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Jumps in commodity prices: New approaches for pricing plain vanilla options[☆]

John Crosby[a], Carme Frau[b,*]

[a] Strome College of Business, Old Dominion University, USA
[b] Business Economics Department, University of Balearic Islands, Spain

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ABSTRACT

We present a new term-structure model for commodity futures prices based on Trolle and Schwartz (2009), which we extend by incorporating multiple jump processes. Our work explores the valuation of plain vanilla options on futures prices when the spot price follows a log-normal process, the forward cost of carry curve and the volatility are stochastic variables, and the spot price and the forward cost of carry allow for time-dampening jumps. We obtain an analytical representation of the characteristic function of the futures prices and, hence, also for plain vanilla option prices using the fast Fourier transform methodology. We price options on WTI crude oil futures contracts using our model and extant models. We obtain higher accuracy than earlier models and save significantly in computing time.

1. Introduction

To obtain accurate estimates of the convenience yield of each commodity, it is crucial to adopt a futures pricing model that is capable of matching the different shapes of the term-structure of commodity futures and can explain a large part of their fluctuations.

There are two major approaches to describe the futures price dynamics for pricing options on commodities, spot models and term-structure models. The spot-based approach relies on specifying the dynamics of a limited set of state variables and deriving futures prices endogenously. According to Schwartz (1997), a single-factor model is not suitable for accurately explaining the variations in futures prices (see, e.g., Brennan and Schwartz (1985)). The inclusion of a second state variable (the convenience yield) substantially enhances the model performance and the model is capable of better describing the forward curve. Typically, two-factor models (see, e.g., Gibson and Schwartz (1990), Brennan (1991) and Schwartz (1997)) let the price of a futures contract depend on the specific dynamics of the spot price and the convenience yield, but they typically assume constant interest rates. Several authors suggest the inclusion of stochastic interest rates as a third state variable. Such three-factor models are discussed in Schwartz (1997), Hilliard and Reis (1998), Miltersen and Schwartz (1998), Cortázar and Schwartz (2003), Nielsen and Schwartz (2004) and Casassus and Collin-Dufresne (2005), among others. Richter and Sørensen (2002), in a model focused on agricultural products, explicitly allow for stochastic volatility. Yan (2002) presents a four-factor model which also allows for stochastic volatility, additionally it allows for jumps in the spot price returns and in the volatility.

The term-structure approach relies on specifying the evolution of the futures curve directly, taking the current market futures prices as given. The futures price is the risk-neutral expectation, conditional

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* Corresponding author.
E-mail addresses: acrosby@odu.edu (J. Crosby), carme.frau@uib.es (C. Frau).

[1] Endogenous conditions are imposed on the drift of the futures price process so that it matches the forward curve.

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on the information of the future spot price available at a given time. Within this approach, models have been focused mostly on deterministic volatility functions, but with a major drawback: they produce flat implied volatility surfaces with respect to strike and time to maturity while we observe smile- and skew-shaped surfaces in the market. The inclusion of stochastic volatility allows us to calibrate the model to option volatility smiles and skews, typically seen in option markets. Models that assume deterministic volatilities are discussed in Reisman (1991), Cortázar and Schwartz (1994), Amin et al. (1995), Hilliard and Reis (1998), Miltersen and Schwartz (1998), Clewlow and Strickland (1999b,a), Miltersen (2003) and Crosby (2008), among others. Andersen (2010) considers a multi-factor diffusion model based on the Heath et al. (1992) (HJM hereafter) framework,[1] paying careful attention to the specification of volatility and to issues of seasonality.

There have been two important extensions in the term-structure literature so far: the inclusion of jumps in the futures dynamics, as in Crosby (2008), and the adaptation of the volatility to be stochastic, as in Trolle and Schwartz (2009) and Trolle (2014). Trolle and Schwartz (2009) is specified under the risk-neutral probability measure and it is based on the HJM framework. Commodity futures prices are driven by two stochastic factors, the spot price and the forward cost of carry curve. They are original in presenting a new stochastic volatility HJM-type model for pricing commodity options. In their three-factor formulation SV1, the volatility of the spot price and that of the cost of carry curve may depend on one volatility factor, whereas in their four-factor general formulation SV2gen, option prices are driven by a short- and a long-term volatility. The advantage of working in an HJM setting is that, as Trolle and Schwartz (2009) point out, unspanned stochastic volatility (USV hereafter) arises naturally. An analytical representation of the characteristic function of the futures price is derived for the computation of standard European options using Fourier transforms. They compute options numerically.

Jumps in the spot price abound in the literature (see, e.g., Merton (1976), Bates (1996) and Trolle (2014)), as do models that present jumps in the variance process (e.g., Duffie et al. (2000) proposes the SVJJ model which extends Bates (1996) by the addition of exponential jumps in the variance). In the scope of commodity models, however, jumps in the cost of carry have scarcely (if ever) been up to date in the literature. Implicitly, they exist in Crosby (2008).[2]

In this work we present a novel term-structure commodity model which is based on the model of Trolle and Schwartz (2009)-SV1. It presents jumps in those factors that affect the futures price, that is, the spot price and the forward cost of carry curve; this idea is inspired by Crosby (2008). We derive an analytical representation of the characteristic function of futures prices, and compute standard European options analytically using the fast Fourier transform algorithm.

The remainder of this article is structured as follows: in Section 2 we present a new three-factor model specification that allows for jumps and describe how to price plain vanilla options on futures contracts; in Section 3 we present an alternative characterisation of the parameters or set-up of our model; in Section 4 we describe the market data set we use and the estimation method; in Section 5 we discuss the values of the calibrated parameters and the pricing performance; and in Section 6 we present our conclusions and ideas for further research.

2. A new three-factor model for futures prices on commodities

Let S_t denote the time-t spot price of the commodity, and let $y(t,T)$ denote the time-t instantaneous forward cost of carry that matures at time T, with $y(t,t) = y_t$ the time-t instantaneous spot cost of carry. We model the evolution of the entire futures curve by specifying one process for S_t and another for $y(t,T)$. Also, let v_t denote the instantaneous variance, which follows a mean-reverting process as in Cox et al. (1985).

Trolle and Schwartz (2009) extends the existing framework to accommodate USV, which we also incorporate in our model. In this work, we introduce simultaneous jumps in S_t returns and in $y(t,T)$, which are uncorrelated with any standard Wiener process present in the equations that describe the factors dynamics.

2.1. The model under the risk-neutral measure $\mathbb{Q}$

Consider the following three-factor model. Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space on which three Brownian motion processes, W_t^S, W_t^y and W_t^v, are defined for all $0 \le t \le T$. On the same probability space, a Poisson process, N_t, is also defined for all $0 \le t \le T$, with a constant intensity parameter $\lambda > 0$. Furthermore, we know that N_t is independent of all Brownian motion processes, W_t^S, W_t^y and W_t^v. Let $\mathcal{F}$ be the filtration generated by these Brownian motions. We also define two random variables, J_S and $J_y(t,T)$, which represent the jump sizes of the Poisson processes in each factor.

The absence of arbitrage implies the existence of a risk-neutral probability $\mathbb{Q}$ under which the drift-adjusted processes followed by $S_t, y(t,T)$ and v_t are governed by the following dynamics

$$\frac{dS_t}{S_t} = \left(y_t - \lambda \mathbb{E}_t^{\mathbb{Q}}\left[e^{J_S} - 1\right]\right) dt + \sigma_S \sqrt{v_t}\, dW_t^S + \left(e^{J_S} - 1\right) dN_t, \quad (2.1)$$

$$dy(t,T) = \left(\mu_y(t,T) - \lambda \mathbb{E}_t^{\mathbb{Q}}\left[J_y(t,T)\right]\right) dt + \sigma_y(t,T)\sqrt{v_t}\, dW_t^y + J_y(t,T)\, dN_t, \quad (2.2)$$

$$dv_t = \kappa\left(\theta - v_t\right) dt + \sigma_v \sqrt{v_t}\, dW_t^v, \quad (2.3)$$

allowing W_t^S, W_t^y and W_t^v to be correlated with ρ_{Sy}, ρ_{Sv} and ρ_{yv}, which denote pairwise correlations. The dynamics in (2.1)–(2.3) can also be expressed in an array form

$$d\begin{pmatrix} S_t \\ y(t,T) \\ v_t \end{pmatrix} = \begin{pmatrix} \left(y_t - \lambda \mathbb{E}_t^{\mathbb{Q}}\left[e^{J_S} - 1\right]\right) S_t \\ \mu_y(t,T) - \lambda \mathbb{E}_t^{\mathbb{Q}}\left[J_y(t,T)\right] \\ \kappa(\theta - v_t) \end{pmatrix} dt + \sqrt{v_t}\begin{pmatrix} \sigma_S S_t & 0 & 0 \\ 0 & \sigma_y(t,T) & 0 \\ 0 & 0 & \sigma_v \end{pmatrix} d\begin{pmatrix} W_t^S \\ W_t^y \\ W_t^v \end{pmatrix} + \begin{pmatrix} \left(e^{J_S} - 1\right) S_t \\ J_y(t,T) \\ 0 \end{pmatrix} dN_t.$$

The forward cost of carry is defined by the difference between the forward interest rate and the forward convenience yield. Our model could be extended with separate processes. Notwithstanding, and as Trolle and Schwartz (2009) indicate, "for pricing most commodity futures contracts, this extension is of minor importance. Furthermore, for pricing short-term or medium-term options on most commodity futures, the pricing error that arises from not explicitly modelling stochastic interest rates is negligible — since the volatility of interest rates is typically orders of magnitudes smaller than the volatility of futures returns, and the correlation between interest rates and futures returns tends to be very low."

Intuitively, the long-term forward cost of carry rates should be less volatile than the short-term ones. Following Trolle and Schwartz (2009), this requirement is satisfied using the following exponentially-dampened specification for the volatility of the forward cost of carry curve[3]

$$\sigma_y(t,T) \equiv \alpha e^{-\gamma(T-t)}, \text{ with } \alpha, \gamma > 0. \quad (2.4)$$

[2] Jumps enter the specification in equal number as futures contracts are considered. Jumps do not enter directly into the spot or cost of carry dynamics, but in the futures dynamics. This model is of the HJM-type but with deterministic volatility.

[3] With this specification, the parameters σ_S, α, θ and σ_v are not simultaneously identified. Trolle and Schwartz (2009) normalise $\eta = \kappa\theta$ to one to achieve identification whereas we, seeking the same objective and inspired by Heston (1993) and Bates (1996), decide to set σ_S to one instead.

The specification followed by the dynamics of the variance v_t guarantees the positiveness of the volatility factor at all times only if the Feller condition is met.[4]

2.1.1. Jump specifications

The jump component will affect mainly the tails of the distribution of futures returns. We assume that the intensity of the jumps λ is constant. We consider the following cases to specify the nature of jump sizes:

- Jumps in S_t are as in Merton (1976)

  Jump assumption (a_1): We assume that $J_S \sim \mathcal{N}(\mu_{J_S}, \sigma^2_{J_S})$ with $\sigma_{J_S} \geq 0$, that is, jumps in S_t are i.i.d. random variables over time.

  Jump assumption (a_2): Jumps in S_t are constant in magnitude, $J_S \equiv \mu_{J_S}$, which is equivalent to imposing $\sigma_{J_S} = 0$ in jump assumption (a_1). This can be seen as an special case of jump assumption (b_2).
- Jumps in $y(t,T)$ are as in Crosby (2008)

$$J_y(t,T) \equiv ae^{-b(T-t)}, \tag{2.5}$$

  Jump assumption (b_1): We assume that jumps in $y(t,T)$ are as in Merton (1976). The jump amplitude parameter a is assumed to be an i.i.d. random variable over time, the distribution of which is defined with respect to $\mathbb{Q}$, satisfying $-\infty < a < \infty$, all of which are independent of the Brownian motions and the Poisson processes. With this, we allow the mean jump to be positive or negative. In this case, the jump-decay parameter b is assumed to be identically equal to zero (i.e., $b \equiv 0$). We assume that $J_y(t,T) \sim \mathcal{N}(\mu_{J_Y}, \sigma^2_{J_Y})$ with $\sigma_{J_Y} \geq 0$.

  Jump assumption (b_2): We assume that jumps in $y(t,T)$ present an exponentially-dampened functional form. The jump amplitude parameter a is assumed to be a finite constant and the jump-decay parameter b is assumed to be any non-negative number. a determines the size of the jump conditional on a jump in N_t, whereas b controls, when jumps occur, how much less long-dated futures contracts jump relative to short-dated futures contracts.
- There can also be a combination of the jumps presented above occurring simultaneously, with the parameters and conditions as previously described (we do not consider mixed jump types):

  Jump assumption (1): We assume i.i.d. jumps in S_t and in $y(t,T)$.

  Jump assumption (2): We assume jumps of constant magnitude in S_t and an exponentially-dampened functional form for jumps in $y(t,T)$.

Their corresponding expressions for expected values and transforms are represented in Table 5.

2.1.2. Futures dynamics

Let $F(t,T)$ denote the time-t price of a futures contract that matures at time T. By definition, we have

$$F(t,T) \equiv S_t e^{\int_t^T y(t,u)du} = S_t e^{Y(t,T)}. \tag{2.6}$$

In the absence of arbitrage opportunities, the process followed by $F(t,T)$ must be a martingale[5] under $\mathbb{Q}$, Duffie (2001). As such, we obtain the following condition for the drift of the forward cost of carry process:

Proposition 1. *The absence of arbitrage implies that the drift term in Eq. (2.2) is given by*

$$\mu_y(t,T) = -v_t\sigma_y(t,T)\Big(\sigma_Y(t,T) + \sigma_S\rho_{Sy}\Big) + \lambda\mathbb{E}_t^{\mathbb{Q}}\left[\left(e^{J_S} + J_y(t,T) - 1\right) - \left(e^{J_S + J_y(t,T)} - 1\right)\right], \tag{2.7}$$

where

$$\sigma_Y(t,T) \equiv \int_t^T \sigma_y(t,u)du = \frac{\alpha}{\gamma}\left(1 - e^{-\gamma(T-t)}\right). \tag{2.8}$$

Proof. See Appendix A.1 for proof. ■

Despite the existence of jumps, this condition is analogous to the drift condition in forward interest rate term-structure models such as Heath et al. (1992).

From applying Itô's Lemma for jump diffusion processes (see Cont and Tankov (2003, Sec. 8.3.2)) to (2.6), given (2.7) and setting the drift to zero, it follows that the dynamics of $F(t,T)$ are given by

$$\frac{dF(t,T)}{F(t,T)} = \sqrt{v_t}\Big(\sigma_S dW_t^S + \sigma_Y(t,T)dW_t^y\Big) - \lambda\mathbb{E}_t^{\mathbb{Q}}\left[e^{J_S + J_Y(t,T)} - 1\right]dt + \left(e^{J_S + J_Y(t,T)} - 1\right)dN_t, \tag{2.9}$$

where

$$J_Y(t,T) \equiv \int_t^T J_y(t,u)du = \frac{a}{b}\left(1 - e^{-b(T-t)}\right). \tag{2.10}$$

The volatility of the futures prices depends on v_t, a factor driven by W_t^v which does not appear in the expression followed by the futures dynamics (2.9). The volatility risk and the options on futures contracts cannot be completely hedged by trading in futures contracts alone, for which reason the model features USV. To the extent that W_t^v is correlated with W_t^S and W_t^y, the variance factor contains a spanned component and volatility risk is partly hedgeable. As special cases, if this correlation is 0, the volatility risk is completely unhedgeable, and if this correlation is ± 1, the volatility risk is completely hedgeable.

In the following proposition, we show that the cost of carry rates are affine jump–diffusion functions of two state variables, namely χ_t and ϕ_t, plus the jump-related terms; the log-futures prices $f(t,T) \equiv \ln F(t,T)$ are also affine jump–diffusion functions of the same variables and terms, plus the log-spot prices $s_t \equiv \ln S_t$:

Proposition 2. *The forward and the instantaneous cost of carry rates and log-futures prices are given by*

$$y(t,T) = y(0,T) - \lambda\int_0^t E_u^{\mathbb{Q}}\left[J_y(u,T)\right]du + \int_0^t J_y(u,T)dN_u + \sigma_y(t,T)\chi_t + \frac{\sigma_y^2(t,T)}{\alpha}\phi_t, \tag{2.11}$$

$$y_t = y(0,t) - \lambda\int_0^t E_u^{\mathbb{Q}}\left[J_y(u,t)\right]du + \int_0^t J_y(u,t)dN_u + \alpha\left(\chi_t + \phi_t\right). \tag{2.12}$$

$$f(t,T) = s_t + f(0,T) - f(0,t) + \sigma_Y(t,T)\chi_t + \frac{\hat{\sigma}_Y(t,T)}{\alpha}\phi_t + \int_0^t\left(y_u - \frac{\sigma_S^2}{2}v_u\right)du + \sigma_S\int_0^t\sqrt{v_u}dW_u^S - \lambda\int_0^t E_u^{\mathbb{Q}}\left[e^{J_S} + J_Y(u,T) - 1\right]du + \int_0^t\left(J_S + J_Y(u,T)\right)dN_u. \tag{2.13}$$

with $\hat{\sigma}_Y(t,T)$ as in (A.19), the dynamics for $f(t,T), \chi_t$ and ϕ_t as in (A.8),(A.13) and (A.14).

[4] In Heston (1993), the parameters obey that $2\kappa\theta > \sigma_v$, which is when the values of v_t are strictly positive. We also consider this restriction is met in Bates (1996), Trolle and Schwartz (2009), Trolle (2014) and in our model.

[5] In this paper, we assume that stochastic processes with zero drift are true martingales and are not merely local martingales. Our assumptions concerning jumps pose no additional issues, in this regard, since we assume that jumps sizes are constant or normally distributed (as in Merton (1976)). There are potential issues concerning our use of Heston (1993) stochastic volatility — issues that are also present in Trolle and Schwartz (2009). For the regularity conditions required for the futures price to be a (true) martingale, under $\mathbb{Q}$, see Theorems 3.5 and 3.6 of Wong and Heyde (2006).

Proof. See Appendix A.2 for proof. ■

In the following proposition we present the expressions followed by the futures and spot prices:

Proposition 3. *With y_u as in (2.12), the expressions followed by the spot and futures prices are given by*

$$F(t,T) = F(0,T)\exp\left\{\int_0^t\left(\sqrt{v_u}\left(\sigma_S dW_u^S + \sigma_Y(u,T)dW_u^y\right) - \frac{v_u}{2}\left(\sigma_S dW_u^S + \sigma_Y(u,T)dW_u^y\right)^2\right)\right\}$$
$$\exp\left\{-\lambda\int_0^t E_u^{\mathbb{Q}}\left[e^{J_S}+J_Y(u,T)-1\right]du + \int_0^t \left(e^{J_S+J_Y(u,T)}-1\right)dN_u\right\}, \quad (2.14)$$

$$S_t = S_0\exp\left\{\int_0^t\left(y_u - \frac{\sigma_S^2}{2}v_u - \lambda E_u^{\mathbb{Q}}\left[e^{J_S}-1\right]\right)du + \sigma_S\int_0^t\sqrt{v_u}dW_u^S + J_S\int_0^t dN_u\right\}, \quad (2.15)$$

Proof. See Appendix A.3 for proof. ■

2.2. Deriving the characteristic function

As with most exchange-traded products, options on oil expire (T_{Opt}) slightly before the expiration date of the underlying futures contract (T). The Fourier transform for the time-t standard European option price can be expressed in terms of the characteristic function (hereafter CF) $\psi(iu,t,T_{Opt},T)$, so it can be obtained by applying the Fourier inversion theorem. We define $\tau \equiv T_{Opt}-t$ and the process $f(T_{Opt},T)\equiv \ln F(T_{Opt},T)$ with dynamics as in (A.8). To price options on futures, we introduce the transform

$$\psi_t(iu,t,T_{Opt},T) \equiv \mathbb{E}_t^{\mathbb{Q}}[e^{iuf(T_{Opt},T)}], \quad (2.16)$$

which has an exponential affine solution as demonstrated in the following proposition:

Proposition 4. *The transform in (2.16) is given by*

$$\psi_t(iu,t,T_{Opt},T) = e^{A(\tau)+B(\tau)v_t+C(\tau)\lambda+iuf(t,T)}, \quad (2.17)$$

with $C(\tau)$ the new term connected with the jumps. $A(\tau)$, $B(\tau)$ and $C(\tau)$ solve the following system of ODEs

$$\frac{\partial A(\tau)}{\partial\tau} = \kappa\theta B(\tau), \quad (2.18)$$
$$\frac{\partial B(\tau)}{\partial\tau} = b_0 + b_1B(\tau) + b_2B^2(\tau), \quad (2.19)$$
$$\frac{\partial C(\tau)}{\partial\tau} = n_{a_jb_j}(\tau) - iu\, m_{a_jb_j}(\tau), \quad (2.20)$$

with

$$b_0 = -\frac{1}{2}(u^2+iu)\left(\sigma_S^2+\sigma_Y^2(t,T)+2\rho_{Sy}\sigma_S\sigma_Y(t,T)\right),$$
$$b_1 = -\kappa + iu\sigma_v\left(\rho_{Sv}\sigma_S + \rho_{yv}\sigma_Y(t,T)\right), \quad (2.21)$$
$$b_2 = \frac{\sigma_v^2}{2},$$

subject to the initial conditions $A(0)=B(0)=C(0)=0$, for $j=1,2$ and following the jump assumptions in Section 2.1.1. The analytical expressions followed by the expectation terms $(m_{a_1b_1}, m_{a_2b_2})$ and the transform terms $(n_{a_1b_1}, n_{a_2b_2})$ are represented in Table 5.

Proof. See Appendix A.4 for proof. ■

The terms $A(\tau)$ and $B(\tau)$ are defined in Trolle and Schwartz (2009). In a recent work, Sitzia (2018) derives an analytical representation followed by the transform of the futures prices $F(t,T)$ for Trolle and Schwartz (2009)-SV1 and Trolle (2014). Eqs. (2.18) and (2.19) have analytical solutions which are given by

$$A(\tau) = \frac{2\kappa\theta}{\sigma_v^2}\left(\beta\gamma\tau - \mu z - \ln g(z)\right) + k_3, \quad (2.22)$$
$$B(\tau) = \frac{2\gamma}{\sigma_v^2}\left(\beta + \mu z + z\frac{g'(z)}{g(z)}\right), \quad (2.23)$$

where $g(z)$ is a linear combination of Kummer's (M) and Tricomi's (U) hypergeometric functions. The expressions followed by $g(z), g'(z), \beta, \mu$ and z can be found in Eqs. (B.4)–(B.7) in Appendix B.1. To match the initial condition $A(0)=0$, we have that

$$k_3 = \frac{2\kappa\theta\mu}{\sigma_v^2\omega}. \quad (2.24)$$

The following proposition provides the analytic expression followed by the term $C(\tau)$ in Eq. (2.17):

Proposition 5. *Eq. (2.20) has an analytical solution which is given by*

$$C(\tau) = n_{A_jB_j}(\tau) - iu\, m_{A_jB_j}(\tau), \quad (2.25)$$

subject to the initial condition $C(0)=0$, for $j=1,2$ and following the jump assumptions in Section 2.1.1. The analytical expressions followed by the expectation terms $(m_{A_1B_1}, m_{A_2B_2})$ and the transform terms $(n_{A_1B_1}, n_{A_2B_2})$ are represented in Table 5.

The inclusion of jumps in the model does not have any impact on the computation of the hypergeometric functions, which are part of the expressions followed by $A(\tau)$ and $B(\tau)$.

2.2.1. Model sub-specifications

We denote our model by SYSVJ.[6] We consider six model sub-specifications to which we refer with an identifier based on the jump assumption made, as described in Section 2.1.1:

- **SYSVJ**^{a_1}: i.i.d. jumps in S_t — equivalent to Trolle (2014)
- **SYSVJ**^{b_1}: i.i.d. jumps in $y(t,T)$
- **SYSVJ**^{1}: i.i.d. jumps in S_t and in $y(t,T)$
- **SYSVJ**^{a_2}: jumps of constant magnitude in S_t
- **SYSVJ**^{b_2}: exponentially-dampened jumps in $y(t,T)$
- **SYSVJ**^{2}: jumps of constant magnitude in S_t and exponentially-dampened jumps in $y(t,T)$

2.2.2. Nested models

Modelling the futures dynamics using jumps and stochastic volatility causes the futures prices to have non-Gaussian returns — a stylised fact in the energy markets. In Table 3 we present the values for the first four moments of the distribution and we perform the Jarque–Bera normality test: Table 3a refers to monthly observations, whereas Table 3b refers to daily observations. We reject the null hypothesis of normality in returns for each of the labelled contracts M2–Q2, and all the contracts taken together. This implies that jumps and stochastic volatility are required, providing skewness and kurtosis to the distribution of returns. Earlier models did not include dynamics of both kinds.

We consider one-, two- and three-factor models, a mix of spot-based models such as Merton (1976), Heston (1993) and Bates (1996), and term-structure models such as Trolle and Schwartz (2009)-SV1 and Trolle (2014). By setting $m_{B_1}, m_{B_2}, n_{B_1}$ and n_{B_2} to zero in the jump term $C(\tau)$ in (2.25), Trolle (2014) is replicated; by setting the jump term $C(\tau)$ to zero, Trolle and Schwartz (2009)-SV1 is replicated. Further modifications to $A(\tau)$, $B(\tau)$ and $C(\tau)$ need to be put in place to replicate the nested models we present.

[6] SYSVJ is the acronym for Stochastic cost of carry $Y(t,T)$, Stochastic Volatility v_t and Jumps.

Table 1
Models dynamics.

Model	Dynamics
Mer76	$\frac{dS_t}{S_t} = \left(y_t - \lambda \mathbb{E}_t^{\mathbb{Q}}\left[e^{J_S} - 1\right]\right)dt + \sigma_S dW_t^S + \left(e^{J_S} - 1\right)dN_t$
	$\frac{dF(t,T)}{F(t,T)} = -\lambda \mathbb{E}_t^{\mathbb{Q}}\left[e^{J_S} - 1\right]dt + \sigma_S dW_t^S + \left(e^{J_S} - 1\right)dN_t$
Hes93	$\frac{dS_t}{S_t} = y_t dt + \sqrt{v_t} dW_t^S$
	$dv_t = \kappa\left(\theta - v_t\right)dt + \sigma_v \sqrt{v_t} dW_t^v$
	$\frac{dF(t,T)}{F(t,T)} = \sqrt{v_t} dW_t^S$
Bat96	$\frac{dS_t}{S_t} = \left(y_t - \lambda \mathbb{E}_t^{\mathbb{Q}}\left[e^{J_S} - 1\right]\right)dt + \sqrt{v_t} dW_t^S + \left(e^{J_S} - 1\right)dN_t$
	$dv_t = \kappa\left(\theta - v_t\right)dt + \sigma_v \sqrt{v_t} dW_t^v$
	$\frac{dF(t,T)}{F(t,T)} = -\lambda \mathbb{E}_t^{\mathbb{Q}}\left[e^{J_S} - 1\right]dt + \sqrt{v_t} dW_t^S + \left(e^{J_S} - 1\right)dN_t$
TS09-SV1	$\frac{dS_t}{S_t} = y_t dt + \sigma_S \sqrt{v_t} dW_t^S$
	$dy(t,T) = \mu_y(t,T)dt + \sigma_y(t,T)\sqrt{v_t} dW_t^y$
	$dv_t = \kappa\left(\theta - v_t\right)dt + \sigma_v \sqrt{v_t} dW_t^v$
	$\frac{dF(t,T)}{F(t,T)} = \sqrt{v_t}\left(\sigma_S dW_t^S + \sigma_Y(t,T)dW_t^y\right)$
TS09-SV1*	$\frac{dF(t,T)}{F(t,T)} = \sqrt{v_t}\sigma_F(t,T)dW_t^F$
Tro14	$\frac{dS_t}{S_t} = \left(y_t - \lambda \mathbb{E}_t^{\mathbb{Q}}\left[e^{J_S} - 1\right]\right)dt + \sigma_S \sqrt{v_t} dW_t^S + \left(e^{J_S} - 1\right)dN_t$
	$dy(t,T) = \mu_y(t,T)dt + \sigma_y(t,T)\sqrt{v_t} dW_t^y$
	$dv_t = \kappa\left(\theta - v_t\right)dt + \sigma_v \sqrt{v_t} dW_t^v$
	$\frac{dF(t,T)}{F(t,T)} = -\lambda \mathbb{E}_t^{\mathbb{Q}}\left[e^{J_S} - 1\right]dt + \sqrt{v_t}\left(\sigma_S dW_t^S + \sigma_Y(t,T)dW_t^y\right) + \left(e^{J_S} - 1\right)dN_t$
Tro14*	$\frac{dF(t,T)}{F(t,T)} = -\lambda \mathbb{E}_t^{\mathbb{Q}}\left[e^{J_S} - 1\right]dt + \sqrt{v_t}\sigma_F(t,T)dW_t^F + \left(e^{J_S} - 1\right)dN_t$
SYSVJ	$\frac{dS_t}{S_t} = \left(y_t - \lambda \mathbb{E}_t^{\mathbb{Q}}\left[e^{J_S} - 1\right]\right)dt + \sigma_S \sqrt{v_t}\, dW_t^S + \left(e^{J_S} - 1\right)dN_t$
	$dy(t,T) = \left(\mu_y(t,T) - \lambda \mathbb{E}_t^{\mathbb{Q}}\left[J_y(t,T)\right]\right)dt + \sigma_y(t,T)\sqrt{v_t}\, dW_t^y + J_y(t,T)\, dN_t$
	$dv_t = \kappa\left(\theta - v_t\right)dt + \sigma_v \sqrt{v_t}\, dW_t^v$
	$\frac{dF(t,T)}{F(t,T)} = -\lambda \mathbb{E}_t^{\mathbb{Q}}\left[e^{J_S + J_Y(t,T)} - 1\right]dt + \sqrt{v_t}\left(\sigma_S dW_t^S + \sigma_Y(t,T)dW_t^y\right) + \left(e^{J_S + J_Y(t,T)} - 1\right)dN_t$
SYSVJ*	$\frac{dF(t,T)}{F(t,T)} = -\lambda \mathbb{E}_t^{\mathbb{Q}}\left[e^{J_F(t,T)} - 1\right]dt + \sqrt{v_t}\sigma_F(t,T)dW_t^F + \left(e^{J_F(t,T)} - 1\right)dN_t$

Notes: This table presents each model dynamics, the last equation refers to its corresponding futures price dynamics. Those models presenting an alternative characterisation of the parameters (i.e., TS09-SV1, Tro14 and SYSVJ) appear with the superscript *.

To compare different models from a commodity perspective, we transform the original spot-based specifications and get the corresponding futures prices dynamics. These models were not originally meant for commodities but rather for equities or exchange rates and they do not consider a stochastic cost of carry rate. We adapt their specifications to commodity assets accordingly. We will hereafter refer to their equivalent term-structure models, though naming them under the original form. Given (2.16), we present the corresponding Fourier transforms to the extant models considered:

i/ Merton (1976) or **Mer76** hereafter:

$$\psi_t(iu, t, T_{Opt}, T) = e^{A(\tau)+C(\tau)\lambda+iuf(t,T)}. \tag{2.26}$$

This model extends Black and Scholes (1973) incorporating jumps in the spot price S_t, where the constant term $A(\tau)$ in both models coincides. The jump-related term $C(\tau)$ corresponds to our jump assumption (a_1) in Section 2.1.1.

ii/ Heston (1993) or **Hes93** hereafter:

$$\psi_t(iu, t, T_{Opt}, T) = e^{A(\tau)+B(\tau)v_t+iuf(t,T)}. \tag{2.27}$$

In this case, a volatility term $B(\tau)$ is necessary as this model extends Black and Scholes (1973) incorporating stochastic volatility v_t; the independent term $A(\tau)$ in both models coincides.

Table 2
Factors and parameter count per model.

Model	Stochastic factors				Jumps			Parameter count	Analytic solution	Charact. function
	S_t	$y(t,T)$	$F(t,T)$	v_t	S_t	$y(t,T)$	$F(t,T)$			
Mer76	✓				✓			4	✓	
Hes93	✓			✓				4	✓	✓
Bat96	✓			✓	✓			7	✓	✓
TS09-SV1	✓	✓	⋆	✓				9(6)	✓	✓
SYSVJ^{a_1}	✓	✓		✓	✓			12	✓	✓
SYSVJ^{b_1}	✓	✓		✓		✓		12	✓	✓
SYSVJ^{1}	✓	✓	⋆	✓	✓	✓	⋆	14(9)	✓	✓
SYSVJ^{a_2}	✓	✓		✓	✓			11	✓	✓
SYSVJ^{b_2}	✓	✓		✓		✓		12	✓	✓
SYSVJ^{2}	✓	✓	⋆	✓	✓	✓	⋆	13(9)	✓	✓

Notes: For each model, this table indicates the factors and jumps considered, the parameter count and if the model allows for an analytical solution for standard European option pricing. In those models presenting alternative set-up, $F(t,T)$ replaces S_t and $y(t,T)$; this is indicated with the symbols ⋆ (alternative set-up) and ✓(original set-up). The parameter count for the alternative set-up is shown in brackets. SYSVJ(a_1) corresponds to Trolle (2014).

iii/ Bates (1996) or **Bat96** hereafter:

$$\psi_t(iu,t,T_{Opt},T) = e^{A(\tau)+B(\tau)v_t+C(\tau)\lambda+iuf(t,T)}. \tag{2.28}$$

This model is a combination of Heston (1993) and Merton (1976). $A(\tau)$ and $B(\tau)$ are as in Heston (1993), and the jump term $C(\tau)$ is as in Merton (1976).

iv/ Trolle and Schwartz (2009)-**SV1** or **TS09-SV1** hereafter:

$$\psi_t(iu,t,T_{Opt},T) = e^{A(\tau)+B(\tau)v_t+iuf(t,T)}. \tag{2.29}$$

This model consists of the extension of Heston (1993) with a stochastic forward cost of carry curve $y(t,T)$, where $A(\tau)$ and $B(\tau)$ follow (2.22) and (2.23), respectively.

v/ Trolle (2014) or **Tro14** hereafter:

$$\psi_t(iu,t,T_{Opt},T) = e^{A(\tau)+B(\tau)v_t+C(\tau)\lambda+iuf(t,T)}. \tag{2.30}$$

This model consists of the extension of Trolle and Schwartz (2009)-SV1 with i.i.d. jumps in the spot price S_t. $A(\tau)$ and $B(\tau)$ are as in Trolle and Schwartz (2009); $C(\tau)$ is as in Merton (1976). This theoretical model has yet to be subjected to an empirical application.

In Table 1 we present the dynamics followed by the models presented in this list together with our model, providing both the spot and the futures price dynamics. Table 2 shows a classification based on the factors and jumps considered. The expressions followed by the ODEs and the solution to the terms in (2.26)–(2.30) and in our model in (2.17) can be found in Tables 4a and 4b, respectively. Table 5 presents the jump assumptions described in Section 2.1.1, their corresponding expressions for expected values and jump transforms. Given that Trolle (2014) is equivalent to our model sub-specification (a_1), it does not explicitly appear in Tables 2, 4 and 6.

Key advantages of the most recent models (Trolle and Schwartz (2009)-SV1, Trolle (2014) and our model) include improved approximation to the real price behaviour and better description of the implied volatility surface. In our case, adding up to five jump parameters provides even more flexibility to replicate the market implied volatilities, allowing for a wider range of possible shapes (e.g., long-dated contracts jump more than those closer to maturity — a stylised fact in the energy markets). Its implementation is not especially difficult, requiring only the addition of one new term $C(\tau)$ to the CF in Trolle and Schwartz (2009)-SV1.[7]

[7] Observe that this new term can present six different forms, as many as the number of model sub-specifications.

2.3. Pricing of standard European options

Let $C(t,T_{Opt},T,K)$ and $\mathcal{P}(t,T_{Opt},T,K)$ denote the time-t prices of a standard European call (hereafter, call) option and a standard European put (hereafter, put) option that expires at time T_{Opt} with strike K on a futures contract that expires at time T, and let $P(T_{Opt},t)$ denote the time-t price of a zero-coupon bond that matures at time T_{Opt}. This option can be priced quasi-analytically within the framework we describe in this section. In our empirical work, we follow the fast Fourier transform (FFT hereafter) methodology.

We use the Carr and Madan (1999) approach for pricing options which permits the use of the computationally efficient FFT algorithm. Its popularity stems from its remarkable speed: while a naive computation needs N^2 operations, the FFT requires only $N\ln(N)$ steps. In the following proposition we present the expression followed by a call and a put option price:

Proposition 6. *The time-t price of a call and a put option that expires at time T_{Opt} with strike K on a futures contract that expires at T is given by*

$$C(t,T_{Opt},T,K) = P(t,T_{Opt})\frac{e^{-\alpha\ln(K)}}{\pi} \times \int_0^\infty \Re\left[\frac{e^{-iu\ln(K)}\psi_t(u-i(1+\alpha),t,T_{Opt},T)}{\alpha(\alpha+1)-u^2+iu(1+2\alpha)}\right]du, \tag{2.31}$$

$$\mathcal{P}(t,T_{Opt},T,K) = P(t,T_{Opt})\frac{e^{-\alpha\ln(K)}}{\pi} \times \int_0^\infty \Re\left[\frac{e^{-iu\ln(K)}\psi_t(u-i(1-\alpha),t,T_{Opt},T)}{\alpha(\alpha-1)-u^2+iu(1-2\alpha)}\right]du, \tag{2.32}$$

where α is the control parameter.[8]

Proof. The proof is in Carr and Madan (1999). ■

This approach presents two advantages: firstly, it permits the use of the computationally efficient FFT; secondly, it requires the evaluation

[8] α has to be chosen to ensure that it makes the modified option price square-integrable and to obtain good numerical accuracy — a sufficient condition for the Fourier transform to exist. This parameter has to be wisely chosen as it might produce very oscillatory arguments of the integral if too big, or it might approach a point mass around 0 if too small. This parameter is often set to 0.75, seeming to achieve very good numerical results on practically all occasions. We also set it to 0.75.

Table 3
Futures returns: moments and Jarque–Bera test.

(a) Monthly observations									
Contract	Maximum	Minimum	Mean	Std. Dev.	Skewness	Kurtosis	JB Stat.	p-Value	Test
M2	0.4949	−0.6062	−0.0050	0.1092	−0.7948	12.2654	9.3739	0.0182	R
M3	0.3957	−0.4878	−0.0050	0.0986	−0.7081	8.4547	10.0370	0.0158	R
M4	0.3232	−0.4176	−0.0050	0.0918	−0.7223	6.4928	10.4806	0.0144	R
M5	0.2816	−0.3766	−0.0051	0.0874	−0.7326	5.6324	10.7459	0.0136	R
M6	0.2516	−0.3486	−0.0051	0.0839	−0.7536	5.1792	10.9373	0.0131	R
Q1	0.2014	−0.2887	−0.0050	0.0764	−0.7752	4.5381	11.2291	0.0123	R
Q2	0.1647	−0.2585	−0.0050	0.0680	−0.8589	4.4968	11.5409	0.0116	R
ALL	0.4949	−0.6062	−0.0050	0.0887	−0.7789	8.7694	2343.644	0.0010	R
(b) Daily observations									
Contract	Maximum	Minimum	Mean	Std. Dev.	Skewness	Kurtosis	JB Stat.	p-Value	Test
M2	0.5812	−0.5686	−0.0002	0.0289	−0.3230	130.9346	194.4978	0.0010	R
M3	0.2400	−0.3408	−0.0002	0.0240	−1.3885	35.7670	208.6014	0.0010	R
M4	0.1858	−0.2771	−0.0002	0.0225	−1.2731	26.5330	217.2025	0.0010	R
M5	0.1714	−0.2478	−0.0002	0.0215	−1.1976	22.4633	222.6880	0.0010	R
M6	0.1564	−0.2365	−0.0002	0.0207	−1.1427	20.0304	226.8677	0.0010	R
Q1	0.1432	−0.2257	−0.0002	0.0196	−1.0530	17.5825	223.3093	0.0010	R
Q2	0.1129	−0.1968	−0.0002	0.0180	−1.0021	14.5854	240.9477	0.0010	R
ALL	0.5812	−0.5686	−0.0002	0.0224	−1.9807	69.5524	42,831.8384	0.0010	R

Notes: JB accounts for the Jarque–Bera normality test. The null hypothesis refers to the normal distribution of futures returns. The statistic critical value related to a significance level of 0.05 is 5.991. Colum Std.Dev. refers to the standard deviations of the returns. The JB Test being R means that we can reject the null hypothesis at 95%, A means that we cannot reject it at 95%. The data sample is May 27th, 2010 to September 30th, 2020 (in-sample period).

of only one integral, as opposed to the two integrals required when using earlier methods such as in Heston (1993) or Duffie et al. (2000), among others.

3. Alternative characterisation

We have reinterpreted some parameters presented in this work for the sake of model simplicity. In fact, it consists of an equivalent way of understanding the nature of the volatilities and the jumps of the spot price S_t and the forward cost of carry curve $y(t,T)$ and, as a result, the futures dynamics too. The option prices are equivalent no matter whether we use this alternative parameter characterisation or the original set.

3.1. Futures dynamics

In this sub-section we present the alternative expressions for the volatility functions, jump expressions and the futures dynamics. The idea beyond this modification consists of considering a single expression for the volatilities of the factors that affect futures prices $F(t,T)$. It is based in the fact that the spot price S_t does not have a maturity whereas the forward cost of carry curve $y(t,T)$ does. As such, when $t=T$, we have that $\sigma_y(t,t)=\alpha$, and by matching the parameters $\sigma_S=0$ we allow a single expression to hold for the volatilities of both factors. We do the same with the jumps. Next, we present the new expressions we refer to and their integrals. We denote the volatility of $F(t,T)$ by $\sigma_f(t,T)$ and the jumps in $F(t,T)$ by $J_f(t,T)$.

Alternative volatility functions We consider that $\sigma_f(t,T)$ follows an exponentially-dampened functional form, with (2.4) and (2.8) becoming

$$\sigma_f(t,T) \equiv \alpha_0 + \alpha e^{-\gamma(T-t)}, \tag{3.1}$$

$$\sigma_F(t,T) \equiv \int_t^T \sigma_f(t,u)du = \alpha_0(T-t) + \frac{\alpha}{\gamma}\left(1-e^{-\gamma(T-t)}\right). \tag{3.2}$$

Observe that in Section 2.1 we imposed $\sigma_S=1$ given that the parameters σ_S, α, θ and σ_v were not simultaneously identified. This implies that the calibrated values for α_0 and α in the alternative set-up will necessarily differ from the values for σ_S and α in the original characterisation of the models.

Alternative jump specifications We consider the jumps in $F(t,T)$ as in Crosby (2008), with expressions (2.5) and (2.10) becoming

$$J_f(t,T) \equiv a e^{-b(T-t)}, \tag{3.3}$$

$$J_F(t,T) \equiv \int_t^T J_f(t,u)du = \frac{a}{b}\left(1-e^{-b(T-t)}\right), \tag{3.4}$$

for which we contemplate two alternatives:

- **Jump assumption (1)**: Jumps as in Merton (1976), i.e., $J_f(t,T) \sim \mathcal{N}(\mu_{J_F}, \sigma^2_{J_F})$ with $\sigma_{J_F} \geq 0$. The jump amplitude parameter a is assumed to be an i.i.d. random variable over time, the distribution of which is defined with respect to $\mathbb{Q}$. The jump-decay parameter b is assumed to be zero.
- **Jump assumption (2)**: Jumps with an exponentially-dampened functional form, where the jump amplitude parameter a is assumed to be a finite constant and the jump-decay parameter b is assumed to be any non-negative number.

Alternative futures dynamics and model sub-specifications As a result, Eq. (2.9) becomes

$$\frac{dF(t,T)}{F(t,T)} = \sqrt{v_t}\sigma_F(t,T)dW_t^F - \lambda\mathbb{E}_t^{\mathbb{Q}}\left[e^{J_F(t,T)}-1\right]dt + \left(e^{J_F(t,T)}-1\right)dN_t. \tag{3.5}$$

In this set-up we consider two model sub-specifications:

- **SYSVJ^1**: i.i.d. jumps in $F(t,T)$
- **SYSVJ^2**: exponentially-dampened jumps in $F(t,T)$

Table 4
Fourier transforms per model — ODEs and solutions.

(a) ODEs			
Model	$\partial A(\tau)/\partial\tau$	$\partial B(\tau)/\partial\tau$	$\partial C(\tau)/\partial\tau$
Mer76	$-\frac{\sigma_S^2}{2}(u^2+iu)$	–	$n_{a_1}-ium_{a_1}$
Hes93	$B(\tau)\kappa\theta$	$-\frac{1}{2}(u^2+iu)+B(\tau)\left(-\kappa+iu\sigma_v\rho_{Sv}\right)+B^2(\tau)\frac{\sigma_v^2}{2}$	–
Bat96	$B(\tau)\kappa\theta$	$-\frac{1}{2}(u^2+iu)+B(\tau)\left(-\kappa+iu\sigma_v\rho_{Sv}\right)+B^2(\tau)\frac{\sigma_v^2}{2}$	$n_{a_1}-ium_{a_1}$
TS09-SV1(⋆)	$B(\tau)\kappa\theta$	$b_0+b_1B(\tau)+b_2B^2(\tau)$	–
SYSVJ	$B(\tau)\kappa\theta$	$b_0+b_1B(\tau)+b_2B^2(\tau)$	$n_{a_jb_j}-ium_{a_jb_j}$
SYSVJ⋆	$B(\tau)\kappa\theta$	$b_0+b_1B(\tau)+b_2B^2(\tau)$	$n_{f_j}-ium_{f_j}$

(b) Solution to ODEs			
Model	$A(\tau)$	$B(\tau)$	$C(\tau)$
Mer76	$-\frac{\sigma_S^2}{2}(u^2+iu)\tau$	–	$n_{A_1}-ium_{A_1}$
Hes93	$\frac{\kappa\theta}{\sigma_v^2}\left((\kappa-iu\sigma_v\rho_{Sv}+d)\tau-2\ln\frac{1-ge^{d\tau}}{1-g}\right)$	$\frac{\kappa-iu\rho_{Sv}\sigma_v+d}{\sigma_v^2}\left(\frac{1-e^{d\tau}}{1-ge^{d\tau}}\right)$	–
Bat96	$\frac{\kappa\theta}{\sigma_v^2}\left((\kappa-iu\sigma_v\rho_{Sv}+d)\tau-2\ln\frac{1-ge^{d\tau}}{1-g}\right)$	$\frac{\kappa-iu\rho_{Sv}\sigma_v+d}{\sigma_v^2}\left(\frac{1-e^{d\tau}}{1-ge^{d\tau}}\right)$	$n_{A_1}-ium_{A_1}$
TS09-SV1(⋆)	$\frac{2\kappa\theta}{\sigma_v^2}\left(\beta\gamma\tau-\mu z-\ln g(z)\right)+k_3$	$\frac{2\gamma}{\sigma_v^2}\left(\beta+\mu z+z\frac{g'(z)}{g(z)}\right)$	–
SYSVJ	$\frac{2\kappa\theta}{\sigma_v^2}\left(\beta\gamma\tau-\mu z-\ln g(z)\right)+k_3$	$\frac{2\gamma}{\sigma_v^2}\left(\beta+\mu z+z\frac{g'(z)}{g(z)}\right)$	$n_{A_jB_j}-ium_{A_jB_j}$
SYSVJ⋆	$\frac{2\kappa\theta}{\sigma_v^2}\left(\beta\gamma\tau-\mu z-\ln g(z)\right)+k_3$	$\frac{2\gamma}{\sigma_v^2}\left(\beta+\mu z+z\frac{g'(z)}{g(z)}\right)$	$n_{F_j}-ium_{F_j}$

Notes: This table presents the expressions followed by each of the ODEs and their solutions as they can be found in the literature. As per our model, these expressions correspond to Eqs. (2.18)–(2.20). The terms $m_{a_1}, m_{a_2}, m_{b_1}, m_{b_2}, m_{a_1b_1}, m_{a_2b_2}, m_{f_1}, m_{f_2}, n_{a_1}, n_{a_2}, n_{b_1}, n_{b_2}, n_{a_1b_1}, n_{a_2b_2}, n_{f_1}$ and n_{f_2} can be found in Eq. (2.20); their expressions are in Table 5. The terms $m_{A_1}, m_{A_2}, m_{B_1}, m_{B_2}, m_{A_1B_1}, m_{A_2B_2}, m_{F_1}, m_{F_2}, n_{A_1}, n_{A_2}, n_{B_1}, n_{B_2}, n_{A_1B_1}, n_{A_2B_2} n_{F_1}$ and n_{F_2} can be found in Eq. (2.25); their expressions are in Table 5. $m_\bullet$ is the expected value at time t under $\mathbb{Q}$ of the jump assuming the jump assumption •; $n_\bullet$ is the transform of the jump assuming the jump assumption •. For Trolle and Schwartz (2009)-SV1 and our model SYSVJ, the values for $k_3, g(z), g'(z), \beta$ and μ are shown in Eqs. (2.24) and (B.4)–(B.6). In each Sub-table, the superscript ⋆ after the model refers to the alternative characterisation of the parameters. Trolle (2014) corresponds to SYSVJ(a_1). For Heston (1993) and Bates (1996), we have that $g=\frac{\kappa-iu\sigma_v\rho_{Sv}+d}{\kappa-iu\sigma_v\rho_{Sv}-d}, \qquad d=\sqrt{(\kappa-iu\sigma_v\rho_{Sv})^2+\sigma_v^2(u^2+iu)}. \qquad (7.1)$.

3.2. Characteristic function

The coefficients in the dynamics of $B(\tau)$ in Eq. (2.19) result in the new expressions

$$b_0=-\frac{1}{2}(u^2+iu)\sigma_F^2(t,T),$$
$$b_1=-\kappa+iu\sigma_v\rho_{Fv}\sigma_F(t,T), \qquad (3.6)$$
$$b_2=\frac{\sigma_v^2}{2},$$

whereas the expressions followed by $A(\tau)$ and $B(\tau)$ in (2.22) and (2.23) remain the same. Their coefficients can be found in Appendices B.1 and B.2. Subject to the initial condition $C(0)=0$, for $j=1,2$ and the jump assumptions presented above, the alternative jump-related term dynamics and solution are

$$\frac{\partial C(\tau)}{\partial\tau}=n_{f_j}(\tau)-ium_{f_j}(\tau), \qquad (3.7)$$

$$C(\tau)=n_{F_j}(\tau)-ium_{F_j}(\tau). \qquad (3.8)$$

The analytical expressions followed by the expectation terms ($m_{f_1}, m_{f_2}, m_{F_1}$ and m_{F_2}) and the transform terms ($n_{f_1}, n_{f_2}, n_{F_1}$ and n_{F_2}) are represented in Table 5.

4. Model estimation

We directly define our model dynamics under $\mathbb{Q}$ and, therefore, the parameter estimation is performed under this measure. We use the least-square fit to calibrate our parameters. This section describes the market data we use for this analysis and the calibration method we follow.

We consider West Texas Intermediate (WTI) light sweet crude oil data listed on the New York Mercantile Exchange (NYMEX), which we obtain from Refinitive Eikon (formerly Thomson-Reuters Datastream). The data set consists of observations of closing prices (quoted in USD) and open interest for futures prices, and market implied (Black) volatilities for the corresponding options.

The period considered spans from May 27th, 2010 to March 2nd, 2022 and is at monthly and daily frequency, making it 142 monthly and 2984 daily observations, respectively. Only ATM and OTM options are utilised. We select ATM options plus those 15 OTM closer to the ATM level, which we label as ATM$\pm 0.5, 1, \dots, 7, 7.5$ USD, with ATM the futures price. This makes 32 options per contract and observation.

The trading of futures contracts terminates three business days prior to the 25th calendar day of the month prior to the contract month (i.e., delivery month). Futures contracts that mature on those termination dates exist for a wide variety of maturities. In deciding which maturity futures contracts to use, we select contracts based on higher open interest, as detailed in Trolle and Schwartz (2009, p. 5,6). This procedure leaves seven generic futures contracts out of the first 60 available nominal ones: the second- to the sixth-month contracts (M2–M6) and the following two with expiration in either March, June, September or December (Q1–Q2). This represents 868 futures contracts in total. The trading of standard European options terminates six business days prior to the 25th calendar day of the month prior to the contract month.

Table 5
Jump assumptions, corresponding expected values and jump transforms.

Jump	Set-up	Expected Value	Transform
(a_1)	✓	$m_{a_1} \equiv \mathbb{E}_t^{\mathbb{Q}}\left[e^{J_S}-1\right] = e^{\mu_{J_S}+\frac{1}{2}\sigma_{J_S}^2}-1$	$n_{a_1} \equiv e^{iu\mu_{J_S}-\frac{u^2}{2}\sigma_{J_S}^2}-1$
		$m_{A_1} \equiv \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T (e^{J_S}-1)\,du\right] = (T-t)\left(e^{\mu_{J_S}+\frac{1}{2}\sigma_{J_S}^2}-1\right) = (T-t)m_{a_1}$	$n_{A_1} \equiv (T-t)\left(e^{iu\mu_{J_S}-\frac{u^2}{2}\sigma_{J_S}^2}-1\right) = (T-t)n_{a_1}$
(b_1)	✓	$m_{b_1} \equiv \mathbb{E}_t^{\mathbb{Q}}\left[e^{J_y(t,T)}-1\right] = e^{\mu_{J_Y}+\frac{1}{2}\sigma_{J_Y}^2}-1$	$n_{b_1} \equiv e^{iu\mu_{J_Y}-\frac{u^2}{2}\sigma_{J_Y}^2}-1$
		$m_{B_1} \equiv \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T (e^{J_y(t,T)}-1)\,du\right] = (T-t)\left(e^{\mu_{J_Y}+\frac{1}{2}\sigma_{J_Y}^2}-1\right) = (T-t)m_{b_1}$	$n_{B_1} \equiv (T-t)\left(e^{iu\mu_{J_Y}-\frac{u^2}{2}\sigma_{J_Y}^2}-1\right) = (T-t)n_{b_1}$
(1)	✓	$m_{a_1b_1} \equiv \mathbb{E}_t^{\mathbb{Q}}\left[e^{J_S+J_y(t,T)}-1\right] = (m_{a_1}+1)(m_{b_1}+1)-1$	$n_{a_1b_1} \equiv (n_{a_1}+1)(n_{b_1}+1)-1$
		$m_{A_1B_1} \equiv \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T (e^{J_S+J_y(t,T)}-1)\,du\right] = (T-t)m_{a_1b_1}$	$n_{A_1B_1} \equiv (T-t)n_{a_1b_1}$
(1)	⋆	$m_{f_1} \equiv \mathbb{E}_t^{\mathbb{Q}}\left[e^{J_f(t,T)}-1\right] = e^{\mu_{J_F}+\frac{1}{2}\sigma_{J_F}^2}-1$	$n_{f_1} \equiv e^{iu\mu_{J_F}-\frac{u^2}{2}\sigma_{J_F}^2}-1$
		$m_{F_1} \equiv \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T (e^{J_f(t,T)}-1)\,du\right] = (T-t)\left(e^{\mu_{J_F}+\frac{1}{2}\sigma_{J_F}^2}-1\right) = (T-t)m_{f_1}$	$n_{F_1} \equiv (T-t)\left(e^{iu\mu_{J_f}-\frac{u^2}{2}\sigma_{J_f}^2}-1\right) = (T-t)n_{f_1}$
(a_2)	✓	$m_{a_2} \equiv \mathbb{E}_t^{\mathbb{Q}}\left[e^{J_S}-1\right] = e^{\mu_{J_S}}-1$	$n_{a_2} \equiv e^{iu\mu_{J_S}}-1$
		$m_{A_2} \equiv \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T (e^{J_S}-1)\,du\right] = (T-t)\left(e^{\mu_{J_S}}-1\right) = (T-t)m_{a_2}$	$n_{A_2} \equiv (T-t)\left(e^{iu\mu_{J_S}}-1\right) = (T-t)n_{a_2}$
(b_2)	✓	$m_{b_2} \equiv \mathbb{E}_t^{\mathbb{Q}}\left[e^{J_y(t,T)}-1\right] = e^{ae^{-b(T-t)}}-1$	$n_{b_2} \equiv e^{iuae^{-iub(T-t)}}-1$
		$m_{B_2} \equiv \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T (e^{J_y(t,T)}-1)\,du\right] = e^{\frac{a}{b}\left(1-e^{-b(T-t)}\right)}-(T-t)$	$n_{B_2} \equiv e^{iu\frac{a}{b}\left(1-e^{-iub(T-t)}\right)}-(T-t)$
(2)	✓	$m_{a_2b_2} \equiv \mathbb{E}_t^{\mathbb{Q}}\left[e^{J_S+J_y(t,T)}-1\right] = (m_{a_2}+1)(m_{b_2}+1)-1$	$n_{a_2b_2} \equiv (n_{a_2}+1)(n_{b_2}+1)-1$
		$m_{A_2B_2} \equiv \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T (e^{J_S+J_y(t,T)}-1)\,du\right] = (T-t)m_{a_2b_2}$	$n_{A_2B_2} \equiv (T-t)n_{a_2b_2}$
(2)	⋆	$m_{f_2} \equiv \mathbb{E}_t^{\mathbb{Q}}\left[e^{J_f(t,T)}-1\right] = e^{ae^{-b(T-t)}}-1$	$n_{f_2} \equiv e^{iuae^{-iub(T-t)}}-1$
		$m_{F_2} \equiv \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T (e^{J_f(t,T)}-1)\,du\right] = e^{\frac{a}{b}\left(1-e^{-b(T-t)}\right)}-(T-t)$	$n_{F_2} \equiv e^{iu\frac{a}{b}\left(1-e^{-iub(T-t)}\right)}-(T-t)$

Notes: The expressions followed by $J_y(t,T)$ and $J_Y(t,T)$ can be found in (2.5) and (2.10), respectively. The expressions followed by $J_f(t,T)$ and $J_F(t,T)$ can be found in (3.3) and (3.4), respectively. The terms $m_{a_1}, m_{a_2}, m_{b_1}, m_{b_2}, m_{a_1b_1}, m_{a_2b_2}, m_{f_1}, m_{f_2}, n_{a_1}, n_{a_2}, n_{b_1}, n_{b_2}, n_{a_1b_1}, n_{a_2b_2}, n_{f_1}$ and n_{f_2} can be found in (2.20). The terms $m_{A_1}, m_{A_2}, m_{B_1}, m_{B_2}, m_{A_1B_1}, m_{A_2B_2}, m_{F_1}, m_{F_2}, n_{A_1}, n_{A_2}, n_{B_1}, n_{B_2}, n_{A_1B_1}, n_{A_2B_2}, n_{F_1}$ and n_{F_2} can be found in (2.25). The term $m_\bullet$ is the expected value at time t under $\mathbb{Q}$ of the jump in •. The term $n_\bullet$ is the transform of the jump in •. When • is a capital letter, it accounts for the integral from t to T of the jump in the corresponding assumption. The different jump assumptions can be found in column Jump. The symbol ⋆ in the column Set-up accounts for the alternative characterisation of the parameters whereas the symbol ✓refers to the original set-up. Trolle (2014) corresponds to SYSVJ(a_1).

The inputs to the calibration algorithm are the underlying futures prices, the option strikes and the discount factors. Additionally, as a proxy for the instantaneous variance, we use the square of ATM volatilities that corresponds to the shorter maturity contract (in our case, the contract labelled M2) — that is, a unique volatility value v_0 per observation date. We use a least-squares fitting with the objective of minimising the mean absolute error in option volatilities MAE(σ)[9] and the root mean squared error in volatilities RMSE(σ).[10] We apply Feller's condition to all models with stochastic volatility, that is, all models considered except Merton (1976). Heston (1993) and Bates (1996) assume that $\sigma_S = 1$. Following this assumption, we also assume that $\sigma_S = 1$ in Trolle and Schwartz (2009)-SV1, Trolle (2014) and our model in the original set-up, $\sigma_S = 0$ in the alternative set-up. We calibrate the parameters for the models listed in Section 2.2.2 and for our model.

Following Carr and Madan (1999), the integral in the pricing functions (2.31)–(2.32) is numerically computed using Simpson's rule (using Matlab's built-in function *simps*). We compare the performance of other integration methods and we find Simpson's rule to be the best. For the sake of brevity, we do not present this analysis in this work, but the results are available upon request. We have carried out an equivalent analysis to choose the optimal value of α to be used in Eqs. (2.31)–(2.32). When performing a numerical calibration, we use a standard fourth order Runge–Kutta algorithm to solve the system of ODEs (2.18)–(2.20) (using Matlab's built-in function *ode*45). We consider an integral step of 1/10 and an upper bound of 60, which implies 600 evaluation points. Experiments are implemented on an HP laptop computer on a CPU Intel Core i7 2.60 GHz 16.0 GB RAM SSD

[9] MAE(σ) represents the absolute mean of differences between the volatilities predicted by the model $\hat{\sigma}_t$ and the implied market values σ_t, with variables observed from $t = 1, \ldots, N$: MAE(σ) = $\frac{\sum_{t=1}^{N} |\hat{\sigma}_t - \sigma_t|}{N}$.

[10] RMSE(σ) represents the square root of the quadratic mean of differences between the volatilities predicted by the model $\hat{\sigma}_t$ and the implied market values σ_t, with variables observed from $t = 1, \ldots, N$: RMSE(σ) = $\sqrt{\frac{\sum_{t=1}^{N}(\hat{\sigma}_t-\sigma_t)^2}{N}}$.

Table 6
Estimated parameters, errors and computation time.

(a) Monthly observations

Model	SYSVJ						TS09-SV1	Bat96	Hes93	Mer76
Jump	2	b_2	a_2	1	b_1	a_1				
σ_S	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	–	–	0.2209
										(0.2011,0.2375)
α	0.7209	0.1087	0.5418	0.4854	0.7692	0.5373	0.4516	–	–	–
	(0.1624,0.9997)	(0.1548,1.0000)	(0.2648,1.0000)	(0.2688,1.0000)	(0.2937,1.0000)	(0.2999,1.0000)	(0.1165,0.6241)			
γ	2.0851	0.4586	1.0796	0.9962	2.0267	2.0065	0.0882	–	–	–
	(0.0903,2.5000)	(0.1032,2.5000)	(0.0545,2.5000)	(0.0528,2.5000)	(0.0580,2.4998)	(0.0577,2.3299)	(0.0775,0.4037)			
κ	1.9029	2.7236	1.9105	1.9478	1.5101	1.0283	0.9387	2.9585	2.2115	–
	(0.9707,4.3141)	(0.7190,3.8870)	(0.2907,2.9677)	(0.2942,3.1439)	(0.2327,3.2720)	(0.1921,3.1356)	(0.5025,2.3494)	(1.4222,4.2549)	(0.6611,3.8921)	
θ	0.0060	0.0070	0.0076	0.0085	0.0074	0.0062	0.1618	0.0028	0.0676	–
	(0.0022,0.0359)	(0.0034,0.0662)	(0.0026,0.0383)	(0.0024,0.0380)	(0.0030,0.0406)	(0.0033,0.0399)	(0.0845,0.1739)	(0.0001,0.0099)	(0.0596,0.0803)	
σ_v	0.1048	0.1869	0.1013	0.1059	0.1037	0.1115	0.2859	0.1209	0.2782	–
	(0.0668,0.4824)	(0.0506,0.5088)	(0.0320,0.2043)	(0.0288,0.2139)	(0.0439,0.3033)	(0.0346,0.2233)	(0.2441,0.3980)	(0.0096,0.1471)	(0.2366,0.3450)	
ρ_{Sy}	–0.7697	–0.6356	–0.8325	–0.8746	–0.9386	–0.7363	–0.9916	–	–	–
	(–1.0000,0.1104)	(–1.0000,–0.0524)	(–1.0000,–0.5090)	(–1.0000,–0.4554)	(–1.0000,–0.6443)	(–1.0000,–0.5179)	(–1.0000,–0.3672)			
ρ_{Sv}	0.9999	0.6595	1.0000	0.9778	0.9851	1.0000	–0.7435	1.0000	–1.0000	–
	(–0.4455,1.0000)	(–0.9773,1.0000)	(–0.9984,1.0000)	(–0.8956,1.0000)	(–0.8011,1.0000)	(–0.9779,1.0000)	(–0.8838,–0.5475)	(–1.0000,1.0000)	(–1.0000,–1.0000)	
ρ_{yv}	0.9714	–0.0492	0.8520	0.9540	0.3150	0.9847	–0.5937	–	–	–
	(–0.2796,1.0000)	(–0.3872,1.0000)	(–0.5755,1.0000)	(–0.1622,0.9999)	(–0.7327,0.9997)	(–0.9968,1.0000)	(–0.8471,–0.2511)			
λ	0.0919	0.0843	0.0875	0.0879	0.0914	0.0896	–	0.1021	–	0.4592
	(0.0270,0.1125)	(0.0380,0.1119)	(0.0339,0.1106)	(0.0367,0.1058)	(0.0408,0.1110)	(0.0398,0.1077)		(0.0471,0.1328)		(0.0938,1.0000)
μ_{J_S}	–0.8664	–	–1.8480	–0.9041	–	–1.8256	–	–1.3045	–	–0.2698
	(–4.7521,–0.6749)		(–12.4688,–1.0101)	(–11.3296,–0.6038)		(–2.5000,–1.2328)		(–2.5000,–1.0299)		(–0.8439,–0.1911)
σ_{J_S}	–	–	–	0.0002	–	0.0002	–	0.0000	–	0.0061
				(0.0000,0.1495)		(0.0000,0.1479)		(0.0000,0.0033)		(0.0000,0.2500)
μ_{J_Y}	–	–	–	–0.9787	–1.7286	–	–	–	–	–
				(–10.9504,–0.6038)	(–2.5000,–1.2581)					
σ_{J_Y}	–	–	–	0.0000	0.0000	–	–	–	–	–
				(0.0000,0.1631)	(0.0000,0.1108)					
a_Y	–0.8664	–1.8608	–	–	–	–	–	–	–	–
	(–4.6192,–0.0950)	(–9.7908,–1.0613)								
b_Y	0.0000	0.0000	–	–	–	–	–	–	–	–
	(0.0000,0.0689)	(0.0000,0.0716)								
Count	13	12	11	14	12	12	9	7	4	4
$MAE^I(\sigma)$	0.0278	0.0279	0.0280	0.0280	0.0278	0.0280	0.0295	0.0277	0.0306	0.0805
$RMSE^I(\sigma)$	0.0547	0.0550	0.0549	0.0548	0.0548	0.0549	0.0590	0.0543	0.0606	0.1262
ACT	0.3926	0.2369	0.2445	0.1873	0.4383	0.2050	0.3266	0.0981	0.0459	0.0941
$MAE^O(\sigma)$	0.0262	0.0267	0.0263	0.0265	0.0263	0.0265	0.0306	0.0286	0.0339	0.1019
$RMSE^O(\sigma)$	0.0343	0.0347	0.0343	0.0346	0.0343	0.0345	0.0384	0.0359	0.0405	0.1210

Notes: MAE(σ) represents the mean absolute pricing error in option volatilities, RMSE(σ) represents the square root of the quadratic mean of errors in option volatilities. For the error statistics, the superscript I refers to the in-sample analysis, whereas the superscript O refers to the out-of-sample analysis. The pricing errors are defined as the differences between fitted volatilities $\hat{\sigma}_t$ and market implied (Black) volatilities σ_t; they are expressed in parts per unit (e.g., 0.0805 means 8.05%). ACT refers to the computation time using the analytical solutions to the models displayed in Table 4b; the values are expressed in hours. Trolle (2014) corresponds to SYSVJ(a_1). Below each estimated parameter value and in brackets we present the confidence interval associated to a percentile of 95%. We have additional analysis available from the authors on request (i.e., CI(97.5%), CI(99%) and plots with the distribution of the estimates' values for different CIs).

(*continued on next page*)

Table 6 (*continued*).

(b) Daily observations

Model	SYSVJ						TS09-SV1	Bat96	Hes93	Mer76
Jump	2	b_2	a_2	1	b_1	a_1				
σ_S	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	–	–	0.2180
α	0.3742	0.3707	0.7727	0.8002	0.9693	0.8935	0.3745	–	–	–
γ	0.6933	0.8335	0.1047	0.1137	0.1180	0.0995	0.1365	–	–	–
κ	2.2080	2.3895	0.6158	0.5724	0.3381	0.4455	0.9943	2.7686	1.9003	–
θ	0.0243	0.0269	0.0265	0.0223	0.0333	0.0292	0.1414	0.0023	0.0683	–
σ_v	0.3228	0.3565	0.1246	0.1275	0.1248	0.1059	0.2775	0.1131	0.2731	–
ρ_{Sy}	−0.4214	−0.2987	−0.9961	−0.9949	−0.9991	−0.9991	−0.9096	–	–	–
ρ_{Sv}	0.0579	0.0653	0.3030	0.4882	0.5686	0.5657	−0.6657	1.0000	−1.0000	–
ρ_{yv}	0.1075	0.0183	0.3505	0.1346	0.6384	0.6391	−0.7219	–	–	–
λ	0.0705	0.0795	0.1000	0.1005	0.1037	0.0997	–	0.0960	–	0.6502
μ_{J_S}	−0.9069	–	−1.2891	−0.6762	–	−1.4818	–	−1.3793	–	−0.2263
σ_{J_S}	–	–	–	0.0256	–	0.1480	–	0.0000	–	0.0001
μ_{J_Y}	–	–	–	−0.6762	−1.4677	–	–	–	–	–
σ_{J_Y}	–	–	–	0.0256	0.0802	–	–	–	–	–
a_Y	−0.7358	−1.2668	–	–	–	–	–	–	–	–
b_Y	0.0680	0.0668	–	–	–	–	–	–	–	–
Count	13	12	11	14	12	12	9	7	4	4
$\text{MAE}^I(\sigma)$	0.0318	0.0321	0.0301	0.0300	0.0297	0.0298	0.0363	0.0250	0.0280	0.0803
$\text{RMSE}^I(\sigma)$	0.0542	0.0540	0.0526	0.0526	0.0519	0.0521	0.0601	0.0489	0.0539	0.1228
ACT	5.1864	3.0470	3.2148	5.2096	6.1521	4.3336	4.2462	8.2889	1.3871	5.4152
$\text{MAE}^O(\sigma)$	0.0278	0.0291	0.0281	0.0277	0.0279	0.0285	0.0284	0.0289	0.0341	0.0964
$\text{RMSE}^O(\sigma)$	0.0363	0.0381	0.0371	0.0366	0.0385	0.0375	0.0392	0.0364	0.0411	0.1117

Notes: MAE(σ) represents the mean absolute pricing error in option volatilities, RMSE(σ) represents the square root of the quadratic mean of errors in option volatilities. For the error statistics, the superscript I refers to the in-sample analysis, whereas the superscript O refers to the out-of-sample analysis. The pricing errors are defined as the differences between fitted volatilities $\hat{\sigma}_t$ and market implied (Black) volatilities σ_t; they are expressed in parts per unit (e.g., 0.0805 means 8.05%). ACT refers to the computation time using the analytical solutions to the models displayed in Sub-Table 4b; the values are expressed in hours. Trolle (2014) corresponds to SYSVJ(a_1).

hard drive machine, running on Windows 10 64 bits, with Matlab version R2020b and Microsoft Office 64 bits.

4.1. In-sample analysis

The period considered spans from May 27th, 2010 to September 30th, 2020 and is at monthly and daily frequency, making it 125 monthly and 2618 daily observations, respectively.

4.1.1. Monthly observations

The observation period considered makes 125 end of month observations, which are taken on the last business day of the month in the period considered. As a result, we consider the seven futures contracts and their 32 corresponding ATM and OTM options for each observation, which represent 28,000 options. Because of data issues, 3570 options are discarded, making it a final number of 24,430 options (87.25% of the total number).

Fig. 1 focuses on the period May 2010 to February 2022, with a vertical line separating the in-sample from the out-of-sample period; it considers the contracts labelled M2, Q1, Q2 only. Fig. 1a shows the evolution of futures prices per contracts, Fig. 1b shows futures returns, Fig. 1c shows Black volatilities for ATM call options, and Fig. 1d presents a histogram of futures returns compared with those normally-distributed (the histogram refers to the in-sample period). In the histogram, the presence of skewness and kurtosis (fat tails) is evident. In Fig. 1c, we also observe that volatility is stochastic. In the period considered, the jump in 2011 corresponds to the Arab spring; the biggest downward jump occurring in 2014 corresponds to the territorial gains made by ISIS in Iraq and Syria, the surprising growth of US shale oil production (fracking) and the decision taken by the OPEC to maintain output. Futures prices plummeted in March 2020, which corresponds to the beginning of the Covid-19 pandemic.

A Jarque–Bera normality test demonstrates that futures returns are not Gaussian. Table 3a presents the results of the test, contract by contract and for all contracts taken together. The values of the skewness and kurtosis signal the returns not to have a normal distribution, which supports the inclusion of stochastic variance as well as jumps.

4.1.2. Daily observations

The observation period considered makes 2618 daily observations. As a result, we consider the seven futures contracts and their 32 corresponding options for each observation, which represent 586,432 options. Because of data issues similar to the monthly observations, 74,194 options are discarded, making it a final number of 512,238 options (87.35% of the total number).

We omit plotting futures prices and returns, and options volatilities on daily observations to reduce clutter on the graphs. Again, a Jarque–Bera normality test demonstrates that futures returns are not Gaussian. Table 3b presents the results.

4.2. Out-of-sample analysis

We perform an out-of-sample analysis using data October 1st, 2020 to March 2nd, 2022. It is at monthly and daily frequency. We consider the seven futures contracts and their 32 corresponding ATM and OTM options for each observation.

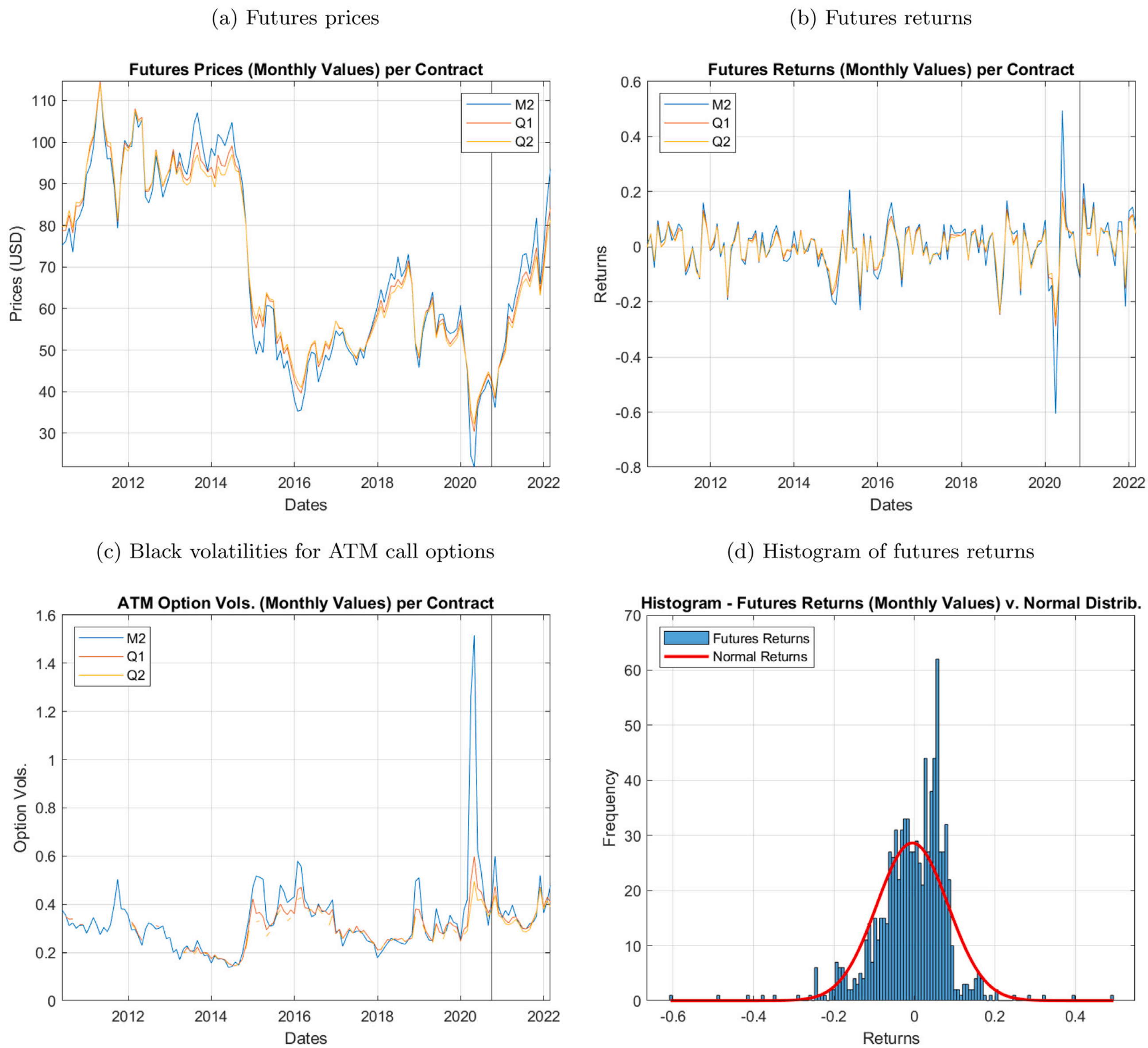
4.2.1. Monthly observations

The observation period considered has 17 monthly observations, which represent 3808 options. Because of data issues, 107 options are discarded, resulting in 3701 options (97.19% of the total number).

In Fig. 1, in particular Sub-figures (a), (b) and (c), the period after the vertical line corresponds to this out-of-sample period. It is a period with upward jumps in prices since the start of Covid-19 pandemic and up to the end of the sample, which coincides with the Russian invasion of Ukraine (February 24th, 2022).

4.2.2. Daily observations

The observation period considered has 366 daily observations, which represent 81,984 options. Because of data issues, 2408 options are discarded, the final number is 79,576 options (97.06% of the total number).



(a) Futures prices

(b) Futures returns

(c) Black volatilities for ATM call options

(d) Histogram of futures returns

Fig. 1. Futures prices and returns, ATM call options implied volatilities — monthly data.

Notes: This figure presents values that correspond to the futures contracts labelled M2, Q1, Q2 for the period May 2010 to March 2022; the vertical line separates the in-sample with the out-of-sample period. Sub-figure (a) presents futures prices, Sub-figure (b) presents futures returns, Sub-figure (c) presents implied volatilities for ATM call options, and Sub-figure (d) presents a histogram for aggregated futures returns. The data sample for the histogram is May 2010 to September 2020 (only the in-sample period).

5. Results

In this section we discuss the empirical pricing performance of our novel model presented in Section 2.1 and each of the extant models listed in Section 2.2.2. Black (1976) is the market model for pricing standard European options on futures prices. In fact, there is one equivalent Black volatility per quoted option; therefore, there is no benefit in calibrating this single-parameter model. Our benchmark model is Trolle and Schwartz (2009)-SV1 (TS09-SV1), which is deployed as a special case of ours, and our sub-specification SVSYJ^{a_1} corresponds to Trolle (2014).

We perform an initial calibration using the in-sample market data set described in Section 4.1, leaving out of the estimation a subset of quoted options with the objective of studying which model(s) perform the best in fitting the prices of the left-out options. Therefore, we calculate a first set of error estimates using only the in-sample data set described in Section 4.2 to see which model performs best in-sample. We finally carry out a pricing exercise of those options left-out with the original parameters' estimates and calculate the error statistics to compare the out-of-sample goodness-of-fit.

5.1. In-sample

We compute the fair value of standard European call and put options contracts on different maturities and strike prices over a period of slightly over 10 years. Numerical results (estimated parameters values, pricing errors and computation time) are reported in Table 6. Table 6a refers to monthly data and Table 6b refers to daily data of futures and option prices. For monthly observations, we additionally present the confidence intervals at a percentile of 95% of each of the parameters estimates. To construct them, we perform a bootstrapping (sample with replacement) exercise which consists of 100 iterations from which we compute the 95% confidence intervals.[11] Errors are expressed in terms of MAE(σ) and RMSE(σ), and computation time in hours.

We want to determine whether the addition of different types of jumps brings any benefit to the pricing performance of our model compared with our benchmark. In this Section we will focus on results generated from daily observations; if results differ between one granularity and the other, we will explicitly discuss it.

Originally, Trolle and Schwartz (2009) was calibrated numerically applying the Fourier inversion theorem as in Duffie et al. (2000). This is the first empirical work in the literature we are aware of which prices plain vanilla options using the analytical solutions in Sitzia (2018). This innovation enables a much faster calibration of TS09-SV1 and our model. The alternative characterisation of the parameters is another novel feature, which reduces calibration time by approximately 50%.[12]

Our model presents the highest performance results (that is, the lowest values in terms of MAE(σ) and RSME(σ)) but also the longest computation times, as it contains the highest number of parameters. There is one exception on monthly (daily) data, where neither the benchmark nor our model can beat Bates (1996) (and Heston (1993)) in terms or errors.[13] So that computational time and calibrated values are comparable, we use the same initial set of (common) parameter values throughout all of our model sub-specifications as well as for our TS09-SV1. We tried many different sets of initial parameter values until we reached the best possible local optimum. In all cases we beat the benchmark; this result is more evident using daily data where, on average, the outperformance reaches 0.57% in terms of MAE(σ) (0.0363 to 0.0306) and 0.72% in terms of RMSE(σ) (0.0601 to 0.0529). These results represent a noteworthy improvement in model performance. Considered along with the reduction in computation time through the analytical solution for the CF and the use of the FFT for option pricing, these improvements provide a significant benefit for practitioners.

In order to compare models and identify superior performance, we analyse the distribution of the errors which can be observed in Tables 7 to 10. These tables present the distribution of MAE(σ) along two dimensions, namely the 32 moneyness levels and the maturity of the seven futures contracts, considered individually and taken together. When the observations are taken monthly (daily), Tables 7 and 8 (9 and 10) present the performances obtained by our benchmark and our model. The first table refers to our i.i.d. sub-specification (1) and the latter to our more advanced model with time-dampening jumps (2). In Sub-tables (a) we present the benchmark errors, Sub-tables (b) present our model errors, and Sub-tables (c) represent how good or bad our model is compared with the benchmark. Where the inclusion of jumps improves the pricing accuracy of our model, the values in Sub-tables (c) are positive.

We subsequently study in detail the information displayed in Tables 9 and 10, which are based on the daily data set. Examining Sub-tables (a) and (b), the first insight is that contracts that report greater errors are found in the shorter maturity contracts (darker red cells). In terms of moneyness, we find that the worst-performing contracts are the ones more distant from the ATM level; the worst of all correspond to options with lower strike level (that is, put options). For larger strike levels, the errors increase again for call options more distant from the ATM level. Furthermore, we observe that error performances are not symmetric. Examining Sub-tables (c) we observe that, on average, our model outperform the benchmark in nearly all contracts — all but one for SYSVJ^1, all but two for SYSVJ^2 (all being $32 \times 7 = 224$). The biggest improvements (darker blue) can be found in shorter contracts with lower strike levels and in OTM call options more distant from the ATM level. Improvements are not symmetric in this case, either. For SYSVJ^1, the model improvement reaches 2.28% and 2.22% for SYSVJ^2, which is noteworthy. The improvement in the aforementioned option contracts is consistent with the inclusion of jumps in the model. A priori, one would expected a positive effect in the short-term goodness of fit given that jumps (as opposed to volatilities) affect prices immediately; this effect is more clear in those contracts closer to maturity — a stylised fact in commodity markets.

In Table 6 it can be observed that in jump models, the jump amplitude μ_J is negative, an empirical fact which can be observed in the heavier left tail in Fig. 1d. Excluding Merton (1976), jump amplitudes (taken together in the case of jump assumptions (1) or (2)) are always higher in magnitude than −1. Their intensity rates are small, implying one jump every 10–12 years (depending on sub-specification and frequency). This results mainly from considering stochastic volatility within the models, which is the main driver of the non-Gaussian returns. Considering models with no jumps such as Heston (1993) and TS09-SV1, the correlation between the spot price and the volatility ρ_{Sv} is negative, whereas it is positive in all others. In models that allow for stochastic cost of carry, correlations between the cost of carry and the volatility ρ_{yv} are positive in most cases. Correlations between the spot price and the cost of carry (convenience yield) ρ_{Sy} are negative (positive) in all cases; this goes in line with the results obtained in prior research.

The introduction of jumps complicates the calibration as parameters values are somewhat unstable; this fact can be observed in Table 6 in either frequency considered. In our model and for monthly data, we observe that with i.i.d jumps, jump volatilities σ_{J_S} and σ_{J_Y} are zero in all sub-specifications — they can all be considered as constant jumps providing similar performance no matter the jump type considered. Something similar occurs with time-dampening jumps, where the dampening factor b_Y hits zero in all sub-specifications (observe that jump assumption (a_2) is equivalent to (2) with $b_Y = 0$). Any type of jump outperforms our benchmark by 0.15%. For daily data and unlike the situation previously described, we can observe that jump parameters do not present null values: jump volatilities in i.i.d. jumps are clearly different from zero although small in magnitude, the jump-decay parameter in time-dependent jumps take reasonable values.

The extant jump models perform in an opposite way to our novel jump model. For monthly observations, the models of Bates (1996) and Merton (1976) have non-zero but quite small jump volatility values. On the contrary, these volatilities are zero for daily observations. In both frequencies, these findings lead us to think that jumps may not precisely be i.i.d.

In terms of model performance, MAE(σ) values are practically indistinguishable between the two sets of sub-specifications: those corresponding to i.i.d. jumps together with (a_2) hit 3.00%; those of (a_2) and (2) stay around 3.20%. Both values clearly outperform our benchmark. An insight discernible only in daily data is that sub-specifications that present time-dampening jumps do not perform better than those related to i.i.d. jumps. Although the time-dependency in jumps is an important stylised fact in commodity markets, we think it relevant to consider jumps variable rather than constant; a mixture of both types might deliver even better results. We conclude that our jump model performs better than our benchmark, especially for short maturity contracts and away from the ATM level.

We now deepen our analysis of model comparisons by computing the MAE and its 95% confidence intervals, splitting the options into 15 buckets. To construct them, we perform a bootstrapping exercise which consists of 100 iterations from which we compute the 95% confidence intervals.[11] These values are calculated subject to the following buckets: deep OTM puts, OTM puts, ATMs, OTM calls and deep OTM calls for moneyness; short, mid, and long for maturities. We present these intervals in Table 11 below the error value in each bucket, which values correspond only to monthly observations.

In Table 12 and following this new granularity, we present models' performances, which are measured as the error difference between

[11] Because of the significant computational burden, we limited ourselves to 100 iterations.

[12] These values are available upon request.

[13] This particular fact is under current review.

Table 7
Model performance of SYSVJ[1] — monthly observations.

(a) TS09-SV1

MAE(σ)	M2	M3	M4	M5	M6	Q1	Q2	ALL
p − 7.5	0.0426	0.0265	0.0289	0.0321	0.0328	0.0428	0.0420	0.0354
p − 7	0.0392	0.0253	0.0285	0.0317	0.0335	0.0422	0.0411	0.0345
p − 6.5	0.0362	0.0238	0.0274	0.0310	0.0336	0.0422	0.0418	0.0337
p − 6	0.0333	0.0229	0.0273	0.0301	0.0330	0.0425	0.0425	0.0331
p − 5.5	0.0307	0.0220	0.0271	0.0307	0.0328	0.0422	0.0411	0.0324
p − 5	0.0281	0.0217	0.0262	0.0301	0.0334	0.0419	0.0416	0.0319
p − 4.5	0.0264	0.0210	0.0254	0.0305	0.0326	0.0413	0.0417	0.0313
p − 4	0.0235	0.0204	0.0251	0.0295	0.0325	0.0414	0.0424	0.0307
p − 3.5	0.0214	0.0197	0.0247	0.0291	0.0322	0.0410	0.0419	0.0300
p − 3	0.0195	0.0187	0.0246	0.0286	0.0320	0.0421	0.0425	0.0297
p − 2.5	0.0185	0.0186	0.0242	0.0284	0.0323	0.0420	0.0421	0.0294
p − 2	0.0164	0.0182	0.0244	0.0287	0.0314	0.0412	0.0412	0.0288
p − 1.5	0.0156	0.0178	0.0243	0.0284	0.0316	0.0399	0.0410	0.0284
p − 1	0.0138	0.0174	0.0237	0.0281	0.0313	0.0414	0.0425	0.0283
p − 0.5	0.0127	0.0172	0.0238	0.0284	0.0313	0.0405	0.0410	0.0278
p	0.0120	0.0166	0.0235	0.0281	0.0312	0.0409	0.0420	0.0277
c	0.0120	0.0169	0.0237	0.0281	0.0313	0.0408	0.0414	0.0277
c + 0.5	0.0112	0.0167	0.0237	0.0282	0.0315	0.0392	0.0416	0.0274
c + 1	0.0109	0.0170	0.0233	0.0280	0.0308	0.0407	0.0417	0.0275
c + 1.5	0.0107	0.0170	0.0241	0.0281	0.0309	0.0389	0.0410	0.0272
c + 2	0.0112	0.0171	0.0234	0.0277	0.0306	0.0395	0.0420	0.0274
c + 2.5	0.0119	0.0176	0.0240	0.0281	0.0312	0.0393	0.0412	0.0276
c + 3	0.0125	0.0175	0.0241	0.0279	0.0310	0.0388	0.0411	0.0276
c + 3.5	0.0133	0.0184	0.0242	0.0284	0.0318	0.0395	0.0406	0.0280
c + 4	0.0138	0.0184	0.0246	0.0282	0.0311	0.0388	0.0410	0.0280
c + 4.5	0.0141	0.0190	0.0246	0.0286	0.0313	0.0389	0.0411	0.0282
c + 5	0.0150	0.0190	0.0251	0.0283	0.0310	0.0400	0.0413	0.0285
c + 5.5	0.0154	0.0195	0.0254	0.0288	0.0315	0.0387	0.0405	0.0285
c + 6	0.0159	0.0199	0.0253	0.0288	0.0322	0.0394	0.0420	0.0291
c + 6.5	0.0166	0.0200	0.0256	0.0294	0.0326	0.0406	0.0418	0.0295
c + 7	0.0173	0.0208	0.0258	0.0295	0.0324	0.0399	0.0417	0.0296
c + 7.5	0.0192	0.0219	0.0264	0.0303	0.0335	0.0402	0.0416	0.0304
ALL	0.0191	0.0195	0.0251	0.0291	0.0319	0.0406	0.0416	0.0295

(b) SYSVJ[1]

MAE(σ)	M2	M3	M4	M5	M6	Q1	Q2	ALL
p − 7.5	0.0464	0.0307	0.0306	0.0322	0.0339	0.0371	0.0406	0.0359
p − 7	0.0416	0.0278	0.0291	0.0307	0.0328	0.0364	0.0376	0.0337
p − 6.5	0.0378	0.0249	0.0271	0.0290	0.0318	0.0364	0.0371	0.0320
p − 6	0.0344	0.0226	0.0260	0.0276	0.0308	0.0376	0.0382	0.0310
p − 5.5	0.0312	0.0209	0.0252	0.0280	0.0307	0.0377	0.0379	0.0302
p − 5	0.0286	0.0201	0.0237	0.0277	0.0311	0.0383	0.0390	0.0298
p − 4.5	0.0268	0.0192	0.0228	0.0278	0.0302	0.0382	0.0384	0.0290
p − 4	0.0237	0.0184	0.0226	0.0270	0.0304	0.0386	0.0388	0.0285
p − 3.5	0.0217	0.0176	0.0223	0.0267	0.0300	0.0387	0.0385	0.0279
p − 3	0.0199	0.0165	0.0221	0.0262	0.0300	0.0403	0.0401	0.0279
p − 2.5	0.0187	0.0164	0.0218	0.0261	0.0302	0.0405	0.0407	0.0278
p − 2	0.0167	0.0159	0.0218	0.0263	0.0296	0.0399	0.0406	0.0272
p − 1.5	0.0160	0.0156	0.0216	0.0206	0.0295	0.0390	0.0397	0.0268
p − 1	0.0146	0.0154	0.0210	0.0256	0.0293	0.0403	0.0407	0.0267
p − 0.5	0.0140	0.0152	0.0211	0.0257	0.0292	0.0396	0.0390	0.0262
p	0.0136	0.0147	0.0208	0.0253	0.0290	0.0400	0.0402	0.0262
c	0.0136	0.0152	0.0210	0.0253	0.0291	0.0400	0.0398	0.0263
c + 0.5	0.0136	0.0149	0.0209	0.0254	0.0290	0.0381	0.0406	0.0261
c + 1	0.0137	0.0153	0.0204	0.0250	0.0284	0.0398	0.0409	0.0262
c + 1.5	0.0142	0.0156	0.0210	0.0248	0.0280	0.0378	0.0401	0.0259
c + 2	0.0151	0.0162	0.0208	0.0242	0.0278	0.0380	0.0398	0.0260
c + 2.5	0.0163	0.0169	0.0212	0.0245	0.0278	0.0374	0.0386	0.0261
c + 3	0.0176	0.0171	0.0216	0.0242	0.0276	0.0366	0.0380	0.0261
c + 3.5	0.0189	0.0180	0.0218	0.0246	0.0278	0.0370	0.0376	0.0265
c + 4	0.0200	0.0187	0.0221	0.0244	0.0277	0.0357	0.0380	0.0266
c + 4.5	0.0209	0.0193	0.0222	0.0248	0.0274	0.0354	0.0380	0.0268
c + 5	0.0218	0.0201	0.0230	0.0243	0.0271	0.0359	0.0383	0.0272
c + 5.5	0.0225	0.0208	0.0236	0.0251	0.0273	0.0343	0.0366	0.0272
c + 6	0.0229	0.0215	0.0234	0.0250	0.0281	0.0344	0.0372	0.0275
c + 6.5	0.0233	0.0219	0.0239	0.0257	0.0285	0.0352	0.0360	0.0278
c + 7	0.0233	0.0228	0.0242	0.0256	0.0280	0.0340	0.0355	0.0276
c + 7.5	0.0235	0.0238	0.0256	0.0264	0.0291	0.0339	0.0351	0.0282
ALL	0.0221	0.0191	0.0230	0.0262	0.0293	0.0376	0.0387	0.0280

(c) Diff. SYSVJ[1] – TS09-SV1

MAE(σ)	M2	M3	M4	M5	M6	Q1	Q2	ALL
p − 7.5	−0.0038	−0.0042	−0.0017	−0.0001	−0.0011	0.0056	0.0015	−0.0005
p − 7	−0.0024	−0.0024	−0.0006	0.0010	0.0007	0.0058	0.0036	0.0008
p − 6.5	−0.0017	−0.0011	0.0003	0.0020	0.0018	0.0058	0.0047	0.0017
p − 6	−0.0010	0.0002	0.0013	0.0025	0.0022	0.0049	0.0043	0.0021
p − 5.5	−0.0006	0.0011	0.0019	0.0026	0.0021	0.0044	0.0031	0.0021
p − 5	−0.0005	0.0016	0.0025	0.0024	0.0023	0.0037	0.0026	0.0021
p − 4.5	−0.0004	0.0019	0.0026	0.0026	0.0023	0.0031	0.0033	0.0022
p − 4	−0.0002	0.0020	0.0025	0.0025	0.0021	0.0028	0.0036	0.0022
p − 3.5	−0.0003	0.0021	0.0024	0.0024	0.0022	0.0022	0.0034	0.0021
p − 3	−0.0003	0.0022	0.0025	0.0024	0.0020	0.0019	0.0024	0.0019
p − 2.5	−0.0003	0.0023	0.0024	0.0024	0.0021	0.0015	0.0014	0.0017
p − 2	−0.0003	0.0023	0.0026	0.0024	0.0019	0.0013	0.0006	0.0015
p − 1.5	−0.0004	0.0022	0.0027	0.0025	0.0021	0.0009	0.0012	0.0016
p − 1	−0.0008	0.0020	0.0027	0.0025	0.0020	0.0010	0.0018	0.0016
p − 0.5	−0.0013	0.0020	0.0027	0.0027	0.0021	0.0010	0.0020	0.0016
p	−0.0017	0.0018	0.0027	0.0028	0.0021	0.0008	0.0018	0.0015
c	−0.0017	0.0018	0.0028	0.0027	0.0022	0.0008	0.0015	0.0014
c + 0.5	−0.0023	0.0018	0.0028	0.0028	0.0025	0.0010	0.0009	0.0014
c + 1	−0.0029	0.0017	0.0028	0.0030	0.0024	0.0008	0.0008	0.0012
c + 1.5	−0.0035	0.0014	0.0031	0.0033	0.0029	0.0011	0.0009	0.0013
c + 2	−0.0039	0.0010	0.0026	0.0034	0.0028	0.0015	0.0022	0.0014
c + 2.5	−0.0045	0.0008	0.0028	0.0036	0.0034	0.0019	0.0026	0.0015
c + 3	−0.0052	0.0004	0.0024	0.0037	0.0035	0.0023	0.0031	0.0015
c + 3.5	−0.0056	0.0005	0.0024	0.0038	0.0040	0.0025	0.0030	0.0015
c + 4	−0.0062	−0.0003	0.0024	0.0038	0.0034	0.0031	0.0030	0.0013
c + 4.5	−0.0068	−0.0002	0.0024	0.0038	0.0039	0.0035	0.0031	0.0014
c + 5	−0.0068	−0.0012	0.0021	0.0039	0.0039	0.0041	0.0030	0.0013
c + 5.5	−0.0070	−0.0014	0.0018	0.0037	0.0041	0.0044	0.0038	0.0013
c + 6	−0.0071	−0.0016	0.0018	0.0039	0.0042	0.0050	0.0048	0.0016
c + 6.5	−0.0067	−0.0019	0.0016	0.0037	0.0042	0.0054	0.0058	0.0017
c + 7	−0.0060	−0.0020	0.0016	0.0038	0.0044	0.0059	0.0063	0.0020
c + 7.5	−0.0043	−0.0019	0.0008	0.0039	0.0043	0.0063	0.0065	0.0022
ALL	−0.0030	0.0005	0.0021	0.0029	0.0027	0.0030	0.0029	0.0016

NOTES: This table reports model accuracy in terms of MAE(σ) within each moneyness-maturity category; the estimations performed on the monthly data set. p – (c +) i refers to put (call) options with strike equal to the ATM strike minus (plus) i USD – the strikes are, therefore, increasing. Only the central rows display ATM options, all others display OTM options. Sub-table (a) refers to TS09-SV1, Sub-table (b) refers to SYSVJ[1] (both models follow the original characterisation of the parameters); observe that the darker the colour of the cell (red), the worse the model performance. Sub-table (c) displays the difference between both models (compares models accuracy), that is, TS09-SV1 – SYSVJ[1]; observe that the darker the colour of the cell (blue), the more accurate our model is (the better the performance of our model).

Table 8
Model performance of $SYSVJ^2$ — monthly observations.

(a) TS09-SV1

MAE(σ)	M2	M3	M4	M5	M6	Q1	Q2	ALL
p − 7.5	0.0426	0.0265	0.0289	0.0321	0.0328	0.0428	0.0420	0.0354
p − 7	0.0392	0.0253	0.0285	0.0317	0.0335	0.0422	0.0411	0.0345
p − 6.5	0.0362	0.0238	0.0274	0.0310	0.0336	0.0422	0.0418	0.0337
p − 6	0.0333	0.0229	0.0273	0.0301	0.0330	0.0425	0.0425	0.0331
p − 5.5	0.0307	0.0220	0.0271	0.0307	0.0328	0.0422	0.0411	0.0324
p − 5	0.0281	0.0217	0.0262	0.0301	0.0334	0.0419	0.0416	0.0319
p − 4.5	0.0264	0.0210	0.0254	0.0305	0.0326	0.0413	0.0417	0.0313
p − 4	0.0235	0.0204	0.0251	0.0295	0.0325	0.0414	0.0424	0.0307
p − 3.5	0.0214	0.0197	0.0247	0.0291	0.0322	0.0410	0.0419	0.0300
p − 3	0.0195	0.0187	0.0246	0.0286	0.0320	0.0421	0.0425	0.0297
p − 2.5	0.0185	0.0186	0.0242	0.0284	0.0323	0.0420	0.0421	0.0294
p − 2	0.0164	0.0182	0.0244	0.0287	0.0314	0.0412	0.0412	0.0288
p − 1.5	0.0156	0.0178	0.0243	0.0284	0.0316	0.0399	0.0410	0.0284
p − 1	0.0138	0.0174	0.0237	0.0281	0.0313	0.0414	0.0425	0.0283
p − 0.5	0.0127	0.0172	0.0238	0.0284	0.0313	0.0405	0.0410	0.0278
p	0.0120	0.0166	0.0235	0.0281	0.0312	0.0409	0.0420	0.0277
c	0.0120	0.0169	0.0237	0.0281	0.0313	0.0408	0.0414	0.0277
c + 0.5	0.0112	0.0167	0.0237	0.0282	0.0315	0.0392	0.0416	0.0274
c + 1	0.0109	0.0170	0.0233	0.0280	0.0308	0.0407	0.0417	0.0275
c + 1.5	0.0107	0.0170	0.0241	0.0281	0.0309	0.0389	0.0410	0.0272
c + 2	0.0112	0.0171	0.0234	0.0277	0.0306	0.0395	0.0420	0.0274
c + 2.5	0.0119	0.0176	0.0240	0.0281	0.0312	0.0393	0.0412	0.0276
c + 3	0.0125	0.0175	0.0241	0.0279	0.0310	0.0388	0.0411	0.0276
c + 3.5	0.0133	0.0184	0.0242	0.0284	0.0318	0.0395	0.0406	0.0280
c + 4	0.0138	0.0184	0.0246	0.0282	0.0311	0.0388	0.0410	0.0280
c + 4.5	0.0141	0.0190	0.0246	0.0286	0.0313	0.0389	0.0411	0.0282
c + 5	0.0150	0.0190	0.0251	0.0283	0.0310	0.0400	0.0413	0.0285
c + 5.5	0.0154	0.0195	0.0254	0.0288	0.0315	0.0387	0.0405	0.0285
c + 6	0.0159	0.0199	0.0253	0.0288	0.0322	0.0394	0.0420	0.0291
c + 6.5	0.0166	0.0200	0.0256	0.0294	0.0326	0.0406	0.0418	0.0295
c + 7	0.0173	0.0208	0.0258	0.0295	0.0324	0.0399	0.0417	0.0296
c + 7.5	0.0192	0.0219	0.0264	0.0303	0.0335	0.0402	0.0416	0.0304
ALL	0.0191	0.0195	0.0251	0.0291	0.0319	0.0406	0.0416	0.0295

(b) $SYSVJ^2$

MAE(σ)	M2	M3	M4	M5	M6	Q1	Q2	ALL
p − 7.5	0.0475	0.0315	0.0311	0.0326	0.0343	0.0374	0.0417	0.0366
p − 7	0.0425	0.0285	0.0294	0.0310	0.0331	0.0365	0.0384	0.0342
p − 6.5	0.0386	0.0256	0.0273	0.0292	0.0319	0.0364	0.0373	0.0323
p − 6	0.0350	0.0233	0.0262	0.0278	0.0308	0.0376	0.0381	0.0313
p − 5.5	0.0318	0.0213	0.0253	0.0282	0.0307	0.0376	0.0376	0.0303
p − 5	0.0291	0.0204	0.0239	0.0278	0.0311	0.0382	0.0386	0.0299
p − 4.5	0.0271	0.0194	0.0229	0.0280	0.0302	0.0380	0.0380	0.0291
p − 4	0.0240	0.0186	0.0227	0.0272	0.0304	0.0385	0.0384	0.0286
p − 3.5	0.0219	0.0177	0.0224	0.0269	0.0299	0.0387	0.0381	0.0280
p − 3	0.0200	0.0166	0.0222	0.0264	0.0300	0.0402	0.0395	0.0279
p − 2.5	0.0188	0.0164	0.0218	0.0262	0.0302	0.0405	0.0399	0.0277
p − 2	0.0167	0.0159	0.0219	0.0264	0.0296	0.0399	0.0396	0.0271
p − 1.5	0.0159	0.0156	0.0217	0.0261	0.0295	0.0390	0.0388	0.0267
p − 1	0.0144	0.0152	0.0211	0.0257	0.0293	0.0403	0.0399	0.0266
p − 0.5	0.0137	0.0149	0.0211	0.0258	0.0292	0.0395	0.0383	0.0261
p	0.0132	0.0144	0.0208	0.0254	0.0290	0.0400	0.0397	0.0261
c	0.0132	0.0148	0.0209	0.0254	0.0291	0.0400	0.0392	0.0261
c + 0.5	0.0130	0.0145	0.0208	0.0254	0.0289	0.0381	0.0401	0.0258
c + 1	0.0132	0.0149	0.0203	0.0251	0.0283	0.0398	0.0404	0.0260
c + 1.5	0.0135	0.0152	0.0208	0.0248	0.0279	0.0378	0.0394	0.0256
c + 2	0.0143	0.0156	0.0205	0.0243	0.0277	0.0379	0.0391	0.0256
c + 2.5	0.0154	0.0163	0.0209	0.0244	0.0276	0.0373	0.0379	0.0257
c + 3	0.0168	0.0165	0.0214	0.0242	0.0274	0.0365	0.0375	0.0257
c + 3.5	0.0181	0.0174	0.0215	0.0245	0.0275	0.0369	0.0370	0.0261
c + 4	0.0192	0.0179	0.0217	0.0242	0.0276	0.0355	0.0373	0.0262
c + 4.5	0.0202	0.0186	0.0218	0.0246	0.0272	0.0352	0.0373	0.0264
c + 5	0.0212	0.0194	0.0226	0.0241	0.0269	0.0357	0.0379	0.0268
c + 5.5	0.0219	0.0201	0.0232	0.0248	0.0271	0.0341	0.0360	0.0267
c + 6	0.0224	0.0208	0.0230	0.0247	0.0279	0.0342	0.0366	0.0271
c + 6.5	0.0228	0.0212	0.0236	0.0254	0.0282	0.0349	0.0355	0.0274
c + 7	0.0229	0.0221	0.0237	0.0254	0.0278	0.0337	0.0349	0.0272
c + 7.5	0.0232	0.0231	0.0251	0.0261	0.0289	0.0336	0.0346	0.0278
ALL	0.0219	0.0189	0.0229	0.0262	0.0292	0.0375	0.0382	0.0278

(c) Diff. $SYSVJ^2$ – TS09-SV1

MAE(σ)	M2	M3	M4	M5	M6	Q1	Q2	ALL
p − 7.5	−0.0049	−0.0050	−0.0022	−0.0004	−0.0015	0.0054	0.0003	−0.0012
p − 7	−0.0033	−0.0032	−0.0009	0.0008	0.0004	0.0057	0.0027	0.0003
p − 6.5	−0.0024	−0.0017	0.0001	0.0017	0.0016	0.0058	0.0045	0.0014
p − 6	−0.0016	−0.0004	0.0011	0.0023	0.0022	0.0049	0.0044	0.0018
p − 5.5	−0.0011	0.0007	0.0018	0.0025	0.0022	0.0046	0.0035	0.0020
p − 5	−0.0010	0.0013	0.0024	0.0023	0.0023	0.0038	0.0030	0.0020
p − 4.5	−0.0007	0.0016	0.0024	0.0025	0.0024	0.0032	0.0037	0.0021
p − 4	−0.0005	0.0018	0.0024	0.0023	0.0021	0.0029	0.0040	0.0021
p − 3.5	−0.0005	0.0020	0.0023	0.0022	0.0022	0.0022	0.0039	0.0020
p − 3	−0.0005	0.0021	0.0024	0.0022	0.0020	0.0019	0.0030	0.0019
p − 2.5	−0.0003	0.0022	0.0023	0.0022	0.0021	0.0015	0.0021	0.0017
p − 2	−0.0003	0.0023	0.0025	0.0022	0.0018	0.0013	0.0016	0.0016
p − 1.5	−0.0003	0.0023	0.0026	0.0023	0.0021	0.0008	0.0022	0.0017
p − 1	−0.0007	0.0022	0.0026	0.0024	0.0020	0.0011	0.0026	0.0017
p − 0.5	−0.0010	0.0023	0.0027	0.0026	0.0020	0.0010	0.0027	0.0017
p	−0.0012	0.0022	0.0028	0.0027	0.0022	0.0008	0.0023	0.0017
c	−0.0012	0.0021	0.0028	0.0027	0.0022	0.0008	0.0022	0.0017
c + 0.5	−0.0018	0.0022	0.0029	0.0028	0.0026	0.0010	0.0015	0.0016
c + 1	−0.0023	0.0021	0.0030	0.0030	0.0025	0.0008	0.0013	0.0015
c + 1.5	−0.0029	0.0018	0.0033	0.0032	0.0030	0.0010	0.0016	0.0016
c + 2	−0.0030	0.0015	0.0029	0.0034	0.0029	0.0016	0.0029	0.0017
c + 2.5	−0.0036	0.0013	0.0031	0.0037	0.0036	0.0020	0.0033	0.0019
c + 3	−0.0043	0.0010	0.0027	0.0037	0.0037	0.0023	0.0037	0.0018
c + 3.5	−0.0048	0.0010	0.0027	0.0040	0.0043	0.0026	0.0036	0.0019
c + 4	−0.0054	0.0004	0.0028	0.0040	0.0036	0.0033	0.0036	0.0018
c + 4.5	−0.0060	0.0005	0.0028	0.0040	0.0041	0.0036	0.0037	0.0018
c + 5	−0.0062	−0.0004	0.0025	0.0042	0.0041	0.0043	0.0034	0.0017
c + 5.5	−0.0065	−0.0006	0.0022	0.0040	0.0044	0.0046	0.0044	0.0018
c + 6	−0.0065	−0.0009	0.0022	0.0041	0.0044	0.0052	0.0054	0.0020
c + 6.5	−0.0062	−0.0012	0.0020	0.0040	0.0044	0.0057	0.0063	0.0021
c + 7	−0.0056	−0.0014	0.0020	0.0041	0.0046	0.0062	0.0068	0.0024
c + 7.5	−0.0040	−0.0013	0.0013	0.0042	0.0046	0.0066	0.0069	0.0026
ALL	−0.0028	0.0007	0.0021	0.0029	0.0027	0.0031	0.0033	0.0017

NOTES: This table reports model accuracy in terms of MAE(σ) within each moneyness-maturity category; the estimations performed on the monthly data set. p – (c +) i refers to put (call) options with strike equal to the ATM strike minus (plus) i USD – the strikes are, therefore, increasing. Only the central rows display ATM options, all others display OTM options. Sub-table (a) refers to TS09-SV1, Sub-table (b) refers to our more advanced model $SYSVJ^2$ (both models follow the original characterisation of the parameters); observe that the darker the colour of the cell (red), the worse the model performance. Sub-table (c) displays the difference between both models (compares models accuracy), that is, TS09-SV1 – $SYSVJ^2$; observe that the darker the colour of the cell (blue), the more accurate our model is (the better the performance of our model).

Table 9
Model performance of SYSVJ[1] — daily observations.

(a) TS09-SV1

MAE(σ)	M2	M3	M4	M5	M6	Q1	Q2	ALL
p − 7.5	0.0770	0.0524	0.0411	0.0372	0.0370	0.0373	0.0392	0.0459
p − 7	0.0732	0.0509	0.0403	0.0367	0.0370	0.0371	0.0390	0.0449
p − 6.5	0.0701	0.0492	0.0394	0.0362	0.0370	0.0365	0.0392	0.0440
p − 6	0.0672	0.0473	0.0385	0.0354	0.0366	0.0365	0.0393	0.0430
p − 5.5	0.0644	0.0457	0.0378	0.0353	0.0364	0.0361	0.0394	0.0422
p − 5	0.0618	0.0441	0.0371	0.0345	0.0363	0.0361	0.0391	0.0413
p − 4.5	0.0594	0.0425	0.0364	0.0339	0.0357	0.0360	0.0397	0.0405
p − 4	0.0569	0.0410	0.0358	0.0332	0.0353	0.0359	0.0398	0.0397
p − 3.5	0.0546	0.0397	0.0350	0.0327	0.0349	0.0356	0.0398	0.0389
p − 3	0.0524	0.0385	0.0342	0.0321	0.0346	0.0359	0.0397	0.0382
p − 2.5	0.0503	0.0373	0.0335	0.0317	0.0339	0.0359	0.0402	0.0375
p − 2	0.0483	0.0362	0.0329	0.0313	0.0335	0.0359	0.0396	0.0368
p − 1.5	0.0465	0.0352	0.0323	0.0310	0.0330	0.0354	0.0398	0.0362
p − 1	0.0447	0.0341	0.0316	0.0305	0.0325	0.0357	0.0397	0.0355
p − 0.5	0.0431	0.0333	0.0309	0.0303	0.0321	0.0353	0.0396	0.0349
p	0.0417	0.0324	0.0304	0.0301	0.0320	0.0349	0.0402	0.0345
c	0.0416	0.0323	0.0304	0.0300	0.0320	0.0352	0.0394	0.0344
c + 0.5	0.0405	0.0317	0.0298	0.0298	0.0316	0.0351	0.0398	0.0340
c + 1	0.0394	0.0311	0.0294	0.0295	0.0314	0.0348	0.0395	0.0336
c + 1.5	0.0387	0.0305	0.0290	0.0294	0.0311	0.0344	0.0392	0.0332
c + 2	0.0380	0.0301	0.0287	0.0292	0.0311	0.0344	0.0391	0.0329
c + 2.5	0.0375	0.0297	0.0284	0.0291	0.0310	0.0342	0.0387	0.0327
c + 3	0.0372	0.0294	0.0282	0.0290	0.0308	0.0340	0.0385	0.0324
c + 3.5	0.0371	0.0293	0.0280	0.0288	0.0309	0.0340	0.0388	0.0324
c + 4	0.0372	0.0293	0.0278	0.0286	0.0308	0.0338	0.0387	0.0323
c + 4.5	0.0375	0.0294	0.0278	0.0285	0.0311	0.0337	0.0385	0.0324
c + 5	0.0380	0.0297	0.0277	0.0285	0.0309	0.0337	0.0384	0.0324
c + 5.5	0.0386	0.0300	0.0278	0.0286	0.0311	0.0336	0.0387	0.0326
c + 6	0.0394	0.0305	0.0280	0.0284	0.0310	0.0336	0.0388	0.0328
c + 6.5	0.0405	0.0310	0.0282	0.0287	0.0312	0.0337	0.0390	0.0332
c + 7	0.0418	0.0317	0.0284	0.0285	0.0314	0.0337	0.0389	0.0335
c + 7.5	0.0437	0.0325	0.0288	0.0287	0.0314	0.0342	0.0393	0.0341
ALL	0.0481	0.0359	0.0320	0.0311	0.0330	0.0351	0.0393	0.0363

(b) SYSVJ[1]

MAE(σ)	M2	M3	M4	M5	M6	Q1	Q2	ALL
p − 7.5	0.0542	0.0348	0.0289	0.0291	0.0303	0.0351	0.0400	0.0361
p − 7	0.0519	0.0336	0.0280	0.0282	0.0298	0.0345	0.0386	0.0350
p − 6.5	0.0502	0.0324	0.0273	0.0276	0.0296	0.0338	0.0379	0.0341
p − 6	0.0489	0.0314	0.0268	0.0271	0.0291	0.0337	0.0374	0.0335
p − 5.5	0.0476	0.0307	0.0266	0.0272	0.0293	0.0334	0.0370	0.0331
p − 5	0.0466	0.0301	0.0264	0.0270	0.0294	0.0334	0.0366	0.0328
p − 4.5	0.0456	0.0294	0.0262	0.0269	0.0292	0.0334	0.0370	0.0325
p − 4	0.0445	0.0287	0.0261	0.0267	0.0292	0.0333	0.0370	0.0322
p − 3.5	0.0434	0.0281	0.0257	0.0266	0.0293	0.0331	0.0369	0.0319
p − 3	0.0424	0.0276	0.0255	0.0264	0.0294	0.0334	0.0369	0.0317
p − 2.5	0.0413	0.0271	0.0252	0.0263	0.0292	0.0334	0.0372	0.0314
p − 2	0.0401	0.0265	0.0251	0.0262	0.0291	0.0334	0.0370	0.0311
p − 1.5	0.0390	0.0260	0.0248	0.0260	0.0291	0.0329	0.0373	0.0307
p − 1	0.0378	0.0254	0.0245	0.0257	0.0288	0.0331	0.0373	0.0304
p − 0.5	0.0366	0.0249	0.0241	0.0256	0.0286	0.0327	0.0374	0.0300
p	0.0355	0.0244	0.0238	0.0253	0.0285	0.0324	0.0377	0.0297
c	0.0354	0.0243	0.0238	0.0253	0.0286	0.0326	0.0374	0.0297
c + 0.5	0.0346	0.0240	0.0234	0.0252	0.0282	0.0326	0.0375	0.0293
c + 1	0.0337	0.0236	0.0231	0.0248	0.0281	0.0323	0.0374	0.0290
c + 1.5	0.0329	0.0232	0.0228	0.0247	0.0277	0.0319	0.0371	0.0286
c + 2	0.0322	0.0228	0.0225	0.0244	0.0276	0.0318	0.0370	0.0283
c + 2.5	0.0317	0.0225	0.0222	0.0244	0.0274	0.0317	0.0365	0.0280
c + 3	0.0312	0.0222	0.0221	0.0242	0.0269	0.0316	0.0361	0.0278
c + 3.5	0.0308	0.0221	0.0220	0.0240	0.0267	0.0315	0.0362	0.0276
c + 4	0.0307	0.0219	0.0219	0.0239	0.0266	0.0313	0.0359	0.0275
c + 4.5	0.0306	0.0219	0.0219	0.0238	0.0265	0.0311	0.0355	0.0273
c + 5	0.0306	0.0219	0.0219	0.0237	0.0262	0.0311	0.0351	0.0272
c + 5.5	0.0307	0.0220	0.0219	0.0241	0.0262	0.0309	0.0351	0.0273
c + 6	0.0310	0.0221	0.0221	0.0238	0.0261	0.0308	0.0349	0.0272
c + 6.5	0.0314	0.0222	0.0222	0.0241	0.0261	0.0307	0.0346	0.0273
c + 7	0.0320	0.0224	0.0223	0.0240	0.0263	0.0305	0.0345	0.0274
c + 7.5	0.0328	0.0227	0.0225	0.0243	0.0262	0.0307	0.0343	0.0277
ALL	0.0381	0.0257	0.0242	0.0255	0.0281	0.0324	0.0367	0.0300

(c) Diff. SYSVJ[1] – TS09-SV1

MAE(σ)	M2	M3	M4	M5	M6	Q1	Q2	ALL
p − 7.5	0.0228	0.0176	0.0122	0.0081	0.0067	0.0022	−0.0008	0.0098
p − 7	0.0213	0.0173	0.0123	0.0085	0.0072	0.0026	0.0004	0.0099
p − 6.5	0.0199	0.0167	0.0121	0.0086	0.0074	0.0027	0.0013	0.0098
p − 6	0.0183	0.0159	0.0117	0.0083	0.0075	0.0028	0.0019	0.0095
p − 5.5	0.0168	0.0150	0.0112	0.0080	0.0072	0.0028	0.0023	0.0090
p − 5	0.0153	0.0141	0.0107	0.0075	0.0069	0.0027	0.0025	0.0085
p − 4.5	0.0138	0.0131	0.0102	0.0070	0.0065	0.0026	0.0027	0.0080
p − 4	0.0124	0.0123	0.0097	0.0065	0.0061	0.0026	0.0028	0.0075
p − 3.5	0.0112	0.0115	0.0092	0.0061	0.0056	0.0025	0.0029	0.0070
p − 3	0.0100	0.0108	0.0087	0.0057	0.0052	0.0025	0.0028	0.0065
p − 2.5	0.0091	0.0102	0.0082	0.0054	0.0047	0.0025	0.0030	0.0061
p − 2	0.0082	0.0097	0.0078	0.0051	0.0043	0.0025	0.0026	0.0058
p − 1.5	0.0075	0.0092	0.0075	0.0050	0.0040	0.0025	0.0025	0.0054
p − 1	0.0069	0.0087	0.0071	0.0048	0.0037	0.0026	0.0023	0.0052
p − 0.5	0.0065	0.0083	0.0068	0.0047	0.0035	0.0026	0.0023	0.0050
p	0.0061	0.0080	0.0066	0.0047	0.0034	0.0025	0.0025	0.0048
c	0.0062	0.0080	0.0066	0.0047	0.0034	0.0026	0.0019	0.0048
c + 0.5	0.0059	0.0077	0.0064	0.0046	0.0034	0.0025	0.0023	0.0047
c + 1	0.0058	0.0075	0.0063	0.0047	0.0034	0.0025	0.0022	0.0046
c + 1.5	0.0057	0.0074	0.0062	0.0047	0.0034	0.0025	0.0021	0.0046
c + 2	0.0058	0.0073	0.0062	0.0048	0.0035	0.0026	0.0021	0.0046
c + 2.5	0.0058	0.0072	0.0061	0.0047	0.0037	0.0025	0.0022	0.0046
c + 3	0.0060	0.0072	0.0061	0.0048	0.0039	0.0024	0.0024	0.0047
c + 3.5	0.0063	0.0072	0.0060	0.0048	0.0041	0.0025	0.0025	0.0048
c + 4	0.0066	0.0073	0.0059	0.0047	0.0042	0.0025	0.0028	0.0049
c + 4.5	0.0069	0.0075	0.0058	0.0047	0.0046	0.0025	0.0030	0.0050
c + 5	0.0074	0.0077	0.0058	0.0047	0.0047	0.0027	0.0032	0.0052
c + 5.5	0.0078	0.0081	0.0058	0.0046	0.0048	0.0027	0.0036	0.0054
c + 6	0.0084	0.0084	0.0059	0.0046	0.0049	0.0028	0.0039	0.0056
c + 6.5	0.0091	0.0088	0.0060	0.0046	0.0051	0.0030	0.0044	0.0058
c + 7	0.0098	0.0093	0.0061	0.0045	0.0051	0.0032	0.0044	0.0061
c + 7.5	0.0109	0.0099	0.0062	0.0044	0.0051	0.0035	0.0050	0.0064
ALL	0.0100	0.0102	0.0078	0.0056	0.0049	0.0026	0.0026	0.0063

NOTES: This table reports model accuracy in terms of MAE(σ) within each moneyness-maturity category; the estimations performed on the daily data set. p – (c +) i refers to put (call) options with strike equal to the ATM strike minus (plus) i USD – the strikes are, therefore, increasing. Only the central rows display ATM options, all others display OTM options. Sub-table (a) refers to TS09-SV1, Sub-table (b) refers to SYSVJ[1] (both models follow the original characterisation of the parameters); observe that the darker the colour of the cell (red), the worse the model performance. Sub-table (c) displays the difference between both models (compares models accuracy), that is, TS09-SV1 – SYSVJ[1]; observe that the darker the colour of the cell (blue), the more accurate our model is (the better the performance of our model).

Table 10
Model performance of SYSVJ² — daily observations.

(a) TS09-SV1

MAE(σ)	M2	M3	M4	M5	M6	Q1	Q2	ALL
p – 7.5	0.0770	0.0524	0.0411	0.0372	0.0370	0.0373	0.0392	0.0459
p – 7	0.0732	0.0509	0.0403	0.0367	0.0370	0.0371	0.0390	0.0449
p – 6.5	0.0701	0.0492	0.0394	0.0362	0.0370	0.0365	0.0392	0.0440
p – 6	0.0672	0.0473	0.0385	0.0354	0.0366	0.0365	0.0393	0.0430
p – 5.5	0.0644	0.0457	0.0378	0.0353	0.0364	0.0361	0.0394	0.0422
p – 5	0.0618	0.0441	0.0371	0.0345	0.0363	0.0361	0.0391	0.0413
p – 4.5	0.0594	0.0425	0.0364	0.0339	0.0357	0.0360	0.0397	0.0405
p – 4	0.0569	0.0410	0.0358	0.0332	0.0353	0.0359	0.0398	0.0397
p – 3.5	0.0546	0.0397	0.0350	0.0327	0.0349	0.0356	0.0398	0.0389
p – 3	0.0524	0.0385	0.0342	0.0321	0.0346	0.0359	0.0397	0.0382
p – 2.5	0.0503	0.0373	0.0335	0.0317	0.0339	0.0359	0.0402	0.0375
p – 2	0.0483	0.0362	0.0329	0.0313	0.0335	0.0359	0.0396	0.0368
p – 1.5	0.0465	0.0352	0.0323	0.0310	0.0330	0.0354	0.0398	0.0362
p – 1	0.0447	0.0341	0.0316	0.0305	0.0325	0.0357	0.0397	0.0355
p – 0.5	0.0431	0.0333	0.0309	0.0303	0.0321	0.0353	0.0396	0.0349
p	0.0417	0.0324	0.0304	0.0301	0.0320	0.0349	0.0402	0.0345
c	0.0416	0.0323	0.0304	0.0300	0.0320	0.0352	0.0394	0.0344
c + 0.5	0.0405	0.0317	0.0298	0.0298	0.0316	0.0351	0.0398	0.0340
c + 1	0.0394	0.0311	0.0294	0.0295	0.0314	0.0348	0.0395	0.0336
c + 1.5	0.0387	0.0305	0.0290	0.0294	0.0311	0.0344	0.0392	0.0332
c + 2	0.0380	0.0301	0.0287	0.0292	0.0311	0.0344	0.0391	0.0329
c + 2.5	0.0375	0.0297	0.0284	0.0291	0.0310	0.0342	0.0387	0.0327
c + 3	0.0372	0.0294	0.0282	0.0290	0.0308	0.0340	0.0385	0.0324
c + 3.5	0.0371	0.0293	0.0280	0.0288	0.0309	0.0340	0.0388	0.0324
c + 4	0.0372	0.0293	0.0278	0.0286	0.0308	0.0338	0.0387	0.0323
c + 4.5	0.0375	0.0294	0.0278	0.0285	0.0311	0.0337	0.0385	0.0324
c + 5	0.0380	0.0297	0.0277	0.0285	0.0309	0.0337	0.0384	0.0324
c + 5.5	0.0386	0.0300	0.0278	0.0286	0.0311	0.0336	0.0387	0.0326
c + 6	0.0394	0.0305	0.0280	0.0284	0.0310	0.0336	0.0388	0.0328
c + 6.5	0.0405	0.0310	0.0282	0.0287	0.0312	0.0337	0.0390	0.0332
c + 7	0.0418	0.0317	0.0284	0.0285	0.0314	0.0337	0.0389	0.0335
c + 7.5	0.0437	0.0325	0.0288	0.0287	0.0314	0.0342	0.0393	0.0341
ALL	0.0481	0.0359	0.0320	0.0311	0.0330	0.0351	0.0393	0.0363

(b) SYSVJ²

MAE(σ)	M2	M3	M4	M5	M6	Q1	Q2	ALL
p – 7.5	0.0548	0.0359	0.0295	0.0290	0.0302	0.0344	0.0404	0.0363
p – 7	0.0533	0.0356	0.0294	0.0288	0.0304	0.0342	0.0394	0.0359
p – 6.5	0.0523	0.0351	0.0292	0.0288	0.0306	0.0338	0.0390	0.0355
p – 6	0.0513	0.0344	0.0291	0.0287	0.0305	0.0339	0.0386	0.0352
p – 5.5	0.0503	0.0340	0.0292	0.0291	0.0308	0.0338	0.0384	0.0351
p – 5	0.0494	0.0336	0.0293	0.0290	0.0311	0.0339	0.0379	0.0349
p – 4.5	0.0485	0.0331	0.0292	0.0291	0.0311	0.0340	0.0382	0.0347
p – 4	0.0475	0.0325	0.0293	0.0291	0.0311	0.0340	0.0380	0.0345
p – 3.5	0.0463	0.0319	0.0290	0.0290	0.0313	0.0338	0.0377	0.0342
p – 3	0.0452	0.0313	0.0288	0.0288	0.0314	0.0341	0.0375	0.0339
p – 2.5	0.0440	0.0308	0.0285	0.0288	0.0312	0.0341	0.0379	0.0336
p – 2	0.0427	0.0301	0.0284	0.0287	0.0312	0.0341	0.0373	0.0332
p – 1.5	0.0415	0.0295	0.0281	0.0285	0.0310	0.0336	0.0375	0.0328
p – 1	0.0402	0.0288	0.0277	0.0282	0.0307	0.0338	0.0374	0.0324
p – 0.5	0.0389	0.0282	0.0272	0.0280	0.0306	0.0334	0.0374	0.0320
p	0.0377	0.0275	0.0269	0.0278	0.0305	0.0330	0.0381	0.0316
c	0.0377	0.0275	0.0269	0.0277	0.0305	0.0333	0.0373	0.0316
c + 0.5	0.0367	0.0270	0.0264	0.0275	0.0301	0.0333	0.0378	0.0312
c + 1	0.0356	0.0265	0.0260	0.0271	0.0300	0.0329	0.0377	0.0308
c + 1.5	0.0348	0.0259	0.0256	0.0269	0.0295	0.0326	0.0376	0.0304
c + 2	0.0340	0.0254	0.0252	0.0264	0.0294	0.0325	0.0375	0.0301
c + 2.5	0.0334	0.0249	0.0247	0.0263	0.0291	0.0324	0.0371	0.0297
c + 3	0.0328	0.0245	0.0244	0.0260	0.0286	0.0323	0.0368	0.0294
c + 3.5	0.0324	0.0242	0.0241	0.0257	0.0284	0.0323	0.0370	0.0291
c + 4	0.0322	0.0239	0.0238	0.0254	0.0281	0.0320	0.0368	0.0289
c + 4.5	0.0320	0.0238	0.0236	0.0253	0.0281	0.0319	0.0366	0.0288
c + 5	0.0321	0.0237	0.0235	0.0251	0.0276	0.0319	0.0363	0.0286
c + 5.5	0.0321	0.0236	0.0233	0.0252	0.0275	0.0317	0.0365	0.0286
c + 6	0.0323	0.0236	0.0233	0.0249	0.0274	0.0316	0.0364	0.0285
c + 6.5	0.0327	0.0236	0.0233	0.0251	0.0273	0.0315	0.0363	0.0286
c + 7	0.0333	0.0238	0.0233	0.0249	0.0274	0.0314	0.0361	0.0286
c + 7.5	0.0340	0.0240	0.0233	0.0251	0.0272	0.0316	0.0363	0.0288
ALL	0.0401	0.0284	0.0265	0.0273	0.0297	0.0330	0.0375	0.0318

(c) Diff. SYSVJ² – TS09-SV1

MAE(σ)	M2	M3	M4	M5	M6	Q1	Q2	ALL
p – 7.5	0.0222	0.0164	0.0116	0.0082	0.0068	0.0029	−0.0012	0.0095
p – 7	0.0199	0.0153	0.0109	0.0079	0.0066	0.0029	−0.0004	0.0090
p – 6.5	0.0179	0.0141	0.0102	0.0075	0.0063	0.0027	0.0002	0.0084
p – 6	0.0159	0.0129	0.0094	0.0068	0.0061	0.0025	0.0007	0.0077
p – 5.5	0.0141	0.0117	0.0086	0.0062	0.0056	0.0023	0.0010	0.0071
p – 5	0.0124	0.0105	0.0078	0.0054	0.0052	0.0021	0.0013	0.0064
p – 4.5	0.0109	0.0094	0.0072	0.0048	0.0047	0.0020	0.0015	0.0058
p – 4	0.0094	0.0085	0.0065	0.0042	0.0042	0.0019	0.0018	0.0052
p – 3.5	0.0083	0.0077	0.0059	0.0037	0.0036	0.0018	0.0021	0.0047
p – 3	0.0072	0.0071	0.0054	0.0032	0.0032	0.0018	0.0022	0.0043
p – 2.5	0.0063	0.0066	0.0049	0.0029	0.0027	0.0018	0.0023	0.0039
p – 2	0.0056	0.0061	0.0045	0.0026	0.0023	0.0018	0.0023	0.0036
p – 1.5	0.0050	0.0057	0.0042	0.0025	0.0020	0.0018	0.0022	0.0033
p – 1	0.0045	0.0053	0.0039	0.0023	0.0017	0.0018	0.0023	0.0031
p – 0.5	0.0042	0.0051	0.0037	0.0023	0.0016	0.0019	0.0022	0.0030
p	0.0039	0.0049	0.0036	0.0023	0.0015	0.0019	0.0021	0.0029
c	0.0039	0.0049	0.0035	0.0023	0.0015	0.0019	0.0021	0.0029
c + 0.5	0.0038	0.0047	0.0035	0.0023	0.0014	0.0019	0.0020	0.0028
c + 1	0.0038	0.0046	0.0034	0.0024	0.0015	0.0019	0.0018	0.0028
c + 1.5	0.0039	0.0046	0.0035	0.0026	0.0016	0.0019	0.0017	0.0028
c + 2	0.0040	0.0047	0.0035	0.0027	0.0017	0.0018	0.0016	0.0029
c + 2.5	0.0041	0.0047	0.0036	0.0028	0.0019	0.0018	0.0016	0.0030
c + 3	0.0044	0.0049	0.0037	0.0030	0.0022	0.0017	0.0017	0.0031
c + 3.5	0.0047	0.0051	0.0039	0.0031	0.0025	0.0017	0.0018	0.0032
c + 4	0.0050	0.0053	0.0040	0.0032	0.0027	0.0017	0.0019	0.0034
c + 4.5	0.0054	0.0056	0.0041	0.0033	0.0030	0.0018	0.0019	0.0036
c + 5	0.0059	0.0060	0.0043	0.0034	0.0033	0.0018	0.0020	0.0038
c + 5.5	0.0064	0.0064	0.0045	0.0034	0.0035	0.0019	0.0022	0.0041
c + 6	0.0070	0.0069	0.0047	0.0035	0.0037	0.0020	0.0024	0.0043
c + 6.5	0.0078	0.0073	0.0049	0.0035	0.0039	0.0022	0.0027	0.0046
c + 7	0.0086	0.0079	0.0052	0.0036	0.0040	0.0023	0.0028	0.0049
c + 7.5	0.0097	0.0086	0.0055	0.0036	0.0042	0.0026	0.0029	0.0053
ALL	0.0080	0.0075	0.0054	0.0038	0.0033	0.0020	0.0017	0.0045

Notes: This table reports model accuracy in terms of MAE(σ) within each moneyness-maturity category; the estimations performed on the daily data set. p – (c +) *i* refers to put (call) options with strike equal to the ATM strike minus (plus) *i* USD – the strikes are, therefore, increasing. Only the central rows display ATM options, all others display OTM options. Sub-table (a) refers to TS09-SV1, Sub-table (b) refers to our more advanced model SYSVJ² (both models follow the original characterisation of the parameters); observe that the darker the colour of the cell (red), the worse the model performance. Sub-table (c) displays the difference between both models (compares models accuracy), that is, TS09-SV1 – SYSVJ²; observe that the darker the colour of the cell (blue), the more accurate our model is (the better the performance of our model).

Table 11
MAE(σ) per bucket of contracts and of moneyness levels — monthly values.

(a) Merton (1976)

MAE(σ)	short	mid	long
+OTM p	0.1007	0.0855	0.0722
	(0.0812,0.1241)	(0.0747,0.0979)	(0.0630,0.0812)
OTM p	0.0981	0.0818	0.0692
	(0.0804,0.1253)	(0.0724,0.0948)	(0.0609,0.0783)
ATM	0.0949	0.0785	0.0655
	(0.0779,0.1177)	(0.0700,0.0905)	(0.0580,0.0744)
OTM c	0.0919	0.0767	0.0639
	(0.0769,0.1102)	(0.0686,0.0873)	(0.0567,0.0722)
+OTM c	0.0906	0.0756	0.0638
	(0.0769,0.1061)	(0.0675,0.0853)	(0.0569,0.0722)

(b) Heston (1993)

	short	mid	long
+OTM p	0.0332	0.0336	0.0402
	(0.0243,0.0474)	(0.0226,0.0436)	(0.0277,0.0519)
OTM p	0.0235	0.0315	0.0394
	(0.0146,0.0334)	(0.0199,0.0427)	(0.0273,0.0517)
ATM	0.0158	0.0298	0.0388
	(0.0079,0.0254)	(0.0177,0.0418)	(0.0263,0.0515)
OTM c	0.0156	0.0295	0.0380
	(0.0099,0.0232)	(0.0182,0.0424)	(0.0258,0.0508)
+OTM c	0.0215	0.0309	0.0383
	(0.0161,0.0290)	(0.0203,0.0435)	(0.0265,0.0512)

(c) Bates (1996)

	short	mid	long
+OTM p	0.0315	0.0297	0.0371
	(0.0211,0.0468)	(0.0204,0.0380)	(0.0254,0.0464)
OTM p	0.0195	0.0277	0.0365
	(0.0128,0.0300)	(0.0186,0.0381)	(0.0257,0.0456)
ATM	0.0127	0.0260	0.0369
	(0.0087,0.0200)	(0.0171,0.0380)	(0.0251,0.0472)
OTM c	0.0155	0.0247	0.0349
	(0.0115,0.0181)	(0.0166,0.0366)	(0.0241,0.0459)
+OTM c	0.0193	0.0251	0.0337
	(0.0142,0.0202)	(0.0172,0.0361)	(0.0238,0.0452)

(d) TS09-SV1

MAE(σ)	short	mid	long
+OTM p	0.0319	0.0309	0.0423
	(0.0213,0.0498)	(0.0207,0.0435)	(0.0297,0.0506)
OTM p	0.0211	0.0290	0.0418
	(0.0132,0.0333)	(0.0187,0.0418)	(0.255,0.0503)
ATM	0.0152	0.0278	0.0411
	(0.0081,0.0231)	(0.0174,0.0397)	(0.0281,0.0498)
OTM c	0.0168	0.0281	0.0404
	(0.0098,0.0209)	(0.0179,0.0390)	(0.0272,0.0493)
+OTM c	0.0191	0.0293	0.0410
	(0.0135,0.0252)	(0.0192,0.0400)	(0.0270,0.0505)

(e) SYSVJ^{a_1}

MAE(σ)	short	mid	long
+OTM p	0.0313	0.0298	0.0379
	(0.0173,0.0442)	(0.0187,0.0374)	(0.0272,0.0474)
OTM p	0.0195	0.0268	0.0393
	(0.0119,0.0270)	(0.0166,0.0352)	(0.0279,0.0483)
ATM	0.0143	0.0253	0.0398
	(0.0093,0.0186)	(0.0159,0.0343)	(0.0265,0.0498)
OTM c	0.0179	0.0249	0.0376
	(0.0121,0.0210)	(0.0160,0.0342)	(0.0252,0.0484)
+OTM c	0.0216	0.0260	0.0358
	(0.0134,0.0252)	(0.0169,0.0348)	(0.0248,0.0468)

(f) SYSVJ^{b_1}

	short	mid	long
+OTM p	0.0324	0.0299	0.0378
	(0.0188,0.0408)	(0.0199,0.0365)	(0.0288,0.0461)
OTM p	0.0196	0.0264	0.0392
	(0.0125,0.0267)	(0.0180,0.0350)	(0.0291,0.0480)
ATM	0.0146	0.0249	0.0396
	(0.0101,0.0181)	(0.0165,0.0355)	(0.0282,0.0502)
OTM c	0.0188	0.0245	0.0371
	(0.0127,0.0204)	(0.0167,0.0356)	(0.0266,0.0485)
+OTM c	0.0227	0.0259	0.0350
	(0.0144,0.0245)	(0.0177,0.0357)	(0.0260,0.0464)

(g) SYSVJ^{1}

	short	mid	long
+OTM p	0.0323	0.0298	0.0377
	(0.0197,0.0438)	(0.0203,0.0374)	(0.0272,0.0456)
OTM p	0.0195	0.0263	0.0391
	(0.0127,0.0278)	(0.0190,0.0360)	(0.0281,0.0472)
ATM	0.0143	0.0248	0.0395
	(0.0100,0.0190)	(0.0171,0.0360)	(0.0273,0.0491)
OTM c	0.0182	0.0245	0.0372
	(0.0124,0.0208)	(0.0171,0.0358)	(0.0260,0.0482)
+OTM c	0.0220	0.0256	0.0352
	(0.0153,0.0255)	(0.0181,0.0362)	(0.0253,0.0468)

(h) SYSVJ^{a_2}

MAE(σ)	short	mid	long
+OTM p	0.0323	0.0298	0.0411
	(0.0173,0.0428)	(0.0193,0.0371)	(0.0270,0.0464)
OTM p	0.0195	0.0264	0.0391
	(0.0116,0.0259)	(0.0171,0.0346)	(0.0270,0.0477)
ATM	0.0144	0.0248	0.0395
	(0.0096,0.0184)	(0.0155,0.0341)	(0.0259,0.0493)
OTM c	0.0184	0.0245	0.0372
	(0.0121,0.0204)	(0.0155,0.0336)	(0.0251,0.0483)
+OTM c	0.0222	0.0254	0.0351
	(0.0135,0.0249)	(0.0163,0.0338)	(0.0249,0.0468)

(i) SYSVJ^{b_2}

	short	mid	long
+OTM p	0.0325	0.0299	0.0381
	(0.0195,0.0448)	(0.0205,0.0383)	(0.0290,0.0476)
OTM p	0.0195	0.0263	0.0390
	(0.0128,0.0296)	(0.0182,0.0379)	(0.0291,0.0490)
ATM	0.0146	0.0247	0.0394
	(0.0092,0.0201)	(0.0167,0.0372)	(0.0275,0.0500)
OTM c	0.0192	0.0244	0.0369
	(0.0108,0.0232)	(0.0169,0.0370)	(0.0259,0.0500)
+OTM c	0.0236	0.0260	0.0346
	(0.0132,0.0276)	(0.0173,0.0385)	(0.0257,0.0501)

(j) SYSVJ^{2}

	short	mid	long
+OTM p	0.0322	0.0298	0.0378
	(0.0210,0.0418)	(0.0213,0.0379)	(0.0288,0.0466)
OTM p	0.0194	0.0263	0.0392
	(0.0138,0.0298)	(0.0193,0.0381)	(0.0294,0.0478)
ATM	0.0148	0.0247	0.0396
	(0.0092,0.0200)	(0.0177,0.0371)	(0.0284,0.0498)
OTM c	0.0195	0.0245	0.0370
	(0.0114,0.0213)	(0.0170,0.0352)	(0.0274,0.0477)
+OTM c	0.0238	0.0261	0.0348
	(0.0133,0.0262)	(0.0177,0.0362)	(0.0270,0.0461)

NOTES: The new contracts/maturities are three: **short** refers to M2 and M3; **mid** refers to M4 to M6; **long** refers to Q1 and Q2. The moneyness levels are five: **+OTM p** refers to those more OTM put options with strikes between the ATM level −7.5 and −5.5; **OTM p** refers to those OTM put options with strikes between the ATM level −5.0 and −1.5; **ATM** refers to those call and put options with strikes between the ATM level ± 1.0; **OTM c** refers to those OTM call options with strikes between the ATM level +1.5 and +5.0; +**OTM c** refers to those more OTM call options with strikes between the ATM level +5.5 and +7.5.

each model and our benchmark TS09: the deeper blue (red) the cells are, the best (worst) the goodness-of-fit compared to our benchmark. The values displayed correspond only to monthly observations. Merton (1976) is the worst performing model, especially for short-term contracts. Heston (1993) outperforms the benchmark for longer maturity contracts only, but on average does worse that the benchmark. Bates (1996) does on average as well as our model SYSVJ (considering all its sub-specifications). Each of our model sub-specifications outperforms TS09, especially for longer maturities and deeper OTM options; also, for mid-term options (except deeper OTM puts).

5.2. Out-of-sample

We compute the fair value of standard European call and put options contracts on different maturities and strike prices over a period of 17 months using the parameters' estimates calculated in-sample. Pricing errors labelled MAE$^O(\sigma)$ and RMSE$^O(\sigma)$ are also reported in Table 6.

We highlight that out-of-sample MAE(σ) values are better (that is, smaller) than the equivalent in-sample ones for each of our model sub-specifications, both for monthly and daily observations. The average value for our model is 2.64% for monthly observations, and we outperform TS09 by 0.42%. For daily observations, the average error value for our model is 2.82%, just marginally better than for TS09. In terms of

Table 12
Model performance per bucket of contracts and of moneyness levels — monthly values.

(a) Merton (1976)

MAE(σ)	short	mid	long
+OTM p	−0.0688	−0.0546	−0.0299
OTM p	−0.0770	−0.0528	−0.0274
ATM	−0.0797	−0.0507	−0.0244
OTM c	−0.0751	−0.0486	−0.0235
+OTM c	−0.0715	−0.0463	−0.0228

(b) Heston (1993)

short	mid	long
−0.0013	−0.0027	0.0021
−0.0024	−0.0025	0.0024
−0.0006	−0.0020	0.0023
0.0012	−0.0014	0.0024
−0.0024	−0.0016	0.0027

(c) Bates (1996)

short	mid	long
0.0004	0.0012	0.0052
0.0016	0.0013	0.0053
0.0025	0.0018	0.0042
0.0013	0.0034	0.0055
−0.0002	0.0042	0.0073

(e) SYSVJ^{a_1}

MAE(σ)	short	mid	long
+OTM p	0.0006	0.0011	0.0044
OTM p	0.0016	0.0022	0.0025
ATM	0.0009	0.0025	0.0013
OTM c	−0.0011	0.0032	0.0028
+OTM c	−0.0025	0.0033	0.0052

(f) SYSVJ^{b_1}

short	mid	long
−0.0005	0.0010	0.0045
0.0015	0.0026	0.0026
0.0006	0.0029	0.0015
−0.0020	0.0036	0.0033
−0.0036	0.0034	0.0060

(g) SYSVJ^{1}

short	mid	long
−0.0004	0.0011	0.0046
0.0016	0.0027	0.0027
0.0009	0.0030	0.0016
−0.0014	0.0036	0.0032
−0.0029	0.0037	0.0058

(h) SYSVJ^{a_2}

MAE(σ)	short	mid	long
+OTM p	−0.0004	0.0011	0.0045
OTM p	0.0016	0.0026	0.0027
ATM	0.0008	0.0030	0.0016
OTM c	−0.0016	0.0036	0.0032
+OTM c	−0.0031	0.0036	0.0059

(i) SYSVJ^{b_2}

short	mid	long
−0.0006	0.0010	0.0042
0.0016	0.0027	0.0028
0.0006	0.0031	0.0017
−0.0024	0.0037	0.0035
−0.0045	0.0033	0.0064

(j) SYSVJ^{2}

short	mid	long
−0.0003	0.0011	0.0045
0.0017	0.0027	0.0026
0.0004	0.0031	0.0015
−0.0027	0.0036	0.0034
−0.0047	0.0032	0.0062

Notes: We define the model performance as the difference between the error statistics associated to our benchmark model TS09 minus those for each model. If the value is negative, TS09 outperforms the model; if the value is positive, the model outperforms the benchmark. The correspondent error statistics we refer to are those displayed in Table 11. Observe that the darker the blue (red) of the cell, the more (less) accurate the model is compared to TS09. The new contracts/maturities are three: **short** refers to M2 and M3; **mid** refers to M4 to M6; **long** refers to Q1 and Q2. The moneyness levels are five: +**OTM p** refers to those more OTM put options with strikes between the ATM level −7.5 and −5.5; **OTM p** refers to those OTM put options with strikes between the ATM level −5.0 and −1.5; **ATM** refers to those call and put options with strikes between the ATM level ± 1.0; **OTM c** refers to those OTM call options with strikes between the ATM level +1.5 and +5.0; +**OTM c** refers to those more OTM call options with strikes between the ATM level +5.5 and +7.5.

out-of-sample RMSE(σ) values, these are better than the equivalent in-sample ones for our model and for each observation frequency. For our model, the average value is 3.44% for monthly observations and 3.74% for daily observations; we outperform TS09 by 0.40% and 0.21%.

From this analysis, we provide evidence that the inclusion of jumps brings a benefit to the goodness-of-fit of our model compared with our benchmark. From Fig. 1a we can clearly observe that there are upward jumps in the out-of-sample period. We think that the better performance of our model comes from the fact that upward jumps were already implicit in the calibrated values of the jump parameters.

6. Conclusions and further research

We have developed a novel term-structure model for commodity futures prices which presents stochastic spot prices, forward cost of carry curves and variance. The novel feature in our model is the presence of simultaneous jumps in the spot prices and the cost or carry, either i.i.d. or following a time-dampening form. We model futures dynamics under $\mathbb{Q}$, compute the CF and price plain vanilla option using the FFT algorithm with an analytic expression. We calibrate parameters for five extant models (Merton (1976), Heston (1993), Bates (1996), Trolle and Schwartz (2009)-SV1 and Trolle (2014)) plus the six sub-specifications of our model SYSVJ, with the objective of analysing pricing performances, both in-sample and out-of sample.

This is the first empirical work in the literature which prices options with Trolle and Schwartz (2009)-SV1 using analytical expressions. We prove that our model produces better results (that is, lower error values in terms of MAE(σ) and RSME(σ)) than our benchmark model TS09-SV1, specially for short maturity contracts and away from the ATM level. In-sample, our model outperforms the benchmark by 0.57% in terms of MAE(σ) and 0.72% in terms of RMSE(σ). Out-of-sample MAE(σ) values are better than the equivalent in-sample ones for each of our model sub-specifications, specially for monthly observations where we outperform TS09 by 0.42%. Considered along with the reduction in computation time due to (i) the analytical solution for the CF, (ii) and the use of the FFT for option pricing and (iii) the alternative set-up, these improvements provide a significant benefit for practitioners.

Contrary to results from prior research, we find that jumps in the WTI crude oil futures market on average tend to be downwards.

Future lines of research will include, firstly, the extension of Trolle and Schwartz (2009)-SV1 to capture the dynamics of seasonal energy assets (e.g. natural gas) in the variance. Secondly, we also aim to price calendar spread options on WTI using a joint CF for the two futures contracts involved in each option, within the framework of the model presented in this work. We expect to also obtain analytical solutions for the new transforms, and analytical expressions to price both types of options.

6.1. Inclusion and diversity statement

One or more of the authors of this paper self-identifies as living with a disability.

CRediT authorship contribution statement

John Crosby: Supervision. **Carme Frau:** Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Data curation, Visualisation, Writing – original draft, Writing – review & editing, Visualisation, Project administration.

Appendix A. Appendix for proofs

A.1. Proof of Proposition 1

From applying Itô's Lemma for jump diffusion processes to $F(t,T)$ in (2.6) we have that

$$\frac{dF(t,T)}{F(t,T)} = \Bigg(\int_t^T \mu_y(t,u)du + v_t\Big(\frac{\sigma_Y^2(t,T)}{2} + \sigma_S\sigma_Y(t,T)\rho_{Sy}\Big) - \lambda\mathbb{E}_t^{\mathbb{Q}}\left[e^{J_S} + J_Y(t,T) - 1\right]\Bigg)dt + \sqrt{v_t}\big(\sigma_S dW_t^S + \sigma_Y(t,T)dW_t^y\big) + \big(e^{J_S+J_Y(t,T)} - 1\big)dN_t. \quad (A.1)$$

In an arbitrage-free framework and given that, by definition, futures prices are martingales, the drift term in Eq. (A.1) must equal zero. Therefore we group together

(i) those terms whose expected value equals zero

$$\sqrt{v_t}\big(\sigma_S dW_t^S + \sigma_Y(t,T)dW_t^y\big) + \big(e^{J_S+J_Y(t,T)} - 1\big)dN_t - \lambda\mathbb{E}_t^{\mathbb{Q}}\left[e^{J_S+J_Y(t,T)} - 1\right]dt,$$

(ii) the remaining terms whose expected value, therefore, also equals zero

$$\Bigg(\int_t^T \mu_y(t,u)du + v_t\Big(\frac{\sigma_Y^2(t,T)}{2} + \sigma_S\sigma_Y(t,T)\rho_{Sy}\Big) - \lambda\mathbb{E}_t^{\mathbb{Q}}\left[e^{J_S} + J_Y(t,T) - 1\right] + \lambda\mathbb{E}_t^{\mathbb{Q}}\left[e^{J_S+J_Y(t,T)} - 1\right]\Bigg)dt.$$

Observe that we had to artificially retrieve a new drift correction term in (i) so that the expected value of the jump terms offset one another. The very same term had to be incorporated in (ii).

For Eq. (A.1) to be a martingale, it must hold that

$$\frac{1}{dt}\mathbb{E}_t^{\mathbb{Q}}\left[\frac{dF(t,T)}{F(t,T)}\right] = \int_t^T \mu_y(t,u)du + v_t\Big(\frac{\sigma_Y^2(t,T)}{2} + \sigma_S\sigma_Y(t,T)\rho_{Sy}\Big) - \lambda\mathbb{E}_t^{\mathbb{Q}}\left[e^{J_S} + J_Y(t,T) - 1\right] + \lambda\mathbb{E}_t^{\mathbb{Q}}\left[e^{J_S+J_Y(t,T)} - 1\right] = 0. \quad (A.2)$$

Setting those terms in (ii) to zero and differentiating with respect to T lets us obtain the expression followed by the drift of $y(t,T)$ in (2.2)

$$\mu_y(t,T) = -v_t\sigma_y(t,T)\big(\sigma_Y(t,T) + \sigma_S\rho_{Sy}\big) + \lambda\mathbb{E}_t^{\mathbb{Q}}\left[\big(e^{J_S} + J_y(t,T) - 1\big) - \big(e^{J_S+J_y(t,T)} - 1\big)\right]. \quad (A.3)$$

Thus, the dynamics of the futures under the martingale condition becomes

$$\frac{dF(t,T)}{F(t,T)} = \sqrt{v_t}\big(\sigma_S dW_t^S + \sigma_Y(t,T)dW_t^y\big) - \lambda\mathbb{E}_t^{\mathbb{Q}}\left[e^{J_S+J_Y(t,T)} - 1\right]dt + \big(e^{J_S+J_Y(t,T)} - 1\big)dN_t. \quad (A.4)$$

From Eq. (A.4), if one were to momentarily assume that there is only one jump in S_t but no jump in $y(t,T)$, $F(t,T)$ would change by $F(t,T)(e^{J_S} - 1)$; with one jump only in $y(t,T)$ but no jump in S_t, $F(t,T)$ would change by $F(t,T)(e^{J_Y(t,T)} - 1)$; with one jump in both S_t and $y(t,T)$, $F(t,T)$ would change by $F(t,T)(e^{J_S+J_Y(t,T)} - 1)$.

We define the processes $s_t \equiv \ln S_t$ and $f(t,T) \equiv \ln F(t,T)$. Therefore, we have

$$df(t,T) = ds_t + dY(t,T). \quad (A.5)$$

From applying Itô's Lemma for jump diffusion processes to s_t and Leibniz's rule to Eq. (2.2), we have

$$ds_t = \frac{\partial s_t}{\partial S_t}dS_t + \frac{1}{2}\frac{\partial^2 s_t}{\partial S_t^2}dS_t^2 + J_S dN_t = \Big(y_t - \frac{\sigma_S^2}{2}v_t - \lambda\mathbb{E}_t^{\mathbb{Q}}\left[e^{J_S} - 1\right]\Big)dt + \sigma_S\sqrt{v_t}dW_t^S + J_S dN_t, \quad (A.6)$$

$$dY(t,T) = \bigg(\int_t^T \mu_y(t,u)du - \lambda\mathbb{E}_t^{\mathbb{Q}}\left[J_Y(t,T)\right] - y_t\bigg)dt + \sigma_Y(t,T)\sqrt{v_t}dW_t^y + J_Y(t,T)dN_t, \quad (A.7)$$

with $\sigma_Y(T,t)$ and $J_Y(t,T)$ as in (2.8) and (2.10), respectively. By substituting (A.6) and (A.7) into (A.5) we have

$$df(t,T) = \bigg(\int_t^T \mu_y(t,u)du - \frac{\sigma_S^2}{2}v_t - \lambda\mathbb{E}_t^{\mathbb{Q}}\left[e^{J_S} + J_Y(t,T) - 1\right]\bigg)dt + \sqrt{v_t}\left(\sigma_S dW_t^S + \sigma_Y(t,T)dW_t^y\right) + \left(J_S + J_Y(t,T)\right)dN_t. \quad (A.8)$$

A.2. Proof of Proposition 2

Integrating (2.2) over t between t and 0, we obtain the expression followed by $y(t,T)$

$$y(t,T) = y(0,T) + \int_0^t \mu_y(u,T)du + \int_0^t \sqrt{v_u}\sigma_y(u,T)dW_u^y - \lambda\int_0^t E_u^{\mathbb{Q}}\left[J_y(u,T)\right]du + \int_0^t J_y(u,T)dN_u, \quad (A.9)$$

and substituting T for t into (A.9) yields the expression followed by y_t

$$y_t = y(0,t) + \int_0^t \mu_y(u,t)du + \int_0^t \sqrt{v_u}\sigma_y(u,t)dW_u^y - \lambda\int_0^t E_u^{\mathbb{Q}}\left[J_y(u,t)\right]du + \int_0^t J_y(u,t)dN_u. \quad (A.10)$$

For all $0 \le s \le t$, we consider the following state variables χ_t and ϕ_t

$$\chi_t = e^{-\gamma(t-s)}\chi_s - \int_s^t v_u\left(\frac{\alpha}{\gamma} + \sigma_S\rho_{Sy}\right)e^{-\gamma(t-u)}du + \int_s^t \sqrt{v_u}e^{-\gamma(t-u)}dW_u^y, \quad (A.11)$$

$$\phi_t = e^{-2\gamma(t-s)}\phi_s + \int_s^t v_u\frac{\alpha}{\gamma}e^{-2\gamma(t-u)}du, \quad (A.12)$$

which dynamics are obtained by applying Itô's Lemma to (A.11) and (A.12) and subject to $\phi_0 = \chi_0 = 0$

$$d\chi_t = -\left(\gamma\chi_t + v_t\left(\sigma_S\rho_{Sy} + \frac{\alpha}{\gamma}\right)\right)dt + \sqrt{v_t}dW_t^y, \quad (A.13)$$

$$d\phi_t = \left(v_t\frac{\alpha}{\gamma} - 2\gamma\phi_t\right)dt. \quad (A.14)$$

Then, (A.9) and (A.10) are then affine jump–diffusion functions of χ_t, ϕ_t and the jump-related terms

$$y(t,T) = y(0,T) + \sigma_y(t,T)\chi_t + \frac{\sigma_y^2(t,T)}{\alpha}\phi_t - \lambda\int_0^t E_u^{\mathbb{Q}}\Big[J_y(u,T)\Big]du + \int_0^t J_y(u,T)dN_u, \quad (A.15)$$

$$y_t = y(0,t) + \alpha(\chi_t + \phi_t) - \lambda\int_0^t E_u^{\mathbb{Q}}\Big[J_y(u,t)\Big]du + \int_0^t J_y(u,t)dN_u. \quad (A.16)$$

A.3. Proof of Proposition 3

Integrating the expression followed by ds_t in Eq. (A.6) and after applying exponentials, we have

$$S_t = S_0 \exp\left\{ \int_0^t \left(y_u - \frac{\sigma_S^2}{2} v_u - \lambda E_u^{\mathbb{Q}}\left[e^{J_S} - 1\right] \right) du + \sigma_S \int_0^t \sqrt{v_u} dW_u^S + J_S \int_0^t dN_u \right\}. \tag{A.17}$$

From Eqs. (2.6), (2.11) and (2.15), we have that the futures price $F(t,T)$ is given by

$$\begin{aligned} F(t,T) &= S_t \exp\left\{ \int_t^T \left(y(0,u) + \sigma_y(t,u)\chi_t + \frac{\sigma_y^2(t,u)}{\alpha}\phi_t - \lambda \int_0^t E_s^{\mathbb{Q}}\left[J_y(s,u)\right] ds + \int_0^t J_y(s,u) dN_s \right) du \right\} \\ &= S_t \frac{F(0,T)}{F(0,t)} \exp\left\{ \sigma_Y(t,T)\chi_t + \frac{\hat{\sigma}_Y(t,T)}{\alpha}\phi_t - \lambda \int_0^t E_u^{\mathbb{Q}}\left[J_Y(u,T)\right] du + \int_0^t J_Y(u,T) dN_u \right\} \\ &= S_0 \frac{F(0,T)}{F(0,t)} \exp\left\{ \sigma_Y(t,T)\chi_t + \frac{\hat{\sigma}_Y(t,T)}{\alpha}\phi_t + \int_0^t \left(y_u - \frac{\sigma_S^2}{2} v_u \right) du + \sigma_S \int_0^t \sqrt{v_u} dW_u^S \right\} \\ &\quad \exp\left\{ -\lambda \int_0^t E_u^{\mathbb{Q}}\left[e^{J_S} + J_Y(u,T) - 1 \right] du + \int_0^t \left(J_S + J_Y(u,T) \right) dN_u \right\}, \end{aligned} \tag{A.18}$$

with

$$\hat{\sigma}_Y(t,T) \equiv \int_t^T \sigma_y^2(t,u) du = \frac{\alpha^2}{2\gamma}\left(1 - e^{-2\gamma(T-t)}\right). \tag{A.19}$$

It is convenient to use $s_t \equiv \ln S_t$ instead of S_t as a state variable. In this case, futures log-prices $f(t,T) \equiv \ln F(t,T)$ are an affine jump–diffusion function of the following for state variables — χ_t, ϕ_t, s_t and the jump-related terms

$$\begin{aligned} f(t,T) &= s_t + f(0,T) - f(0,t) + \sigma_Y(t,T)\chi_t + \frac{\hat{\sigma}_Y(t,T)}{\alpha}\phi_t + \int_0^t \left(y_u - \frac{\sigma_S^2}{2} v_u \right) du + \sigma_S \int_0^t \sqrt{v_u} dW_u^S \\ &\quad - \lambda \int_0^t E_u^{\mathbb{Q}}\left[e^{J_S} + J_Y(u,T) - 1 \right] du + \int_0^t \left(J_S + J_Y(u,T) \right) dN_u. \end{aligned} \tag{A.20}$$

Rewriting Eq. (A.20) in its integral form lets us see the evolution of the price of a futures contract

$$\begin{aligned} F(t,T) &= F(0,T) \exp\left\{ \int_0^t \left(\sqrt{v_u}\left(\sigma_S dW_u^S + \sigma_Y(u,T) dW_u^y \right) - \frac{v_u}{2}\left(\sigma_S dW_u^S + \sigma_Y(u,T) dW_u^y \right)^2 \right) \right\} \\ &\quad \exp\left\{ -\lambda \int_0^t E_u^{\mathbb{Q}}\left[e^{J_S} + J_Y(u,T) - 1 \right] du + \int_0^t \left(e^{J_S + J_Y(u,T)} - 1 \right) dN_u \right\}, \end{aligned} \tag{A.21}$$

which shows that $F(t,T)$ and $F(t,t) \equiv S_t$ are Markov in a finite number of state variables.

A.4. Proof of Proposition 4

We find the expressions followed by the terms $A(\tau), B(\tau)$ and $C(\tau)$ similarly to Duffie et al. (2000). The proof consists of showing that the CF $\psi(t) \equiv \psi(iu, t, T_{Opt}, T)$ is a martingale under $\mathbb{Q}$. To this end, we conjecture that $\psi(iu, t, T_{Opt}, T)$ is of the form in expression (2.17).

From applying Itô's Lemma for jump diffusion processes to $\psi(t)$, we obtain the following PIDE

$$\begin{aligned} \frac{d\psi(t)}{\psi(t)} &= \left(-\frac{\partial A(\tau)}{\partial \tau} - \frac{\partial B(\tau)}{\partial \tau} v_t - \frac{\partial C(\tau)}{\partial \tau}\lambda \right) dt \\ &\quad + B(\tau) dv_t + iu \frac{dF(t,T)}{F(t,T)} + \frac{B^2(\tau)}{2} dv_t^2 - \frac{u^2 + iu}{2}\left(\frac{dF(t,T)}{F(t,T)} \right)^2 \\ &\quad + iuB(\tau) dv_t \frac{dF(t,T)}{F(t,T)} + \lambda \int_{-\infty}^{+\infty} [\psi(t,J) - \psi(t)] \varpi(J)\, dN_t, \end{aligned} \tag{A.22}$$

where $\tau \equiv T_{Opt} - t$, $J \equiv J_S + J_y(t,T)$ is the jump size, $\varpi(J)$ is the distribution function of the random variable J, and $\lambda > 0$ is the constant intensity parameter of the Poisson process N_t.

Based on the fact that the jump size J is independent of $f(t,T)$, we consider the jump integral term in (A.22)

$$\begin{aligned} \int_{-\infty}^{+\infty} [\psi(t,J) - \psi(t)]\, \varpi(J)\, dN_t &= \int_{-\infty}^{+\infty} \left(\mathbb{E}_t^{\mathbb{Q}}[e^{iu(f(t,T)+J)}] - \mathbb{E}_t^{\mathbb{Q}}[e^{iuf(t,T)}] \right) \varpi(J)\, dN_t \\ &= \int_{-\infty}^{+\infty} \mathbb{E}_t^{\mathbb{Q}}[e^{iuf(t,T)}] \mathbb{E}_t^{\mathbb{Q}}[e^{iuJ} - 1] \times \varpi(J)\, dN_t \\ &= n_{a_j b_j}(\tau) - iu\; m_{a_j b_j}(\tau). \end{aligned} \tag{A.23}$$

For $\psi(t)$ to be a martingale, it must hold that

$$\begin{aligned} \frac{1}{dt}\mathbb{E}_t^{\mathbb{Q}}\left[\frac{d\psi_t}{\psi_t} \right] &= \left(-\frac{\partial A(\tau)}{\partial \tau} + B(\tau)\kappa\theta \right) + \left(-\frac{\partial C(\tau)}{\partial \tau} + n_{a_j b_j}(\tau) - iu\; m_{a_j b_j}(\tau) \right)\lambda \\ &\quad + \left(-\frac{\partial B(\tau)}{\partial \tau} + b_0 + B(\tau) b_1 + B^2(\tau) b_2 \right) v_t = 0, \end{aligned} \tag{A.24}$$

with b_0, b_1 and b_2 as in (2.21) and subject to the initial conditions $A(0) = B(0) = C(0) = 0$. Since Eq. (A.24) holds for all $t, f(t,T), v_t$ and λ, then the terms in each parentheses must vanish, reducing the problem to solving three much simpler ODEs

$$\frac{\partial A(\tau)}{\partial \tau} = B(\tau)\kappa\theta, \tag{A.25}$$

$$\frac{\partial B(\tau)}{\partial \tau} = b_0 + B(\tau) b_1 + B^2(\tau) b_2, \tag{A.26}$$

$$\frac{\partial C(\tau)}{\partial \tau} = n_{a_j b_j}(\tau) - iu\; m_{a_j b_j}(\tau), \tag{A.27}$$

for $j = 1, 2$ and following the jump assumptions listed in Section 2.1.1. The expressions followed by the terms $n_{a_1 b_1}, n_{a_2 b_2}, m_{a_1 b_1}$ and $m_{a_2 b_2}$ are in Table 5.

Appendix B. Appendix for analytic expressions

B.1. Analytic expression for $B(\tau)$

We apply the following change of variable $B(\tau) = -\frac{y'(\tau)}{y(\tau) b_2(\tau)}$ to Eq. (2.19) so that it becomes

$$\left(-\frac{y'(\tau)}{y(\tau) b_2} \right)' = b_0(\tau) + b_1(\tau)\left(-\frac{y'(\tau)}{y(\tau) b_2} \right) + b_2\left(-\frac{y'(\tau)}{y(\tau) b_2} \right)^2. \tag{B.1}$$

Given that b_2 is constant, it simplifies and we get to the following homogeneous second order ODE

$$y''(\tau) - \left(c_0(\tau) + c_1(\tau)e^{-\gamma\tau}\right)y'(\tau) + \left(d_0(\tau) + d_1(\tau)e^{-\gamma\tau} + d_2(\tau)e^{-2\gamma\tau}\right)y(\tau) = 0. \tag{B.2}$$

Eq. (2.19) has an analytical solution which is given by

$$B(\tau) = \frac{2\gamma}{\sigma_v^2}\left(\beta + \mu z + z\frac{g'(z)}{g(z)}\right), \tag{B.3}$$

where the function $g(z)$ is a linear combination of Kummer's (M) and Tricomi's (U) hypergeometric functions

$$g(z) = k_1 M(a, b, z) + k_2 U(a, b, z), \tag{B.4}$$

$$g'(z) = \frac{a}{b}k_1 M(a+1, b+1, z) - ak_2 U(a+1, b+1, z), \tag{B.5}$$

with coefficients

$$\begin{aligned} a &= -\mu b - \beta c_1\frac{\omega}{\gamma} - d_1\frac{\omega}{\gamma^2}, & b &= 1 + 2\beta + \frac{c_0}{\gamma}, \\ \mu &= -\frac{1}{2}\left(1 + \frac{c_1\omega}{\gamma}\right), & \beta &= \frac{-c_0 \pm \sqrt{c_0^2 - 4d_0}}{2\gamma}, \\ \omega &= \pm\frac{\gamma}{\sqrt{c_1^2 - 4d_2}}, & z &= \frac{e^{-\gamma\tau}}{\omega}, \end{aligned} \tag{B.6}$$

$$\begin{aligned} c_0 &= -\kappa + iu\sigma_v\left(\rho_{Sv}\sigma_S + \rho_{yv}\frac{\alpha}{\gamma}\right), & d_0 &= -\frac{\sigma_v^2(u^2+iu)}{4}\left(\sigma_S^2 + \frac{\alpha^2}{\gamma^2} + 2\sigma_S\rho_{Sy}\frac{\alpha}{\gamma}\right), \\ c_1 &= -iu\sigma_v\rho_{yv}\frac{\alpha}{\gamma}e^{-\gamma(T-T_{Opt})}, & d_1 &= +\frac{\sigma_v^2(u^2+iu)}{2}\frac{\alpha}{\gamma}\left(\frac{\alpha}{\gamma} + \rho_{Sy}\sigma_S\right)e^{-\gamma(T-T_{Opt})}, \\ & & d_2 &= -\frac{\sigma_v^2(u^2+iu)}{4}\frac{\alpha^2}{\gamma^2}e^{-2\gamma(T-T_{Opt})}. \end{aligned} \tag{B.7}$$

In particular, if the initial condition is $B(0) = 0$, we have

$$k_1 = \frac{-\beta\omega - \mu + a\frac{U(a+1,b+1,\frac{1}{\omega})}{U(a,b,\frac{1}{\omega})}}{\frac{a}{b}M(a+1,b+1,\frac{1}{\omega}) + aM(a,b,\frac{1}{\omega})\frac{U(a+1,b+1,\frac{1}{\omega})}{U(a,b,\frac{1}{\omega})}}, \qquad k_2 = \frac{1 - k_1 M(a,b,\frac{1}{\omega})}{U(a,b,\frac{1}{\omega})}. \tag{B.8}$$

The proof is in Sitzia (2018).

B.2. Analytic expression for the alternative $B(\tau)$

Given that $\sigma_S = 0$ and $\rho_{yv} = \rho_{Fv}$ and alternatively to (B.7), the coefficients in $B(\tau), g(z)$ and $g'(z)$ become[14]

$$\begin{aligned} c_0 &= -\kappa + iu\sigma_v\rho_{Fv}\frac{\alpha}{\gamma}, & d_0 &= -\frac{\sigma_v^2(u^2+iu)}{4}\frac{\alpha^2}{\gamma^2}, \\ c_1 &= -iu\sigma_v\rho_{Fv}\frac{\alpha}{\gamma}e^{-\gamma(T-T_{Opt})}, & d_1 &= +\frac{\sigma_v^2(u^2+iu)}{2}\frac{\alpha^2}{\gamma^2}e^{-\gamma(T-T_{Opt})}, \\ & & d_2 &= -\frac{\sigma_v^2(u^2+iu)}{4}\frac{\alpha^2}{\gamma^2}e^{-2\gamma(T-T_{Opt})}. \end{aligned} \tag{B.9}$$

Appendix C. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.eneco.2022.106302.

[14] The expressions followed by a, b, μ, β, ω and z remain the same as in (B.6) under the original set-up.

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