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Are fixed point theorems in G-metric spaces an authentic generalization of their classical counterparts?

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Abstract Many G-metric fixed point results can be retrieved from classical ones given in the (quasi-)metric framework. Indeed, many G-contractive conditions can be reduced to a quasi-metric counterpart assumed in the statement of celebrated fixed point results. In this paper, we show that the existence of fixed points for the most part in the aforesaid G-metric fixed point results are guaranteed by a very general celebrated result by Park, even when the G-contractive condition is reduced to a quasi-metric one which is not considered as a contractive condition in any celebrated fixed point result. Moreover, in all those cases in which a quasi-metric contractivity can be raised, we show that the uniqueness of the fixed point is also derived from it. Despite our finding, we also show that there are a few G-metric fixed point results in the literature whose contractive condition cannot be reduced to any (quasi-)metric counterpart. In these cases, however we have seen that Park's result applies again for all cases analyzed, the G-metric techniques become essential in order to yield the uniqueness of the fixed point and, even, to check that the self-mapping under study satisfies some requirements of the Park result. Therefore, it seems natural to encourage the researchers in this field to only study new results in this direction, i.e., in which the aforesaid Park's result cannot be applied.

Keywords (quasi-)metric · G-metric · self-mapping · fixed point

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1 Introduction

In the last few years there has been intense research activity in the field of a new kind of metric type space, the so-called G-metric spaces in the sense of [14]. In particular, a large number of fixed point theorems for self-mappings in G-metric spaces have been given in numerous recent works (see, for instance, [1,3, 7,9–13] and the references therein). The importance of this research subject is such that a monograph has been written recently with the target of collecting the basics and advanced topics about G-metric spaces and fixed point theory (see [2]). In all aforementioned references the fixed point results assume in their statement some type of generalized G-metric contractive condition for the self-mapping. However, it was proved that most of the aforesaid fixed point results in G-metric spaces can be deduced immediately from known and celebrated fixed point theorems in (quasi-)metric spaces. In fact, in the aforesaid references, many G-metric contractive conditions under consideration can be reduced to a (quasi-)metric contractive condition and, thus, the classical results can be applied in order to get the existence and uniqueness of the self-mappings (we refer the reader to [1,5,7,17,18] for a wider treatment). Hence, the natural question about whether most fixed point results in G-metric spaces could be deduced from a classical one stated in (quasi-)metric spaces arose in a natural way. In this direction, a few G-metric contractive conditions for self-mappings that cannot be reduced to certain classical (quasi-)metric contractive conditions were presented in [11,12]. Thus, the authors of the last references used such an argument to justify the importance and need of both, the fixed point theory in G-metric spaces and the use of purely G-metric techniques in such a theory. Nevertheless, we have detected that even in that cases such fixed point results in the G-metric approach can be deduced from a very general fixed point result in (quasi-)metric spaces, which was obtained by S. Park. Hence the main objective of this paper is to provide a positive answer to the above posed question and to prove that most fixed point results for self-mappings in G-metric spaces that we have found in the literature so far, are not real generalization of classical ones because they can be derived from Park's result. Despite our finding, there are a few special cases in which the G-metric fixed point results whose contractive condition really cannot be reduced to any (quasi-)metric one. In these cases, although we have seen that Park's results applies again for all cases analyzed, the G-metric techniques become essential in order to yield the uniqueness of the fixed point and, even, to check that the self-mapping under study satisfies some requirements of the Park result. We present the last fact as a motivation for redirecting the G-metric fixed point theory research towards the right direction, i.e., towards those fixed point results where the quasi-metric techniques are not useful and the Park result fails to be applicable.

2 Reducing G-metric fixed point theorems to Park's fixed point result

In order to archive our main objective, let us recall a few basic aspects about G-metric spaces. According to [14] (see also [2]), a G-metric space is a pair (X, G)

such that X is a nonempty set and $G : X \times X \times X \to [0, \infty[$ is a mapping satisfying the following axioms for all $x, y, z, a \in X$:

$(G1)$ $G(x, y, z) = 0$ if and only if $x = y = z$;
$(G2)$ $G(x, x, y) > 0$ if $x \neq y$;
$(G3)$ $G(x, x, y) \leq G(x, y, z)$ if $z \neq y$;
$(G4)$ $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetric in all three variables);
$(G5)$ $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$;

It must be pointed out that in the original definition of G-metric space given in [14], the condition (G1) asserts that $G(x, y, z) = 0$ if $x = y = z$. However, we have altered the aforesaid condition in the preceding definition because Proposition 1 in [14] guarantees that if (X, G) is a G-metric space and, in addition, $G(x, y, z) = 0$, then $x = y = z$.

Following [14], each G-metric G on a nonempty set X induces a topology τ_G on X such that the family of open balls $B_G(x, \varepsilon)$ forms a neighbourhood system at $x \in X$, where $B_G(x, \varepsilon) = \{y \in X : G(x, y, y) < \varepsilon\}$ and $\varepsilon \in]0, \infty[$.

In [2] a relationship between G-metric spaces and quasi-metric spaces has been stated. With the aim of recalling such a relationship let us compile the pertinent notions about quasi-metric spaces that will be crucial in our subsequent discussion.

On account of [8], a quasi-metric space is a pair (X, d) such that X is a nonempty set and $d : X \times X \to [0, \infty[$ is a mapping satisfying the following axioms for all $x, y, z \in X$:

$(Q1)$ $d(x, y) = 0 \Leftrightarrow x = y$;
$(Q2)$ $d(x, z) \leq d(x, y) + d(y, z)$.

Observe that this notion of quasi-metric is called T_1 quasi-metric in [8]. Moreover, a metric d on a nonempty set X is a quasi-metric which fulfills additionally the property for all $x, y \in X$:

$(Q3)$ $d(x, y) = d(y, x)$.

Each quasi-metric d on a set X induces a T_1 topology τ_d on X such that the family of open balls $B_d(x, \varepsilon)$ forms a neighbourhood system at $x \in X$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ and $\varepsilon \in]0, \infty[$. Notice that in general the topology τ_d induced by a quasi-metric d is only T_1 but not T_2 (Hausdorff) in general.

According to [2], there exists a deep connection between both of a G-metric spaces and quasi-metric. By Lemma 3.3.1 in [2] we have the next concrete relationship.

If (X, G) is a G-metric space, then the pair (X, d_G) is a quasi-metric space with the quasi-metric d_G defined on X as follows: $d_G(x, y) = G(x, y, y)$ for all $x, y \in X$. It is obvious that $\tau_G = \tau_{d_G}$. Moreover, by Theorem 3.2.1 (see also Section 3 in [14]), the topology τ_G is Hausdorff and, thus, τ_{d_G} is Hausdorff. Observe that when the G-metric is symmetric, according to [14], the quasi-metric d_G is exactly a metric.

In G-metric fixed point theory the completeness plays a central role. On account of [14], a G-metric space (X, G) is complete provided that every Cauchy sequence is convergent with respect to τ_G, where a sequence $(x_n)_{n \in \mathbb{N}}$ is said to be Cauchy when, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq n_0$.

Taking into account the exposed facts about G-metric spaces and quasi-metric spaces we are able to show that most fixed point results obtained in G-metric spaces can be deduced from a fixed point result stated in quasi-metric spaces obtained by S. Park in [16]. To this end, let us recall such a result.

Theorem 1 *Let (X, τ) be a topological space, let $d : X \times X \to [0, \infty[$ be a continuous mapping such that $d(x, y) = 0 \Leftrightarrow x = y$ and let $f : X \to X$ be a mapping. Suppose that there exist $x, x_0 \in X$ such that the following conditions hold:*

1. *$lim_{n\to\infty} d(f^n(x_0), f^{n+1}(x_0)) = 0$,*
2. *$(f^n(x_0))_{n\in\mathbb{N}}$ converges to x with respect to τ,*
3. *f is orbitally continuous at x with respect to τ.*

Then $x \in Fix(f) = \{y \in X : f(y) = y\}$.

It must be stressed that the original Park's version of the preceding result was stated for lower semicontinuous mappings d. However, we have focused our attention on continuous ones because it is enough for our announced purpose. Recall that, given a topological space (X, τ) and $x \in X$, a self-mapping $f : X \to X$ is said to be orbitally continuous at x with respect to τ provided that the sequence $(f^{n+1}(x_0))_{n\in\mathbb{N}}$ converges to $f(x)$ with respect to τ whenever $(f^n(x_0))_{n\in\mathbb{N}}$ converges to x with respect to τ (see [4]).

When a G-metrics is under consideration, the quasi-metric induced by it enjoys the next property which will be crucial for our subsequent purpose.

Proposition 1 *Let (X, G) be a G-metric space. Then the induced quasi-metric d_G is continuous with respect to τ_{d_G}.*

Proof Let $x, y \in X$ and consider the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ which converge to x and y with respect to τ_{d_G}, respectively. Since $\tau_{d_G} = \tau_G$ we have that

$$\lim_{n\to\infty} G(x_n, x_n, x) = \lim_{n\to\infty} G(x_n, x, x) = \lim_{n\to\infty} G(y_n, y_n, y) = \lim_{n\to\infty} G(y_n, y, y) = 0.$$

Since each G-metric G is continuous with respect to τ_G (see [14, Proposition 8], then

$$\lim_{n\to\infty} d_G(x_n, y_n) = \lim_{n\to\infty} G(x_n, y_n, y_n) = G(x, y, y) = d_G(x, y).$$

Hence, G is a continuous.

In the light of the preceding auxiliary result we obtain from Theorem 1 the following corollary.

Corollary 1 *Let (X, G) be a G-metric space and let $f : X \to X$ be a mapping. Suppose that there exist $x, x_0 \in X$ such that the following conditions hold:*

1. *$lim_{n\to\infty} d_G(f^n(x_0), f^{n+1}(x_0)) = 0$,*
2. *$(f^n(x_0))_{n\in\mathbb{N}}$ converges to x with respect to τ_{d_G},*
3. *f is orbitally continuous at x with respect to τ_{d_G}.*

Then $x \in Fix(f) = \{y \in X : f(y) = y\}$.

In the statement of fixed point results for self-mappings in complete G-metric spaces G-contractive conditions are assumed which hold for all elements in the space. Nevertheless, in order to get the existence of a fixed point, the contractive condition under consideration is imposed on a sequence of elements. These elements are induced by a Picard sequence generated by the self-mapping in such a way that the next facts are in general proved in results collected in the cited references:

1. Fixing $x_0 \in X$ and imposing the contractive condition under consideration on the sequence of triplets $(f^n(x_0), f^{n+1}(x_0), f^{n+1}(x_0))_{n\in\mathbb{N}}$ leads to $lim_{n\to\infty} G(f^n(x_0), f^{n+1}(x_0), f^{n+1}(x_0)) = 0$.
2. From the fact that $lim_{n\to\infty} G(f^n(x_0), f^{n+1}(x_0), f^{n+1}(x_0)) = 0$ is showed that the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is Cauchy and, thus, the completeness of the G-metric space yields the existence of $x \in X$ such that $(f^n(x_0))_{n\in\mathbb{N}}$ converges to x with respect to τ_G, i.e., $lim_{n\to\infty} G(x, f^n(x_0), f^n(x_0)) = 0$.

In addition to the preceding two conditions, the following common fact can be deduced from the G-contractive conditions assumed in the aforementioned fixed point results:

$$lim_{n\to\infty} G(f(x), f^{n+1}(x_0), f^{n+1}(x_0)) = 0.$$

It must be pointed out that the above condition is equivalent to the orbital continuity of the self-mapping f at x with respect to τ_G. Notice that this notion has been considered recently in [9]. Moreover, we want to stress that the preceding fact is not always explicitly proved in the G-metric fixed point theorems. Indeed, this fact is in general glossed over by many G-metric fixed point researchers.

Next, observe that the preceding three conditions implies the following ones:

1. $lim_{n\to\infty} d_G(f^n(x_0), f^{n+1}(x_0)) = 0$,
2. $lim_{n\to\infty} d_G(x, f^n(x_0)) = 0$, i.e., $(f^n(x_0))_{n\in\mathbb{N}}$ converges to x with respect to τ_{d_G},
3. $lim_{n\to\infty} d_G(f(x), f^{n+1}(x_0)) = 0$, i.e., f is orbitally continuous at x with respect to τ_{d_G}.

Therefore the existence of a fixed point in most fixed point theorems for self-mappings defined on complete G-metric spaces found in the quoted references are immediately deduced from Corollary 1. As an illustrative example of the power of Corollary 1 we discuss the following result that was recently proved in [9]:

Theorem 2 *Let (X, G) be a G-complete G-metric space and let $f : X \to X$ be a mapping. Suppose that f satisfies the following condition for all $x, y, z \in X$:*

$$G(f(x), f(y), f(z)) \leq$$

$$\left(\frac{G(f(x), y, z) + G(x, f(y), z) + G(x, y, f(z))}{2G(x, f(x), f(x)) + G(y, f(y), f(y)) + G(z, f(z), f(z)) + 1} \right) G(x, y, z).$$

Then there exists $x \in X$ such that:

1. *$x \in Fix(f)$.*
2. *The sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges to x with respect to τ_G for any $x_0 \in X$.*

Next we show that actually the preceding result can be derived from Corollary 1. Indeed, in [9] the following facts were proved:

1. $G(f^n(x_0), f^{n+1}(x_0), f^{n+1}(x_0)) \leq \alpha^n G(x_0, f(x_0), f(x_0))$ for all $n \in \mathbb{N}$,
2. $G(f^n(x_0), f^m(x_0), f^m(x_0)) \leq \left(\alpha^n + \alpha^{n+1} + \ldots + \alpha^m\right) G(x_0, f(x_0), f(x_0)) \leq \frac{\alpha^n}{1-\alpha} G(x_0, f(x_0), f(x_0))$ for all $m, n \in \mathbb{N}$,

where $\alpha = \left(\frac{2G(x_0,f(x_0),f(x_0)+2G(f(x_0),f^2(x_0),f^2(x_0))}{2G(x_0,f(x_0),f(x_0)+2G(f(x_0),f^2(x_0),f^2(x_0))+1}\right)$.

Since $\alpha \in [0,1[$, we immediately deduce that

$$lim_{n\to\infty} d_G(f^n(x_0), f^{n+1}(x_0)) = 0$$

and that

$$lim_{n,m\to\infty} d_G(f^n(x_0), f^m(x_0)) = 0.$$

Hence, $(f^n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence and there exists $x \in X$ such that $lim_{n\to\infty} d_G(x, f^n(x_0)) = 0$. So we have verified two of the required conditions in Corollary 1. It remains to check that f is orbitally continuous at x. However, in [9] it was shown that

$$G(f(x), f^{n+1}(x_0), f^{n+1}(x_0)) \leq$$

$$\left(\frac{G(f(x),f^n(x_0),f^n(x_0))+2G(x,f^n(x_0),f^{n+1}(x_0))}{2G(x,f(x),f(x))+2G(f^n(x_0),f^{n+1}(x_0),f^{n+1}(x_0))+1}\right) G(x, f^n(x_0), f^n(x_0)).$$

It follows that $\lim_{n\to\infty} G(f(x), f^{n+1}(x_0), f^{n+1}(x_0)) = 0$, since G is continuous and $\lim_{n\to\infty} G(x, f^n(x_0), f^n(x_0)) = 0$. So, $\lim_{n\to\infty} d_G(f(x), f^{n+1}(x_0)) = 0$. Consequently, f is orbitally continuous at x with respect to τ_{d_G} and, thus, Corollary 1 applies.

The same arguments work for the following result also proved in [9].

Theorem 3 *Let (X, G) be a G-complete G-metric space and let $f : X \to X$ be a mapping. Let $a_1, a_2, a_3, a_4, a_5, a_6, a_7 : X \times X \times X \to [0, \infty[$ be mappings such that there exists $\lambda \in]0, 1[$ with*

$$a_1(x, y, z) + 3a_2(x, y, z) + 4a_3(x, y, z) + a_4(x, y, z) + a_6(x, y, z) \leq \lambda.$$

Suppose that f satisfies the following condition for all $x, y, z \in X$:

$$\begin{aligned}
&G(f(x), f(y), f(z)) \leq a_1(x, y, z)G(x, y, z)+\\
&a_2(x, y, z)\left[G(x, f(x), f(x)) + G(y, f(y), f(y)) + G(z, f(z), f(z))\right]+\\
&a_3(x, y, z)[G(f(x), y, z) + G(x, f(y), z) + G(x, y, f(z))]+\\
&a_4(x, y, z)\min\{G(y, f(y), f(y)), G(z, f(z), f(z))\}\tfrac{[1+G(x,f(x),f(x))]}{1+G(x,y,z)}+\\
&a_5(x, y, z)G(f(x), y, z)\tfrac{[1+G(x,f(y),z)+G(x,y,f(z))]}{1+G(x,y,z)}+\\
&a_6(x, y, z)\tfrac{[1+G(x,f(x),f(x))+G(f(x),y,z)]}{1+G(x,y,z)} + a_7(x, y, z)G(f(x), y, z).
\end{aligned}$$

Then there exists $x \in X$ such that:

1. *$x \in Fix(f)$.*
2. *The sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges to x with respect to τ_G for any $x_0 \in X$.*

It must be stressed that in [9], Theorem 3 was stated assuming that the self-mapping is orbitally continuous. However, it is not hard to verify that this fact can be derived from the G-contractive condition.

Notice that Park's result only guarantees the existence of fixed point but not the uniqueness.

In those cases in which the G-contractivity can be directly reduced to a (quasi-)metric contractive condition valid for all elements of the G-metric space, then the uniqueness can be directly deduced from it. In many cases, in addition, teh resulting (quasi-)metric contractivity matches up with any one assumed in known results. This occurs with many published G-metric fixed point theorems. Examples of this cases are those results given in [1,5,7,18,17] and all results proved in Chapter 4 of [2].

Again, we illustrate our argument by showing the next result provided in [9, Theorem 2.4].

Theorem 4 *Let (X,G) be a G-complete G-metric space and let $f : X \to X$ be a mapping. Let $\alpha, \beta \in [0,1[$ such that $\alpha + \beta < 1$. Suppose that f satisfies the following condition for all $x, y, z \in X$:*

$$G(f(x), f(y), f(z)) \leq$$

$$\alpha \left(\frac{\min\{G(y, f(y), f(y)), G(z, f(z), f(z))\}[1 + G(x, f(x), f(x))]}{1 + G(x, y, z)} \right) + \beta G(x, y, z).$$

Then there exists $x \in X$ such that:

1. *$Fix(f) = x$.*
2. *The sequence $(f^n(x_0))_{n \in \mathbb{N}}$ converges to x with respect to τ_G for any $x_0 \in X$.*

The above contractive condition can be reduced to the following one in the quasi-metric framework, when we take $y, z \in X$ such that $y = z$.

$$d_G(f(x), f(y)) \leq \alpha \left(\frac{d_G(y, f(y))[1 + d_G(x, f(x))]}{1 + d_G(x, y)} \right) + \beta d_G(x, y),$$

for each $x, y \in X$.

Following a similar scheme to that given in the proof of Theorem 2 and by the facts proved in [9, Theorem 2.4] we deduce:

1. $d_G(f^n(x_0), f^{n+1}(x_0)) \leq \left(\frac{\beta}{1-\alpha}\right)^n d_G(x_0, f(x_0))$ for all $n \in \mathbb{N}$,
2. $d_G(f^n(x_0), f^m(x_0)) \leq \left(\frac{\beta}{1-\alpha}\right)^n \left(\frac{1-\alpha}{1-\beta-\alpha}\right) d_G(x_0, f(x_0))$ for all $m, n \in \mathbb{N}$.

Since $\left(\frac{\beta}{1-\alpha}\right) \in [0,1[$, we immediately deduce that

$$lim_{n \to \infty} d_G(f^n(x_0), f^{n+1}(x_0)) = 0$$

and that

$$lim_{n,m \to \infty} d_G(f^n(x_0), f^m(x_0)) = 0.$$

Hence, $(f^n(x_0))_{n \in \mathbb{N}}$ is a Cauchy sequence and there exists $x \in X$ such that $\lim_{n \to \infty} d_G(x, f^n(x_0)) = 0$. So we have checked two of the required conditions in Corollary 1. It remains to prove that f is orbitally continuous at x.

By the above contractive condition, we have that

$$d_G(f(x), f^{n+1}(x_0)) \leq$$

$$\alpha \left(\frac{d_G(f^n(x_0), f^{n+1}(x_0))[1 + d_G(x, f(x))]}{1 + d_G(x, f^n(x_0))} \right) + \beta d_G(x, f^n(x_0)).$$

So, $\lim_{n\to\infty} d_G(f(x), f^{n+1}(x_0)) = 0$ since both $lim_{n\to\infty} d_G(f^n(x_0), f^{n+1}(x_0)) = 0$ and $\lim_{n\to\infty} d_G(x, f^n(x_0)) = 0$. Then, f is orbitally continuous at x with respect to τ_{d_G} and, thus, Corollary 1 applies.

To show the uniqueness of the fixed point, we assume that $x, y \in Fix(f)$, then we deduce that

$$d_G(x, y) \leq \beta d_G(x, y)$$

which implies $d_G(x, y) = 0$ since $\beta \in [0, 1[$. Therefore, $x = y$.

Similar reasoning applies to the following result that also was proved in [9].

Theorem 5 *Let (X, G) be a G-complete G-metric space and let $f : X \to X$ be a mapping. Let $a_1, a_2, a_3 : X \times X \times X \to [0, \infty[$ be mappings such that there exists $\lambda \in]0, 1[$ with $\sum_{i=1}^{3} a_i(x, y, z) \leq \lambda$. Suppose that f satisfies the following condition for all $x, y, z \in X$:*

$$G(f(x), f(y), f(z)) \leq a_1(x, y, z) \left(\tfrac{G(y, f(y), f(y))[1+G(x, f(x), f(x))]}{1+G(x,y,z)} \right) +$$

$$a_2(x, y, z) \left(\tfrac{G(z, f(z), f(z))[1+G(x, f(x), f(x))]}{1+G(x,y,z)} \right) + a_3(x, y, z) G(x, y, z).$$

Then there exists $x \in X$ such that:

1. *$Fix(f) = x$.*
2. *The sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges to x with respect to τ_G for any $x_0 \in X$.*

In this case, by the same argument in Theorem 4, the contractive condition is reduced to the following one in the quasi-metric context. For each $x, y \in X$,

$$d_G(f(x), f(y)) \leq \alpha(x, y) \left(\frac{d_G(y, f(y))[1 + d_G(x, f(x))]}{1 + d_G(x, y)} \right) + \beta(x, y) d_G(x, y),$$

where $\alpha(x, y) = a_1(x, y, y) + a_2(x, y, y)$ and $\beta(x, y) = a_3(x, y, y)$, for each $x, y \in X$.

Observe that the G-contractive condition required in the statement of Theorems 2 and 3 cannot be reduced to a quasi-metric contractive condition and this is the reason for which we only can, in general, ensure the existence of a fixed point.

In [11, 12], G-contractive conditions for self-mappings were given in order to justify the need for using G-metric techniques to obtain a fixed point. Indeed, it was pointed out that metric techniques do not work to this end. In particular, it was claimed that Hardy's Theorem (see [6, Theorem 1]) cannot be applied to get the aforesaid fixed point. However, Corollary 1 again is able to retrieve them as a particular case. In fact, an easy computation shows that in these cases the assumed G-contractive conditions are reduced to quasi-metric contractive ones. In order to avoid repetition of the exposed argument to show that Corollary 1 can be applied, we only focus our attention on one instance in order to show how the G-metric condition can be reduced to a quasi-metric one. Of course, similar techniques can be adapted to the remainder of the results collected in [11, 12].

The following result was proved in [11].

Theorem 6 *Let (X,G) be a G-complete G-metric space and let $f : X \to X$ be a mapping. Let $\alpha \in [0,1[$ and $\beta \in [0, \frac{1}{3}[$. Suppose that f satisfies the following condition for all $x, y, z \in X$:*

$$G(f(x), f(y), f(y)) \leq \max\{\alpha G(x,y,y), \beta[G(x, f(x), f(x)) + 2G(y, f(y), f(y))], \\ \beta[G(x, f(y), f(y)) + G(y, f(y), f(y)) + G(y, f(x), f(x))]\}.$$

Then there exists $x \in X$ such that:

1. *$Fix(f) = x$.*
2. *the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges to x with respect to τ_G for any $x_0 \in X$.*

As we have pointed out, the preceding G-contractive condition can be reduced directly to the following quasi-metric one by proceeding as in the case of Theorem 4.

$$d_G(f(x), f(y)) \leq \max\{\alpha d_G(x,y), \beta[d_G(x, f(x)) + 2d_G(y, f(y))], \\ \beta[d_G(x, f(y)) + d_G(y, f(y)) + d_G(y, f(x))\}.$$

It is easily seen that if $x, y \in Fix(f)$, then we have that

$$d_G(x,y) \leq \max\{\alpha d_G(x,y), 2\beta d_G(x,y), \beta[d_G(x,y) + d_G(y,x)\}.$$

Observe that following similar arguments to those exposed before, Theorem 6 can be proved by applying Park's theorem. To avoid repetition we omit the argument.

Despite the arguments put forward in favor of the feasibility of reducing most of G-contractive conditions assumed in fixed point theory to quasi-metric contractive conditions, one can find a few results for which this approach fails. Theorems 6.1.1, 6.1.2 and 6.1.3, Corollaries 6.1.1., 6.1.2, 6.1.3 and 6.1.4 in [2] are instances of this kind of situation. Nevertheless, in all of them Corollary 1 can be applied in order to get the existence of a fixed point. Since the G-contractive condition cannot be reduced to a quasi-metric contractive condition, one discovers that actually in these cases the purely G-metric techniques reveal themselves as useful for getting, on the one hand, the orbitally continuity of the self-mapping and, on the other hand, the uniqueness of the fixed point.

Next we illuminate our preceding dissertation by discussing a prototypical example. The next result was given in [3] (see also [2, Theorem 6.1.2.]).

Theorem 7 *Let (X,G) be a G-complete G-metric space and let $f : X \to X$ be a mapping. Let $\alpha, \beta, \gamma, \delta \in [0,1[$ such that $\alpha + \beta + \gamma + \delta < 1$. Suppose that f satisfies the following condition for all $x, y, z \in X$:*

$$G(f(x), f(y), f^2(y)) \leq$$

$$\alpha G(x, f(x), f^2(x)) + \beta G(y, f(y), f^2(y)) + \gamma G(x, f(x), f(y)) + \delta G(y, f(y), f^3(x))$$

Then there exists $x \in X$ such that:

1. *$Fix(f) = \{x\}$.*
2. *The sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges to x with respect to τ_G for any $x_0 \in X$.*

Next we show that Corollary 1 is useful for proving the above result. First, let us recall that the next fact was showed in [2]:

$$G(f^n(x_0), f^{n+1}(x_0), f^{n+1}(x_0)) \leq \left(\frac{\alpha+\gamma}{1-\beta-\delta}\right)^n G(x_0, f(x_0), f^2(x_0)),$$

for all $n \in \mathbb{N}$.

By axiom $(G5)$ we have that $G(x_0, f(x_0), f^2(x_0)) \leq G(x_0, f(x_0), f(x_0)) + G(f(x_0), f(x_0), f^2(x_0)))$ and so, the above inequality can be transformed into the follows quasi-metric one:

$$d_G(f^n(x_0), f^{n+1}(x_0)) \leq k^n \cdot A, \tag{1}$$

where $k = \left(\frac{\alpha+\gamma}{1-\beta-\delta}\right)$ and $A = d_G(x_0, f(x_0)) + d_G(f^2(x_0), f(x_0))$.

Note that $k \in [0,1[$, since $\alpha+\beta+\gamma+\delta \in [0,1[$. Then,

$$lim_{n\to\infty} d_G(f^n(x_0), f^{n+1}(x_0)) = 0,$$

and so the first requirement in the statement of Park's Theorem holds.

Moreover, by inequality (1) and axiom $(G5)$ we deduce the next fact, for each $n, m \in \mathbb{N}$ with $m > n$:

$$d_G(f^n(x_0), f^m(x_0)) \leq A \cdot \sum_{i=n}^{m-1} k^i.$$

Again, since $k \in [0,1[$ we conclude that $\lim_{n,m\to\infty} d_G(f^n(x_0), f^m(x_0)) = 0$. Hence, $(f^n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence and there exists $x \in X$ such that $\lim_{n\to\infty} d_G(x, f^n(x_0)) = 0$. So, we have checked the second required condition in Corollary 1.

It remains to prove that f is orbitally continuous at x with respect to τ_{d_G}. However, notice that we cannot achieve such a target by means of purely quasi-metric techniques because the term $G(f(x), f(y), f^2(y))$ in the G-contractive condition cannot be transformed into any quasi-metric value due to its dependence on three different elements. This shows that in this distinguished case, the G-metric methods play a central and relevant role. Let us recall, for the sake of completeness, the way in which we can show the desired orbital continuity taking advantage of the arguments used in [2].

Following [2], we have that if a sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is convergent to x with respect to τ_G, then $f^2(x) = f(x)$. Indeed, on the one hand,

$$\begin{aligned}
&G(f^{n+1}(x_0), f(x), f^2(x)) \leq \\
&\alpha G(f^n(x_0), f^{n+1}(x_0), f^{n+2}(x_0)) + \beta G(x, f(x), f^2(x)) + \\
&\gamma G(f^n(x_0), f^{n+1}(x_0), f(x)) + \delta G(x, f(x), f^{n+3}(x_0))
\end{aligned}$$

for all $n \in \mathbb{N}$. On the other hand, since the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is Cauchy, by Corollary 1 in [14] we have that $lim_{n\to\infty} G(f^n(x_0), f^{n+1}(x_0), f^{n+2}(x_0)) = 0$. So, taking limits in both sides of the above inequality we obtain, by continuity of G, that

$$G(x, f(x), f^2(x)) \leq \frac{\gamma+\delta}{1-\beta} G(x, x, f(x)).$$

If we suppose that $f^2(x) \neq f(x)$, then we deduce by axiom $(G3)$ that

$$G(x, f(x), f^2(x)) \leq \frac{\gamma + \delta}{1 - \beta} G(x, x, f(x)) < G(x, f(x), f^2(x))$$

which is a contradiction. So $f^2(x) = f(x)$ and we obtain, on the one hand,

$$G(x, f(x), f(x)) = G(x, f(x), f^2(x)) \leq \frac{\gamma + \delta}{1 - \beta} G(x, x, f(x)) \tag{2}$$

and, on the other hand,

$$\begin{aligned} &G(f(x), f^{n+1}(x_0), f^{n+2}(x_0)) \leq \\ &\alpha G(x, f(x), f^2(x)) + \beta G(f^n(x_0), f^{n+1}(x_0), f^{n+2}(x_0)) + \\ &\gamma G(x, f(x), f^{n+1}(x_0)) + \delta G(f^n(x_0), f^{n+1}(x_0), f^3(x)) \end{aligned}$$

for all $n \in \mathbb{N}$.

Observe that $f^3(x) = f(x)$. Then, since $lim_{n\to\infty} G(f^n(x_0), f^{n+1}(x_0), f^{n+2}(x_0)) = 0$, taking limits in both sides of the above inequality we obtain, by continuity of G, that

$$G(x, x, f(x)) \leq \frac{\alpha}{1 - \gamma - \delta} G(x, f(x), f(x)). \tag{3}$$

Now, by inequalities (2) and (3) we have that

$$G(x, f(x), f(x)) \leq \frac{\gamma + \delta}{1 - \beta} G(x, x, f(x)) \leq \frac{\gamma + \delta}{1 - \beta} \cdot \frac{\alpha}{1 - \gamma - \delta} G(x, f(x), f(x))$$

and sine $\frac{\gamma+\delta}{1-\beta}, \frac{\alpha}{1-\gamma-\delta} \in [0, 1[$ we deduce that $G(x, f(x), f(x)) = 0$. Thus, $x = f(x)$ and so f is orbitally continuous because $(f^{n+1}(x_0))_{n\in\mathbb{N}}$ converges to $f(x)$.

Finally, the uniqueness can be easily deduced from the G-contractive condition, since, for all $x, y \in Fix(F)$, we have

$$G(x, y, y) \leq \frac{\gamma}{1 - \delta} G(x, x, y) \leq \left(\frac{\gamma}{1 - \delta}\right)^2 G(x, y, y),$$

which implies $G(x, y, y) = 0$ and so $x = y$.

3 Conclusions

In fixed point theory, many results in the G-metric framework have been obtained. However, most of them can be retrieved from well-known classical fixed point results given in the (quasi-)metric context due to the possibility of reducing the G-contractive condition to a quasi-metric one. Moreover, we have shown that the most part of the existence of a fixed point in the aforesaid G-metric fixed point results is a consequence of Park's celebrated result, even when the G-contractive condition is reduced to a quasi-metric one that is not considered as a contractive condition in the statement of any known fixed point result. In those cases in which a quasi-metric contractivity can be raised, the uniqueness of the fixed point is also derived from it. All these facts seem to be overlooked, in general, by the G-metric fixed point theory community. In counterpart, there are G-metric fixed point results whose contractive condition cannot be reduced to any (quasi-)metric one. In these cases, we have nonetheless seen that Park's result applies for all cases analyzed, the G-metric techniques become essential in order to yield the uniqueness of the fixed point and, even, to check that the self-mapping under study satisfies some requirements of the Park result. Therefore, it seems natural to encourage researchers in this field to restrict attention to those new results in which Park's result cannot be applied.

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