f(x), \varepsilon, t) \in \mathcal{T}_\alpha^N$, where $\alpha = \varphi(t)$ and f is continuous, we have that $f^{-1}(B^N(f(x), \varepsilon, t)) \in \mathcal{T}_\alpha^M$. Therefore, for each $y \in f^{-1}(B^N(f(x), \varepsilon, t))$ we can find $r \in (0,1)$ such that $B^M(y, r, t) \subseteq f^{-1}(B_N(f(x), \varepsilon, t))$. In particular, given $x \in f^{-1}(B^N(f(x), \varepsilon, t))$ there exists $\delta \in (0,1)$ such that $B^M(x, \delta, t) \subseteq f^{-1}(B^N(f(x), \varepsilon, t)$, and hence, if $M(x, y, t) > 1 - \delta$, that is $y \in B^M(x, \delta, t)$, then $y \in f^{-1}(B^N(f(x), \varepsilon, t))$. Thus $N(f(x), f(y), t) > 1 - \varepsilon$. Therefore, the mapping $f : (X, M, *_M) \rightarrow (Y, N, *_N)$ is strongly continuous at a point x, and, since $x \in X$ is arbitrary, the mapping $f : (X, M, *_M) \rightarrow (Y, N, *_N)$ is strongly continuous

    Conversely, suppose that a mapping $f : (X, M, *_M) \rightarrow (Y, N, *_N)$ is strongly continuous, but $f : (X, \mathcal{T}^M) \rightarrow (Y, \mathcal{T}^N)$ is not continuous. Then, we can find $V \in 2^Y$ such that $\mathcal{T}^M(f^{-1}(V)) < \mathcal{T}^N(V)$. Let $\gamma = \mathcal{T}^M(f^{-1}(V))$ and $\beta = \mathcal{T}^N(V)$. Since, by our assumption, $0 \leq \gamma < \beta \leq 1$, we can find $\alpha \in (\gamma, \beta)$ such that $f^{-1}(V) \notin \mathcal{T}_\alpha^M$, but $V \in \mathcal{T}_\alpha^N$.

The inequality $f^{-1}(V) \notin \mathcal{T}_\alpha^M$ means that there exists $x_0 \in f^{-1}(V)$ such that $A \nsubseteq f^{-1}(V)$ for each $A \in \mathcal{T}_\alpha^M$ containing point x_0 and, in particular, $B^M(x_0, r, t) \nsubseteq f^{-1}(V)$ for each $r \in (0,1)$, where $t = \varphi^{-1}(\alpha)$.

On the other hand, since $f(x_0) \in V \in \mathcal{T}_\alpha^N$, we can find $\varepsilon_0 \in (0,1)$ such that $B^N(f(x_0), \varepsilon_0, t) \subseteq V$. Therefore, we have found $x_0 \in X$ and $\varepsilon_0 \in (0,1)$ such that for each $r \in (0,1)$ it holds $B^M(x_0, r, t) \nsubseteq f^{-1}(B_N(f(x_0), \varepsilon_0, t))$ where $t = \varphi^{-1}(\alpha)$. In its turn this means that for each $r \in (0,1)$ there exists a point $y \in X$ such that $M(x_0, y, t) > 1 - r$, but $N(f(x_0), f(y), t) \leq 1 - \varepsilon_0$ and hence the mapping $f : (X, M, *_M) \to (Y, N, *_N)$ is not strongly continuous at the point x_0. The obtained contradiction completes the proof. $\square$

In the paper [11] the following statement is proved:

Proposition 3.19. *A mapping* $f : (X, M, *_M) \to (Y, N, *_N)$ *is strongly continuous at* x_0 *if and only if*

$$\lim_n M(x_n, x_0, t) = 1 \implies \lim_n N(f(x_n), f(x_0), t) = 1$$

for each sequence $\{x_n\}_{n \in \mathbb{N}}$ *in* X.

Note that the above proposition just means that $f : (X, M, *_M) \to (Y, N, *_N)$ is strongly continuous if and only if it transforms each p-convergent sequence for $t \in \mathbb{R}^+$ in X into a p-convergent sequence for the same $t \in \mathbb{R}^+$ in Y. Taking into account this fact, and referring to the above Theorem 3.15 and Theorem 3.18 we have the next corollary.

Corollary 3.20. *A mapping* $f : (X, \mathcal{T}^M) \to (Y, \mathcal{T}^N)$ *is fuzzy continuous if and only if* f *transforms* λ-*fuzzy convergent sequences from* $(X, \mathcal{T}^M)$ *into* λ-*fuzzy convergent sequences in* $(Y, \mathcal{T}^N)$.

4. Product of fuzzy metric spaces

When studying products of fuzzy metric spaces[3] we distinguish the case of two, and hence, by induction, of a finite number of factors from the case of a countable number of factors. The reason for this is that in the first case we deal with an arbitrary continuous t-norm while in the second case we have to assume additional restrictions on the t-norm in the definition of a fuzzy metric. A consequence of this is also that the proofs in these cases are different.

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[3]In this section we do not assume the strongness of fuzzy metrics

4.1. The case of two factors

Let $(X_1, M_1, *)$ and $(X_2, M_2, *)$ be two fuzzy metric spaces on the basis of the same t-norm $*$. Borrowing the idea from the construction given in Proposition 3.5 in [25], we define fuzzy sets $M, N : (X_1 \times X_2) \times (X_1 \times X_2) \times \mathbb{R}^+ \to [0, 1]$ by setting

$$M((x_1, x_2), (y_1, y_2), t) = M_1(x_1, y_1, t) * M_2(x_2, y_2, t),$$

$$N((x_1, x_2), (y_1, y_2), t) = M_1(x_1, y_1, t) \wedge M_2(x_2, y_2, t).$$

Proposition 4.1.

(i) $(X_1 \times X_2, M, *)$ *is a fuzzy metric space in case $*$ has no zero divisors (i.e. $a * b \neq 0$ whenever $a, b \neq 0$).*
(ii) $(X_1 \times X_2, N, *)$ *is a fuzzy metric space.*

Proof It is straightforward. $\square$

To avoid the restriction that the t-norm has no zero divisors one can modify the above construction of the product $M((x_1, x_2), (y_1, y_2), t) = M_1(x_1, y_1, t) * M_2(x_2, y_2, t)$ for an arbitrary continuous t-norm $*$ as follows.

First, we recall the next A. Sapena's result:

Proposition 4.2. [25]. *Let $(X, M, *)$ be a fuzzy metric space and let $k \in (0, 1)$. Define*

$$M^k(x, y, t) = \max\{M(x, y, t), k\} \text{ for each } x, y \in X, t \in \mathbb{R}^+.$$

Then $(X, M^k, *)$ is a fuzzy metric space, which generates the same topology on the set X as the topology generated by the fuzzy metric M.

Proposition 4.3. *Let $(X_1, M_1, *)$ and $(X_2, M_2, *)$ be two fuzzy metric spaces. Since $*$ is continuous, there exists $k \in (0, 1)$ such that $k * k > 0$. We define a fuzzy set $\bar{M}^k : (X_1 \times X_2) \times (X_1 \times X_2) \times \mathbb{R}^+$ by setting*

$$\bar{M}^k((x_1, x_2), (y_1, y_2), t) = M_1^k(x_1, y_1, t) * M_2^k(x_2, y_2, t).$$

Then $(X_1 \times X_2, \bar{M}^k, *)$ is a fuzzy metric space

Proof   The proof can be done patterned after the proof of Proposition 4.1, taking into account that $\bar{M}^k(x, y, t) \geq k * k > 0$ and hence $\bar{M}^k$ satisfies $(GV1)$. □

Our next aim is to show that fuzzy metrics N, M and $\bar{M}^k$ induce the same topology on the set $X_1 \times X_2$ as the product of the topologies T^{M_1} and T^{M_2} induced by fuzzy metrics M_1 and M_2, respectively. We start with the case of the fuzzy metric N.

Theorem 4.4. *The fuzzy metric N induces on the set $X_1 \times X_2$ the same topology T^N as the product of the topologies T^{M_1} and T^{M_2} induced by fuzzy metrics M_1 and M_2, respectively.*

Proof It is straightforward. □

As a corollary of the last theorem we obtain:

Corollary 4.5. *If fuzzy metrics M_1 and M_2 are strong, then for each α the fuzzy metric N induces on the set $X_1 \times X_2$ the same topology T_α^N as the product of the topologies $T_\alpha^{M_1}$ and $T_\alpha^{M_2}$ induced by the fuzzy metrics M_1 and M_2 respectively.*

Following the lines of the proof of Proposition 3.5 in [25], one can easily obtain the following:

Proposition 4.6. *Fuzzy metrics N, M (in case $*$ has no zero-divisors) and $\bar{M}^k$ induce on the product space $X_1 \times X_2$ the same topology $T^N = T^M = T^{\bar{M}_k}$. In case fuzzy metrics N, M, and $\bar{M}^k$ are strong (and this is happens if fuzzy metrics M_1 and M_2 are strong), they induce the same topology $T_\alpha^N = T_\alpha^M = T_\alpha^{\bar{M}_k}$ on the product space $X_1 \times X_2$ for every $\alpha \in [0,1]$.*

Now, applying this proposition, we get the following corollaries from Theorem 4.4 and Corollary 4.5:

Corollary 4.7. *Topologies T^N, T^M and $T^{\bar{M}^k}$ induced by fuzzy metrics N, M (if $*$ has no zero-divisors) and $\bar{M}^k$ coincide on the product $(X_1, M_1) \times (X_2, M_2)$ with the product topology $T^{M_1} \times T^{M_2}$.*

Corollary 4.8. *If fuzzy metrics M_1 and M_2 are strong, then for each $\alpha \in [0,1]$ topologies T_α^N, T_α^M and $T_\alpha^{\bar{M}^k}$ induced by fuzzy metrics N, M (if $*$ has no zero-divisors) and M^k on the product of fuzzy metric spaces $(X_1, M_1) \times (X_2, M_2)$ coincide with the product topology $T_\alpha^{M_1} \times T_\alpha^{M_2}$.*

4.2. The case of a countable number of factors

Let $\{(X_i, M_i, *) : i \in \mathbb{N}\}$ be a countable family of fuzzy metric spaces basing on the same t-norm $*$. Fix $k \in (0, 1)$ and consider the family $\{(X_i, M_i^k, *) : i \in \mathbb{N}\}$ defined in the same way as in Proposition 4.2. We set $N(x, y, t) = \bigwedge_{i \in \mathbb{N}} M_i^k(x_i, y_i, t)$. Then, taking into account that $N(x, y, t) \geq k$ and $\wedge$ has no zero divisors, we can prove the following.

Proposition 4.9. $(X, N, *)$ *is a fuzzy metric space, where* $X = \prod_{i \in \mathbb{N}} X_i$ *and* $N(x, y, t) = \bigwedge_{i \in \mathbb{N}} M_i^k(x_i, y_i, t)$ *for each* $x, y \in X$ *and each* $t \in \mathbb{R}^+$ *(where* $x = \{x_i\}_{i \in \mathbb{N}}, y = \{y_i\}_{i \in \mathbb{N}}$*).*

Unfortunately, the topology generated by N on X does not agree with the product topology $\prod_{i \in \mathbb{N}} T^{M_i}$, in general. Indeed, consider the countable family of fuzzy metric spaces $\{(X_i, M_i, *) : i \in \mathbb{N}\}$, where $X_i = [0, 1]$ and $M_i = M_d$ for each $i \in \mathbb{N}$, where d is the euclidean metric on the real line $\mathbb{R}$, i.e. $d(x_i, y_i) = |x_i - y_i|$ (see Example 2.6). Let $k \in (0, 1)$. Then, given $x, y \in X$ and $t \in \mathbb{R}^+$ we have that,

$$N(x, y, t) = \bigwedge_{i \in \mathbb{N}} \left\{ \max \left\{ \frac{t}{t + |x_i - y_i|}, k \right\} \right\} = \max \left\{ \left(\bigwedge_{i \in \mathbb{N}} \frac{t}{t + |x_i - y_i|} \right), k \right\}$$

$$= \max \left\{ \left(\frac{t}{t + \bigvee_{i \in \mathbb{N}} |x_i - y_i|} \right), k \right\} = M_{d_\infty}(x, y, t) \vee k = M_{d_\infty}^k(x, y, t),$$

where $d_\infty(x, y) = \bigvee_{i \in \mathbb{N}} |x_i - y_i|$.

Therefore, N generates the same topology on $[0, 1]^{\mathbb{N}}$ as the fuzzy metric M_{d_∞} does, that is $T^{M_{d_\infty}}$, which is equal to T_{d_∞}.

On the other hand, the product topology $\prod_{i \in \mathbb{N}} T^{M_i}$ is $\prod_{i \in \mathbb{N}} T_d = \prod_{i \in \mathbb{N}} T_{U_{|[0,1]}}$, where $T_{U_{|[0,1]}}$ denotes the ordinary topology of $\mathbb{R}$ restricted to $[0, 1]$.

Clearly, T_{d_∞} does not agree with the product topology $\prod_{i \in \mathbb{N}} T_U$ on $[0, 1]^{\mathbb{N}}$. Indeed, $\prod_{i \in \mathbb{N}} [0, \frac{1}{2}) \in T_{d_\infty}$, but $\prod_{i \in \mathbb{N}} [0, \frac{1}{2}) \notin \prod_{i \in \mathbb{N}} T_{U_{|[0,1]}}$ □

Now, under some assumptions on the t-norm used in the definition of fuzzy metrics, we show how a fuzzy metric can be defined on the product of a countable family of fuzzy metric spaces, in such a way that the induced topology agrees with the product topology.

Let $\{(X_i, M_i, *) : i \in \mathbb{N}\}$ be a countable family of fuzzy metric spaces and take $k \in (0, \frac{1}{2})$. For each $i \in \mathbb{N}$ let $k_i = 1 - k^i$. We use this constant to modify the fuzzy metric space (X_i, M_i) as in Proposition 4.2. Namely,

replace the original fuzzy metric M_i on the set X_i by a modified fuzzy metric $M_i^{k_i}$ where $M_i^{k_i}(x_i, y_i, t) = \max\{M(x_i, y_i, t), k_i\}$ for all $x_i, y_i \in X_i, t \in \mathbb{R}^+$. From Proposition 4.2 we know that for each $i \in \mathbb{N}$ fuzzy metric $M_i^{k_i}$ generates the same topology $T_i^{M_i}$ on X_i, as the original fuzzy metric M_i.

Given a countable family of numbers $a_i \in [0, 1], i \in \mathbb{N}$ we define their $*$-product by

$$\prod_{i \in \mathbb{N}}^{*} a_i = a_1 * a_2 * \cdots * a_n * \ldots = \lim_{n \to \infty} \prod_{i=1,\ldots,n}^{*} a_i.$$

Since the family of partial products is non-increasing, the limit exists and hence this definition is correct. As the next theorem states, under assumptions that $*$ dominates the Łukasiewicz t-norm $*_L$ (that is $a *_L b = \max\{a + b - 1, 0\}$) the operation $\prod_{i \in \mathbb{N}}^{*} a_i$ allows to define a coherent fuzzy metric on the countable product of fuzzy metric spaces.

Theorem 4.10. *Let $\{(X_i, M_i, *) : i \in \mathbb{N}\}$ be a countable family of fuzzy metric spaces where $* \geq *_L$, and let $X = \prod_{i \in \mathbb{N}} X_i$ be the product of the underlying sets. Let the mapping $M : X \times X \times \mathbb{R}^+ \to [0, 1]$ be defined by*

$$M(x, y, t) = \prod_{i \in \mathbb{N}}^{*} M_i^{k_i}(x_i, y_i, t).$$

Then $(X, M, *)$ is a fuzzy metric space and the topology T^M generated by M on X agrees with the product topology $T = \prod_{i \in \mathbb{N}} T^{M_i}$.

Proof First, we will show that M is a fuzzy metric on X. To do this we have to verify only that it fulfills the condition $(GV1)$, since it is straightforward to prove that M satisfies conditions (GV2) – (GV5).

Let $x, y \in X$ and $t \in \mathbb{R}^+$. Then,

$$M(x, y, t) = \prod_{i \in \mathbb{N}}^{*} M_i^{k_i}(x_i, y_i, t) \geq \prod_{i \in \mathbb{N}}^{*} (1 - k^i)$$

$$\geq (1 - k) *_L (1 - k^2) *_L (1 - k^3) *_L \cdots = 1 - \sum_{i \in \mathbb{N}} k^i = 1 - \frac{k}{1 - k} = \frac{1 - 2k}{1 - k} > 0,$$

(recall that k is chosen in $(0, \frac{1}{2})$).

20

Now, we will see that the topology T^M on X agrees with the product topology $T = \prod_{i \in \mathbb{N}} T^{M_i}$.

Suppose that $A \in T^M$. Then, for each $x = \{x_n\} \in A$ there exist $r \in (0,1)$ and $t \in \mathbb{R}^+$ such that $B^M(x,r,t) \subseteq A$. Given $r \in (0,1)$, we choose $r_1, r_2 \in (0,1)$ in such a way, that $(1-r_1)*(1-r_2) > 1-r$.

Since $1 - \sum_{i \in \mathbb{N}} k^i > 0$ we can find $n_0 \in \mathbb{N}$ such that $1 - \sum_{i \geq n_0} k^i > 1 - r_2$. Further, given r_1 and $n_0 \in \mathbb{N}$, we can find numbers $\{\delta_1, \delta_2, \ldots, \delta_{n_0}\} \subset (0,1)$ such that

$$(1 - \delta_1) *_L (1 - \delta_2) *_L \cdots *_L (1 - \delta_{n_0}) \geq 1 - r_1.$$

Therefore,

$$(1 - \delta_1) * (1 - \delta_2) * \cdots * (1 - \delta_{n_0}) * \prod_{n > n_0}^{*} (1 - k^i) \geq (1 - r_1) * (1 - r_2) > 1 - r.$$

Consider the family $U = \{U_n : n \in \mathbb{N}\}$, where $U_i = B^{M_i^{k_i}}(x_i, \delta_i, t)$ for each $i \in \{1, \ldots, n_0\}$ and $U_i = X_i$ for each $i > n_0$. Then, by the definition of the product topology T, we have $U = \prod_{i \in \mathbb{N}} U_i \in T$.

On the other hand $U \subset B^M(x,r,t)$. Indeed, let $y = \{y_n\} \in U$, then

$$M(x,y,t) = \prod_{i \in \mathbb{N}}^{*} M_i^{k_i}(x_i, y_i, t) > (1 - \delta_1) * \cdots * (1 - \delta_{n_0}) * \prod_{n > n_0}^{*} (1 - k^i) > 1 - r.$$

This means that $A \in T$ and hence $T^M \subseteq T$.

Conversely, suppose that $A \in T$. Without loss of generality we may assume that A is taken from the standard base of the product topology. This means that there exists $n_0 \in \mathbb{N}$ such that $A = \prod_{i \in \mathbb{N}} A_i$, where $A_i \in T^{M_i}$ for each $i \in \{1, \ldots, n_0\}$ and $A_i = X_i$ for each $i > n_0$.

Let $x = \{x_n\} \in A$, then there exist $\{r_1, \ldots, r_{n_0}\} \subset (0,1)$ such that $B^{M_i^{k_i}}(x_i, r_i, t) \subset A_i$, for each $i \in \{1, \ldots, n_0\}$. Take $r \in (0,1)$ such that $r < \min\{r_1, \ldots, r_{n_0}\}$. To complete the proof we show that $B^M(x,r,t) \subset A$.

Indeed, let $y = \{y_i\} \in B^M(x,r,t)$, then $M(x,y,t) = \prod_{i \in \mathbb{N}}^{*} M_i^{k_i}(x_i, y_i, t) > 1-r$, and hence $M_i^{k_i}(x_i, y_i, t) > 1-r$, for each $i \in \mathbb{N}$. Therefore, $M_i^{k_i}(x_i, y_i, t) > 1 - r_i$, for each $i \in \{1, \ldots, n_0\}$. However, this means that $y_i \in A_i$ for each $i \in \{1, \ldots, n_0\}$ and hence $y \in A$. $\qquad\square$

Assume now that all fuzzy metrics are strong. Then, it is straightforward to verify that at each level $\alpha \in (0,1)$ we obtain the following.

21

Corollary 4.11. *Let $\{(X_i, M_i, *) : i \in \mathbb{N}\}$ be a countable family of strong fuzzy metric spaces where $* \geq *_L$, and let $X = \prod_{i \in \mathbb{N}} X_i$ be the product of the underlying sets. Let the mapping $M : X \times X \times \mathbb{R}^+ \to (0, 1]$ be defined by*

$$M(x, y, t) = \prod_{i \in \mathbb{N}}^{*} M_i^{k_i}(x_i, y_i, t).$$

Then $(X, M, *)$ is a strong fuzzy metric and at each level $\alpha \in (0, 1]$ the fuzzy metric M induces on X the same topology T_α^M as the product topology $T_\alpha = \prod_{i \in \mathbb{N}} T_\alpha^{M_i}$.

Remark 4.12. We have found a theorem similar to our Theorem 4.10 in the paper [1] (see Lemma 2.4 in [1]). However, we have noticed some essential defects in the proof. In particular, in [1] the authors do not assume that the t-norm used in the base of the fuzzy metrics dominated the Łukasiewicz t-norm, and we do not know whether the statement is true without this assumption. Besides, as different from [1] we need the level-vise version of this theorem. Therefore we include our proof of this, fundamental for us, result.

4.3. Fuzzifying topologies of a product of fuzzifying topologies induced by strong fuzzy metrics

Let $(X_1, M_1, *), (X_2, M_2, *)$ be strong fuzzy metric spaces on the base of the same t-norm $*$ and let $(X_1 \times X_2, N), (X_1 \times X_2, M)$ and $(X_1 \times X_2, \bar{M}^k)$ be their products defined in Subsection 4.1. Then, applying construction developed in Section 3.1 to each of them, we get fuzzifying topologies. The first two of these fuzzy metrics induce fuzzifying topologies $\mathcal{T}^{M_1}$ and $\mathcal{T}^{M_2}$ on the sets X_1 and X_2 respectively, while the rest three induce fuzzifying topologies $\mathcal{T}^N, \mathcal{T}^M$ and $\mathcal{T}^{\bar{M}_k}$ on the product $X_1 \times X_2$.

Theorem 4.13. *Fuzzifying topologies induced by fuzzy metrics N, M and $\bar{M}^k$ on the product $(X_1 \times X_2)$ coincide and are equal to the product of the fuzzifying topologies $\mathcal{T}^{M_1}$ and $\mathcal{T}^{M_2}$:*

$$\mathcal{T}^N = \mathcal{T}^M = \mathcal{T}^{\bar{M}^k} = \mathcal{T}^{M_1} \times \mathcal{T}^{M_2}$$

Proof Recalling the construction of a fuzzifying topology from a non-increasing family of usual topologies we have:

$$\mathcal{T}^N : 2^{X_1 \times X_2} \to [0, 1] : \quad \mathcal{T}^N(A) = \bigvee \left\{ \alpha : A \in T_\alpha^N \right\}, A \in 2^{X_1 \times X_2};$$

$$\mathcal{T}^{M_1} : 2^{X_1} \to [0,1] : \quad \mathcal{T}^{M_1}(A_1) = \bigvee \left\{ \alpha : A_1 \in T_\alpha^{M_1} \right\}, A_1 \in 2^{X_1};$$

$$\mathcal{T}^{M_2} : 2^{X_2} \to [0,1] : \quad \mathcal{T}^{M_2}(A_2) = \bigvee \left\{ \alpha : A_2 \in T_\alpha^{M_1} \right\}, A_2 \in 2^{X_2}.$$

The equality $\mathcal{T}^N = \mathcal{T}^{M_1} \times \mathcal{T}^{M_2}$ follows from the equality $T_\alpha^N = T_\alpha^{M_1} \times T_\alpha^{M_2}$, which in its turn is guaranteed by Corollary 4.5.

The proof of the equalities $\mathcal{T}^M = \mathcal{T}^{M_1} \times \mathcal{T}^{M_2}$ and $\mathcal{T}^{M^k} = \mathcal{T}^{M_1} \times \mathcal{T}^{M_2}$ can be done in the same way. $\qquad\square$

Coming now to the case of the product of countably many fuzzy metric spaces, we have the following:

Theorem 4.14. *Let $\{(X_i, M_i, *) : i \in \mathbb{N}\}$ be a countable family of strong fuzzy metric spaces with the same t-norm $*$ dominating Łukasiewicz t-norm (that is $* \geq *_L$) and let $(X, M) = \left(\prod X_i, \prod M_i^{k_i} \right)$ be their product (defined in Theorem 4.10). Then the fuzzifying topology $\mathcal{T}^M$ induced by the fuzzy metric M coincides with the product of fuzzifying topologies $\mathcal{T}^{M_i}$.*

Proof The proof can be done in a similar way as the proof of the Theorem 4.13. However now, instead of applying Corollary 4.5 one has to refer to Corollary 4.11. $\qquad\square$

5. Conclusions

In our paper we have developed a construction of a fuzzifying topology from a strong fuzzy metric, and studied some properties of this construction.

As the first prospective for the future work we see the problem to extend the construction of induced fuzzifying topologies for the case of general, that is not only strong, fuzzy metrics.

To explain the problem in more details, let us consider a fuzzy metric space $(X, M, *)$ and let, as above, $\varphi : \mathbb{R}^+ \to (0,1)$ be a strictly increasing continuous bijection. Fix $\alpha \in (0,1)$ and consider the family $B_\alpha^M = \{B^M(x, r, t) : x \in X, r \in (0,1)\}$, where $t = \varphi^{-1}(\alpha)$. To develop a construction of a fuzzifying topology our first question is:

Question 5.1. *Is B_α^M a base of a topology on X?*

From the definition of a principal fuzzy metric it is clear that the answer is positive for such fuzzy metrics:

Proposition 5.2. *If $(X, M, *)$ is principal, then for each $\alpha \in (0,1)$, B_α^M is a base of a topology T_α^M on X.*

Considering further the above question, it is clear that in any case, B_α is a subbase for some topology T_α^M on X and hence we can define a mapping $\mathcal{T}^M : 2^X \to [0,1]$ by setting $\mathcal{T}^M(A) = \sup\{\alpha : A \in T_\alpha^M\}$. However, we do not know whether the family $\{T_\alpha^M : \alpha \in (0,1)\}$ is non-increasing (that is whether $\alpha < \beta \Rightarrow \mathcal{T}_\alpha^M \supseteq \mathcal{T}_\beta^M$). Hence, when defining the mapping $\mathcal{T}^M : 2^X \to [0,1]$, we cannot refer to the construction described in Subsection 2.3 and hence to conclude that $\mathcal{T}^M$ is a fuzzifying topology. Therefore we have:

Question 5.3. *Is $\mathcal{T}^M$ a fuzzifying topology on X?*

We conclude with some remarks concerning relations between our research and the work done in [35] and [20].

The context of the work in [35], as well as ours, are fuzzy(-pseudo) metric spaces in the sense of George-Veeramani [7] [4]. However, as different from the level-wise construction of a fuzzifying topology presented in this paper, Y. Yue and F-G Shi realize a fuzzy pseudometric as a certain "measure of closeness" between points of the underlying set in order to use it for the construction of a generalized neighborhood system [36]. A generalized neighborhood system, in its turn, gives rise to a fuzzifying topology on the underlying set. In order to fulfill the whole construction the authors restrict to the minimum t-norm $\wedge$.

Although the authors of [20] work in the context of KM-fuzzy metrics (a modified version of fuzzy metrics in the sense of I. Kramosil and J. Michalek [18] defined in [21]), their idea of "level-wise" approach for the construction of a fuzzifying topology is closer to the one developed in this paper. Using the representation theorem [21] that allows to represent a fuzzy pseudometric by a non-decreasing family of ordinary pseudometrics, the authors obtain a non-increasing family of topologies. These topologies are used to construct a fuzzifying topology in a way, similar to ours. The difference between the both approaches, in addition to the context of the work *KM-fuzzy pseudo-metrics vs George-Veeramani fuzzy pseudo-metrics*, is that the representation theorem applied in [21] requests the use of the minimum t-norm $\wedge$ in [20]. On the other hand within our approach, we have to restrict to strong fuzzy metrics in order to obtain a non-increasing family of level topologies.

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[4]The fact that we restrict ourselves to fuzzy-metric spaces is not essential: most of the results presented here can be "automatically" reformulated for pseudo-metric spaces

Acknowledgments

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