# The noisy voter model with contrarian agents: a theoretical and computational study

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Contents

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# 1 Introduction

The computational social science is a challenging and quite recent interdisciplinary field of study, aimed at studying social phenomena by merging the traditional research of social science with the use of a computational and data-driven approach [1, 2]. In the conceptual framework of the complex systems science the environment where these phenomena live and act can be seen as a social system, composed of many elements, the agents, which form a social complex network: thus, the agents can be represented by the nodes of a network, whose reciprocal positions determine the resulting network structure. Therefore, the complex systems science perspective proves particularly useful in order to study the interactions occurring in a social environment: as a matter of fact, these mechanisms give rise to emergent collective behaviors at the system level. Thus, some property of the whole network stems from the various relationships occurring among the nodes. Being a complex system, the emergent behaviors can not be inferred from the single agents behaviors or properties, according to the Aristotle's quote "The whole is more than the sum of the parts".

Hence, the aim of the computational social science is to study the different interaction mechanisms so as to understand their role in the rise of the collective social phenomena. In order to perform effective and robust studies, plenty of models have been proposed, ranging from general social dynamics [3, 4] and social spreading [5, 6] to crowd behavior [7] and opinion [8, 9, 10, 11, 12, 13, 14, 15], cultural [16] or language dynamics [17, 18], or hierarchy [19] or segregation formation. The general approach often used is focusing on very few and simple mechanisms in each model, so as to study the variation of the social dynamics in response to basic interaction rules, thus eliciting the single contribution of a specific phenomenon to the collective behavior of the social system.

After the theoretical formulations of the model, the second fundamental methodological step is the comparison between the model predictions of the system behavior and the actual dynamics emerging in the computer simulations. Real data can help improve statistics and show the evolution of social systems in time. Moreover, the powerful technological and computational means that are available nowadays allow researchers to manage huge amounts of data regarding large social systems, with a wide number of interactions among their agents, as well as to perform advanced simulations. Opinion dynamics, social interactions, media influences can be monitored and analyzed even real-time, thus leading of a progressive comprehension of the way these big data shape the social world and determine its complexity.

In this work, we propose and analyze an agent-based model for the dynamics of opinions with three important mechanisms: herding, noisy, and contradiction. In order to better understand it we first need to put it in a proper context by presenting a short review of the most common models.

## 1.1 Opinion dynamics and social consensus models

By everyday experience, one of the most important social phenomenon is the agreement: especially when it comes to a discussion involving different opinions, the process of reaching at least a partial agreement proved to be a basic catalyst of the evolution of the human thinking during time. Its importance was already pointed out by philosophy and logic, from Heraclitus to Hegel, with the "thesis-antithesis" conceptualizations. There are many situations in the social and political life in which a community needs to reach a consensus. Through a process of progressive agreements on different themes, some opinions spread, some others disappear and some others evolve through merging and non-linear memetic processes: in this way, the dominant macro-culture is formed and evolves in a society.

In many situations, this dynamics involves the shortest possible range of opinions, a binary choice: choosing between yes or not, left or right, candidate "A" or candidate "B". Binary options can easily be represented by numerical variables, thus allowing to study the social mechanisms with mathematical models. In spite of the simplicity of a mere binary choice, the resulting opinion dynamics and the system behaviors turn out to show the typical features of emergent phenomena, like in[27], as previously described.

In the computational social sciences, plenty of models are aimed at studying opinion dynamics: often each model proposed can be modified with various variants. That is exactly what happens with the voter model, the one considered in this work. The voter model [8, 9, 11, 12, 15] describes a social stochastic process where each agent, represented by a node in a network, has to make a binary choice, randomly interacting with his neighbors and emulating them: the model is studied in order to understand, among other aspects, if the "voters" can eventually reach an agreement and how. In this work, we are going to present a model for opinion dynamics: a variant version of the noisy voter model, with the presence of contrarian agents, namely agents taking opposite positions respect to the other voters. This name was invented by Serge Galam, who firstly studied this typology of agents: original studies are in [31, 32]. Then, other studies involving contrarians regard the Sznajd model, in [33], the voter model itself, in [34, 35], and also the so-called q-voter model, in [36]. The latter [37] is a variant version of the voter model, firstly presented in, in which $q$ neighbors (with possible repetition) are consulted for a voter to change opinion: if they agree, the voter takes their opinion, otherwise still a voter can flip its state with a certain probability. As previously said, we are following a typical methodology of computational social science research: we focus on enlarging the typologies of agents present in the system, rather than including other interaction rules beyond the classical one covered by the voter model, namely the imitation mechanism.

In general, a consensus model allows to understand if a set of interacting agents can reach a consensus when choosing among several options: political vote, opinions, cultural features are well-known examples of them. Opinion dynamics models, then, can be seen as a sub-set of consensus models, when the options to choose from are opinions and the possible social consensus to be reached is the opinion agreement.

Beside the famous voter model, there are various other models studied in the opinion dynamics branch of the computational social science. The models can differ in the interaction mechanisms considered or in the features of the network structure of the agents involved. As we will see in detail, in the voter model the social mechanism examined is the imitation of peers. In the Sznajd model [20, 21], instead, the dynamics of the system is determined by two different rules: "social validation" and "discord destroys". The former represents a partial agreement in a total neighborhood of agents when two (or more, depending on possible variants) adjacent nodes agree on one chosen opinion, while the latter represents the local division and arguing process when a block of adjacent neighbors disagree.

As the methodological way of building up these models directly come from statistical mechanics, also some variants of the well-known Ising model can be used in order to study the social dynamics in response to phenomena like the social pressure. Other examples of consensus models were not born specifically in order to study opinion dynamics, but they (or their variants) are also used for this purpose. In the Axelrod's model [16] the stochastic process considered is the cultural dissemination: in this case, each node is characterized by a set of $F$ cultural features, each of which can assume $q$ states. Similarly to the voter model, the social influence is considered, but another mechanism studied in this model is homophily, which is relevant in the culture dynamics process and in the community formations. Other models, like Granovetter's [22], investigate the possible presence of a critical threshold in the number of agreeing decision-makers, which, if reached, determines the realization of a consensus.

### 1.1.1 Agent-based models

All these models are part of another fundamental class of models, the so-called agent-based models. The main elements of such models are the agents, whose states are defined by one or more variables. As we can imagine, in the social environment described the variables are very often discrete or even binary. For each time step, the state of the agents evolves according to the model specific update rules, which generally affect only the neighborhood of each agent, determining local interactions. These mechanisms, then, can lead to possible global emergent behaviors at the system's level [28].

Beside the interactions among the agents, also the topology of the network structure considered proves fundamental in order to understand the properties of the system and its global state during the whole evolution process. The topology can be fixed, as in the d-dimensional lattice case, or not, in random networks. The advantages of the agent-based models range from the possibility to contextualize and study them with the conceptual instruments of statistical mechanics, to the simplicity of their implementations for numerical simulations, up to their usefulness in understanding the relationships between the microscopical properties of the agents, their interactions, the influence of the topology and the macroscopical and emergent behaviors of the system [28, 29].

### 1.1.2 The voter model

As an important agent-based model, very often used as a model of opinion dynamics and social consensus, we present the voter model. In its original formulation [8, 14], the voter model describes a system of a set of agents with positions in an integer lattice of dimension $d$. Each agent $i$ has its state, defined as a binary variable representing the opinion of the agent, with the density $\sigma_i = \pm 1$. Using the language of the statistichal mechanics, this state can be seen as a spin. At each time step, an agent $i$ of the system and one of its neighbours, the $j$ one, are randomly selected; the update rule of the state mathematically represents one of the simplest and mostly diffused social interaction mechanisms, wich is known as unconditional imitation: the agent $i$ merely copies $j$, meaning that the opinion of $i$ will become equal to the opinion of $j$, namely $\sigma_i = \sigma_j$ .

The main quantities considered in the model are:

1) The average interface density, $\varrho$, which is the density of the active links, namely the links connecting nodes in different states. It will be $\varrho = 0$ at the absorbing states.

2) The time $\tau$ to reach one of the two possible absorbing states.

In general, in the numerical simulations of the original voter model these two parameters are studied for different system sizes, so as to understand the modification of the collective behavior for different numbers of agents composing the network structure of the system. When $t \longrightarrow \infty$, and for $d \leq 2$, the stationary collective state reached is always ordered, in that simplest version of the voter model: one of the two options will totally disappear, while the other one will dominate in the system. Then, in this case the social network eventually agrees with total consensus. In the statistical physics language, this state can also be seen as an absorbing state. Otherwise, it is possible to show that for $d \geq 3$ the two options keep coexisting for a time that scales as the system size and hence diverges the case of infinite systems.

We have seen that the key elements that influence and can determine the collective properties of the system are the network topology, the way the microscopic states of the agents are defined, and the interaction mechanisms considered, represented by the update rules of the process. Several variants can be created with modifications of such aspects, enlarging the variety of the phenomena considered or of the agents included. A fundamental variant version of the voter model is in works of [9, 13], in which a noisy voter model is studied. Respect to the original version the interaction mechanism between the agents and their neighbors is modified: instead of the unconditional imitation of the neighbor's state, any agent will perform an imperfect imitation process at each step of the temporal evolution. The agent will copy the neighbor's state with probability $\lambda$, and it will adopt a random value with probability $1 - \lambda$. The noisy term perfectly fits, from the methodological and mathematical point of view: as a matter of fact, in the original literature the model is studied with the wide formalism of the random walkers and stochastic processes, then the imperfect imitation mechanism is implemented trough the addition of the noise term. Another interesting modification to the original version of the voter model was proposed in [14]: in this case, each agent has a ternary state, instead of binary. The three options represent leftist, centrist, and rightist ideologies. Such modification enlarges the width of the range of possible final absorbing states, but it also implies a slower collective dynamics at the system's level.

## 2 The noisy voter model

In this section we focus on a variant version of the voter model, relevant for our work: the noisy voter model, already briefly presented in the previous section. As said, in this variant version of the voter model the emulation process considered is not perfect: as the noise term is included, the imitation mechanism is imperfect and the process represented by the model is stochastic. The inclusion of the noise term is aimed at representing mathematically the contribution of the free will of the single agents to the change of their state: then, the agents will change their state both because of their herding behavior and because of an autonomous choice.

Thus, as stated in [23], the stochastic process studied and the behavior of this model are characterized by the competition between two opposing mechanisms. On the one hand, the emulation (herding) process is a pairwise interaction mechanism related to a copying process and tends to order the system, which is lead towards a homogeneous configuration, with all agents (spins) in the same state (up or down). When no other mechanism is present, the model reduces to the case of the original voter model: then, the homogeneous configurations become absorbing states of the dynamics. On the other hand, the random change mechanism caused by the free-will term can be seen as a "noise" which tends to disorder the system, driving it away from the homogeneous configurations.

The noisy voter model has been studied in regular networks with all-to-all interactions and a finite number of agents, but also in complex networks. In both these macro-typologies of configurations also some modifications to the classic noisy voter model have been studied, for example in [23]: in particular, as in [24], the inclusion of the intransigent agents, also called "zealots".

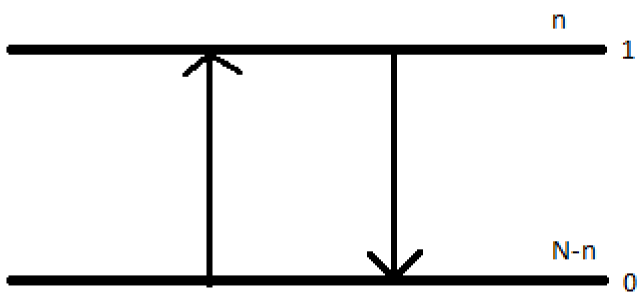


Figure 1: Qualitative representation: the two states and the switches of the agents between them.

Being a voter model, the system of $N$ voters considered has agents which can be in one of two possibile states, 0 ("pessimistic") and 1 ("optimistic"). At each temporal step, there are $n$ optimistic voters at a certain moment and $N-n$ pessimistic ones, as schematically represented in Figure 1. As there are two mechanisms in the model, the emulation among agents and the noise of the single voters, the two main parameters are:

1) $a$: the noise parameter, which represents the free will;
2) $h$: the herding parameter, which represents the imitation mechanism.
Hence, the rate at which a given voter jumps from state 0 to state 1 is

$$\omega(0 \to 1) = a + h\frac{n}{N}, \tag{1}$$

while the rate at which a given voter jumps from state 1 to state 0 is

$$\omega(1 \to 0) = a + h\frac{N-n}{N}. \tag{2}$$

Then, the rate at which each voter jumps from state 0 to state 1 is proportional to the number of voters $n$ already in the state 1, while the rate at which each voter jumps from state 1 to state 0 is proportional to the number of voters $N-n$ already in the state 0.

In general, the rates for transitions $n \to n+1$ and $n \to n-1$ are

$$\begin{cases} \Omega(n \to n+1) = \left(a + h\frac{n}{N}\right)(N-n) \\ \Omega(n \to n-1) = \left(a + h\frac{N-n}{N}\right)n \end{cases} \tag{3}$$

namely the rate at which a each voter jumps from state 0 to state 1 (or from 1 to 0), multiplied by the number of the pessimistic (or optimistic) voters.

The already mentioned competition between the two mechanisms included in the model leads to the disappearance of consensus, the typical absorbing states of the voter model. As reported in [23], the main consequence of this process is the presence of a noise-induced transition between two different behavioral regimes: a mostly ordered regime, dominated by the voters' emulation interactions, and a mostly disordered regime dominated by noise. Then, a bimodal-unimodal phase transition by changing the noise to herding ratio can be detected. It is characterized by a qualitative change of the steady-state probability distribution $P(x)$ of observing a magnetization $x = \frac{2n}{N} - 1$. Figure 2 shows typical trajectories $x(t)$ and their respective steady probability functions $P(x)$. The critical value $(a/h)_c = 1/N$ separates the two phases. For $a/h \leq (a/h)_c$ the system is in the bimodal phase where voters share the same opinion most of the time, having the two opinions the same overall probabilities in the long run: at any point in time, the most likely outcome of a static observation is to find a large majority of nodes in the same state; hence the steady probability function $P(x)$ accumulates around the extremes and becomes bimodal with symmetric maxima at $x_m = \pm 1$. For $a/h \geq (a/h)_c$ the system is in the unimodal phase where probabilities accumulate around $x_m = 0$, meaning that a coexistence of opinions is present. In the border case $a/h = (a/h)_c$ the probability function $P(x)$ is uniform in the $x \in [-1, 1]$ space, corresponding to an equal probability for any fractions of optimistic voters.

The first moments of the distribution also provide useful information of the behaviour of the system. Due to symmetry considerations, the steady-state value for the global magnetization is zero

$$\langle x \rangle_{st} = 0, \tag{4}$$

while for the second moment it can be proved that

$$\left\langle x^2 \right\rangle_{st} = \frac{2a+h}{2Na+h}, \tag{5}$$

which, in turns, can be used to compute the critical point of the unimodal-bimodal transition, simply using the fact that $\langle x^2 \rangle_{st} = (N+2)/(3N)$ for a discrete random variable uniformly distributed in $\{-1, -1+2/N, ..., 1-2/N, 1\}$.

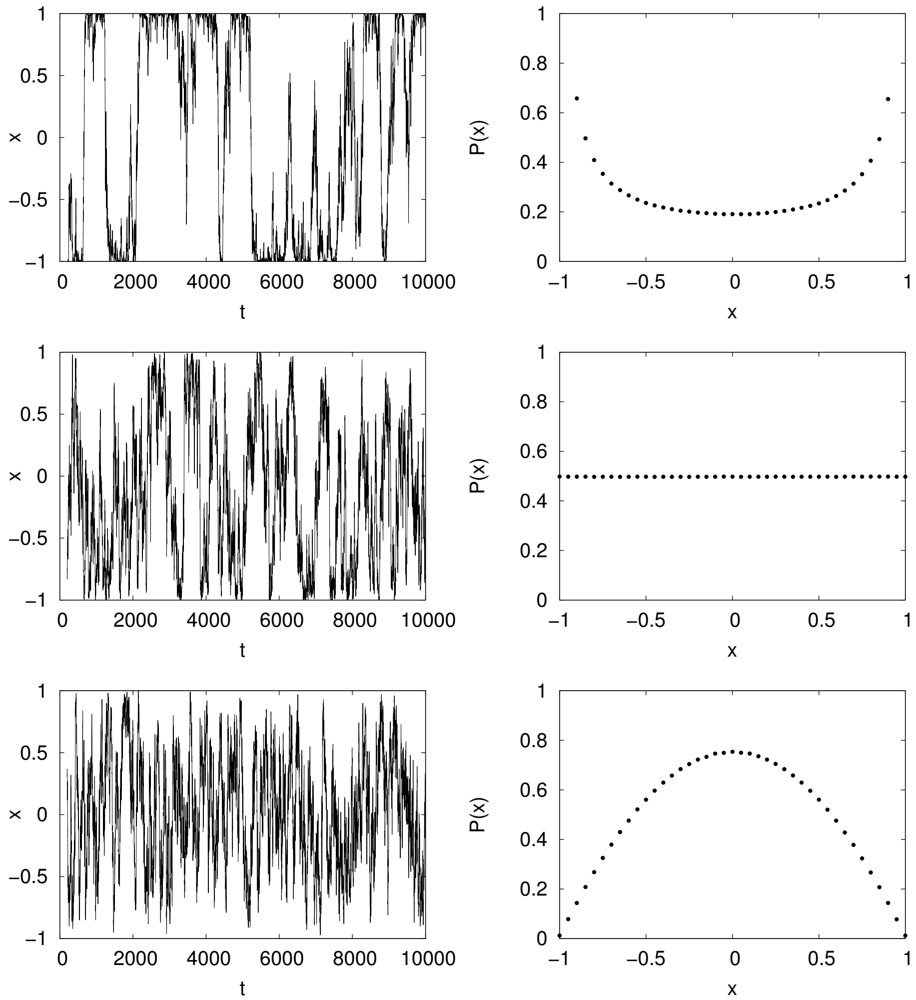


Figure 2: Simulation results for the trajectory $x(t)$ and probability function $P(x)$ of a system of $N = 200$ agents for $a/h \ll (a/h)_c$ (top), $a/h = (a/h)_c$ (middle), and $a/h \gg (a/h)_c$ (bottom). Time is measured in unit of $h^{-1}$.

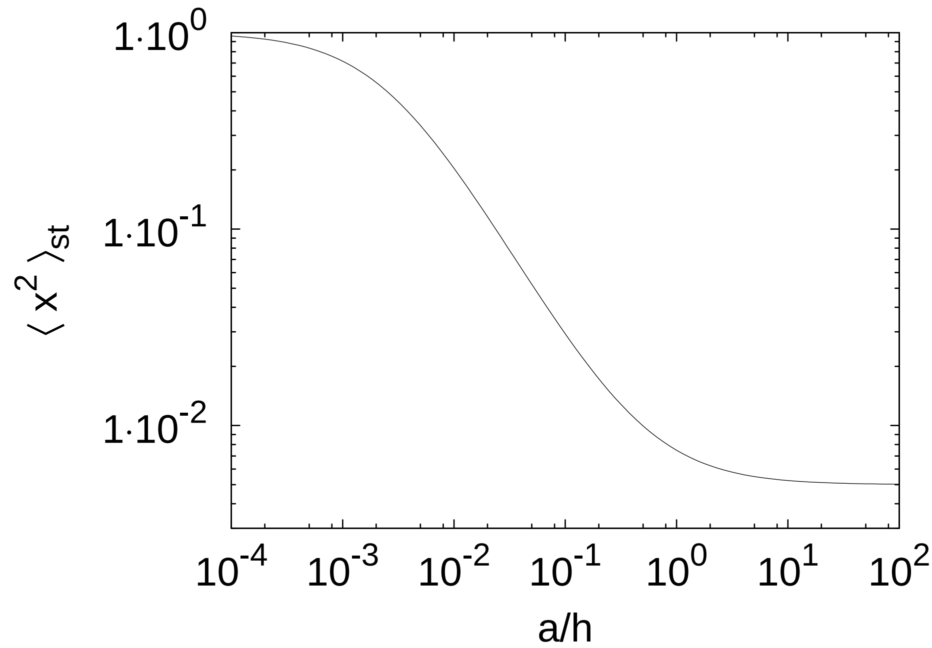


Figure 3: Second moment of the global magnetization given by Eq. (5) for $N = 200$ noncontrarians.

Of course, the network topology is a fundamental aspect that influences the results obtained in literature: for example in [23] the model is studied for different network structures and it is shown that the topology and the degree distribution influence different parameters governing the behavior of the system. In particular they study the influence of the degree distribution of the underlying network on the critical value of the noise parameter $a$, on the steady state variance of $n$ in different network topologies, as well as the steady state of the average interface density as a function of the noise parameter $a$ for different types of networks.

Among the various applications of the noisy voter model in different disciplinary contexts, it is worthy to mention the usefulness of the model upon describing financial markets, as first used by Kirman [25] and later analyzed by others, as in [26]. Inspired by different experiments with ant colonies, pursued by some entomologists, Kirman proposed a stochastic model using the herding and noise formalism to represent and study decision making among financial agents. In the experiments with ants, two asymmetric collective behaviors emerged, starting from an apparently symmetric situation: when ants had to choose between two identical food sources, a majority of the population tended to exploit only one of them at a given time. Then most ants alternatively turned their attention to the other source every, after some time. In order to explain this behavior, Kirman developed a stochastic model where the probability for an ant to change its foraging source results from a combination of two mechanisms: the presence of a herding propensity among the ants and an autonomous switching tendency, basically a free will contribution. According to the Kirman's proposal, the ants present both a tendency to follow the crowd, which requires the existence of some kind of interaction mechanism with information transmission, and an instinct to randomly explore their neighborhood looking for new food sources.

Then Kirman applied this herding model to the study of the market behavior. In this case, instead of an ant's binary choice between food sources, there is a market agent's choice between two different trading strategies, related to the ways the agents' expectations about the future evolution of prices can form, or resulting from the interpretation of present and past information about the market state: for example the choice between an optimistic or a pessimistic tendency. In this situation the Kirman model is applied to the decision-making mechanism of the financial agents: they have to decide whether to buy or sell in a given situation, implying market switches between a dominance of one or the other strategy. Intuitively, in this scenario the possible consensus state is a complete dominant strategy at the steady state.

In [26] the authors applied the Kirman model in the financial market, taking into account the presence of a source of external information like a time-varying advertising, or rumor, orienting the market in favor or against one of two possible trading behaviors. Then they studied the three different regimes that emerge when an external information source affects the market: amplification, precise assimilation and undervaluation of the incoming information.

## 2.1 The noisy voter model with contrarians

We aim at studying the the effect of the so-called contrarian agents under the influence of noise. As the name suggests, the contrarians are characterized by the tendency to take opposite positions respect to the other ones. Furthermore, we also want to study the effect of contrarians in a population of noisy voters: then, we have to include both contrarians and non-contrarians in the system.

Let us consider a system of $N$ voters: agents or particles, each one capable to be in one of two possible states, 0 and 1. For the sake of clarity, we can think of the two different categories of agents as "pessimistic" and "optimistic" ones. We introduce the occupation number $n$ of the state 1. In the whole system we have $n$ optimistic voters at a certain moment and $N - n$ pessimistic ones, as schematically represented in Figure 1. Moreover, we consider two communities: for subindex 1 we denote the community of non-contrarian voters, while for the subindex 2 we have the community of the contrarians. In this case the total number of agents is $N = N_1 + N_2$, with the total number of agents in state 1 being $n = n_1 + n_2$. Hence, differently from the original and simplest formulation of the voter model, here there are two more elements: the presence of noise and the presence of a fraction of contrarian agents.

The dynamics is fully characterized by all transition rates. The rate at which a given non-contrarian agent jumps from state 0 to state 1 is chosen as

$$\omega_1 \left(0 \to 1\right) = a + h\frac{n}{N}, \tag{6}$$

while the rate at which a given non-contrarian agent jumps from state 1 to state 0 is

$$\omega_1 \left(1 \to 0\right) = a + h\frac{N-n}{N}, \tag{7}$$

that is, nothing but the rates of the noisy voter model, Eq. (1) and (2). In both cases, $a$ is the noise parameter and $h$ is the herding parameter, as described in Eq. (1). Similarly, for the contrarians community we have

$$\omega_2 \left(0 \to 1\right) = a + h\frac{N-n}{N}, \tag{8}$$

for the rate at which a given contrarian agent jumps from state 0 to state 1, and

$$\omega_2 \left(1 \to 0\right) = a + h\frac{n}{N} \tag{9}$$

for the transitions from state 1 to state 0. For the latter two rates, the coefficient $a$ still plays the same role of a noisy term, while the coefficient $h$ now is associated with some sort of anti-herding mechanism.

Since all agents are equivalent inside their communities, the relevant rates are the ones that increase or decrease $n_1$and $n_2$. The rates for transitions $n_k \to n_k + 1$, for $k = 1, 2$, are

$$\Omega_k \left(n_k \to n_k + 1\right) = \omega_k \left(0 \to 1\right) \left(N_k - n_k\right), \tag{10}$$

and for transitions $n_k \to n_k - 1$,

$$\Omega_k \left(n_k \to n_k - 1\right) = \omega_k \left(1 \to 0\right) n_k. \tag{11}$$

Having the rates for the transitions occurring in the system, we can build up the master equation for the probability $P(n_1, n_2; t)$ of finding values of $n_1$ and $n_2$ at time $t$ as

$$\frac{\partial P(n_1, n_2; t)}{\partial t} = \sum_k \sum_{l=\pm 1} (E_k^l - 1) \left[ \Omega_k(n_k \to n_k - l) P(n_1, n_2; t) \right], \qquad (12)$$

where $E_k^{\pm}$ are operators acting on any function $f(n_k)$ such as $E_k^{\pm} f(n_k) = f(n_k \pm 1)$.

# 3 The only-contrarians model

Before considering the most general case of a system made of both contrarians and non-contrarians, we focus first on the case of only contrarians. The aim is to describe the global behaviour of the system, as compared to that of the only non-contrarians or noisy voters, described in the previous sections. First, we report some theoretical results and then compare them with numercial simulations of the master equation.

## 3.1 Theoretical results

Using the notation of section 2.1, here we study the case of $n_1 = 0$, $n_2 = n$, and $N_2 = N$. In this case, the most general quantity is the probability $P(n;t)$ of finding a value $n$ at time $t$, which obeys the following master equation

$$\frac{\partial P(n;t)}{\partial t} = \sum_{l=\pm 1} \left(E^l - 1\right)\left[\Omega_2(n \rightarrow n-l) P(n;t)\right], \tag{13}$$

where the operators $E^{\pm}$act on $n$, and $\Omega_2$ is given by Eqs. (8) and (9).

Instead of solving Eq. (13), we focus on partial information by deriving equation for the first moments of the probability distribution. They can be computed from (13) by multiplying by $n^k$ and summing over all possible values of $n$, and finally using the following general result

$$\left\langle n^k \left(E^{\pm} - 1\right) f(n)\right\rangle = \left\langle f(n)\left(E^{\mp} - 1\right) n^k\right\rangle, \text{ for } k = 0, 1, 2, ...., \tag{14}$$

that holds for any function $f(n)$, satisfying $f(n) = 0$ for$n = -1$ and $n = N+1$. For the case of the mean number of optimistic agents $\langle n \rangle$, we get

$$\frac{d\langle n \rangle}{dt} = -\sum_{l=\pm 1} \langle l\Omega_2(n \rightarrow n-l)\rangle = -2(a+h)\left(\langle n \rangle - \frac{N}{2}\right), \tag{15}$$

whose general solution is

$$\langle n \rangle = \frac{N}{2} + \left(\langle n \rangle_0 - \frac{N}{2}\right)\exp\left[-2(a+h)t\right], \tag{16}$$

with $\langle n \rangle_0$ being the initial value of $\langle n \rangle$. In terms of the global magnetization $x \equiv \frac{2n}{N} - 1$, the latter equation can also be written as

$$\langle x \rangle = \langle x \rangle_0 \exp\left[-2(a+h)t\right]. \tag{17}$$

That is, for any initial condition, and for any values of $a$ and $h$, provided $a+h > 0$, the mean number of optimistic agents tends to $\langle n \rangle_{st} = N/2$, or equivalentely the magnetization to zero. This result could also be inferred using symmetry considerations, as in the case of only noncontrarians.

For the second moment $\langle n^2 \rangle$we have

$$\frac{d}{dt}\left\langle n^2\right\rangle = N(a+h) + 2\left[Na + (N-1)h\right]\langle n \rangle - 2\left[2a + \frac{2N-1}{N}h\right]\left\langle n^2\right\rangle, \tag{18}$$

which has to be solved using Eq. (15). The general solution exhibits an exponential decay towards the steady state value $\left\langle n^2\right\rangle_{st}$ given by

$$\left\langle n^2 \right\rangle_{st} = \frac{(N+1)\,a + Nh}{2\left[2a + h\,(2N-1)\,/N\right]} N. \tag{19}$$

In terms of the magnetization, we have

$$\left\langle x^2 \right\rangle_{st} = \frac{2a + h}{2aN + (2N-1)\,h}. \tag{20}$$

The latter result implies that the magnetization keeps of the order $\left\langle x^2 \right\rangle_{st} \sim 1/N \ll 1$, meaning, together with $\langle x \rangle_{st} = 0$, that the ensemble of contrarian agents keeps always close to the coexistence of opinions in the long-time limit, contrary to what happens in general with a collection of noisy voters. Another interesting difference between the system made of only noncontrarians and the one with only contrarians is in the dependence of the second moment on the noise parameter $a$ : for the former case, $\left\langle x^2 \right\rangle_{st}$ is a decresing function of $a$, while for the latter case the second moment is an increasing function of the noise. For $a \gg h$, both systems behave in a similar way, both the first and second moments coincide: $\langle x \rangle_{st} = 0$, $\left\langle x^2 \right\rangle_{st} \sim \frac{1}{N}$.

### 3.2 Numerical results

We have used the first reaction method for solving the master equation, as in [30]. Moreover we have used $N = 200$ agents. In the sequel we show the comparison of the numerical results with the main theoretical parts.

As it is apparent from Fig. 4 (top plot), there is a very good agreement between theory and simulations for the second moment. This was expected since the theoretical results are exact. As previously analyzed, the second moment of the magnetization is an increasing function of the noise to anti-herding ratio $a/h$. That means that the antiherding mechanism of contrarians, as opposite to noncontrarians, tries to keep the system in a situation of coexistence of opinions. This tendency is also induced by the noise mechanism in both the noisy voters (noncontrarians) and all-contrarians case (recall that for $a \gg h$ both cases coincide). Moreover, the increasing character of the second moment shows that the contrarian mechanism of coexistence is more efficient than the noise, although the difference is of order $1/N$.

In Fig. 4 (bottom plot) we also show the probability distribution of the magnetization measured in simulations for different values of $a/h$. Since the system always keeps in the unimodal phase, and the fluctuation of the magnetization is very small, the scaled distributions are very good fixed by a Gaussian distribution.

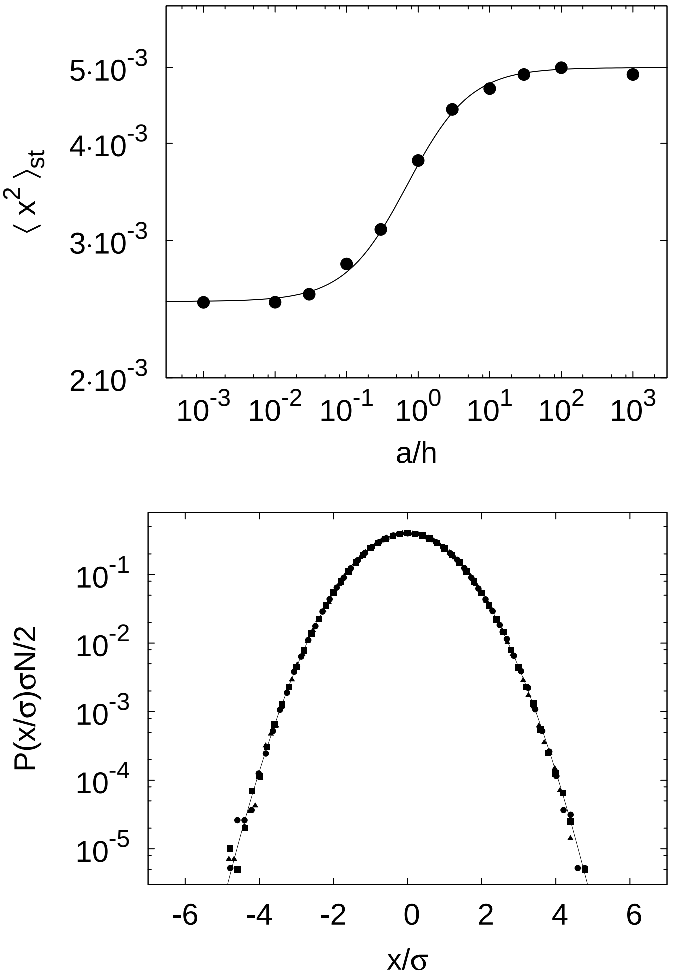


Figure 4: Numerical simulations for a system of $N = 200$ contrarians. Top: numerical simulations (symbols) and theory (line) for the second moment of the magnetization as a function of $a/h$. Bottom: probability distribution (symbols) for the magnetization divided by $\sigma \equiv \sqrt{\langle x^2 \rangle_{st}}$ for $a/h = 10^{-3}$(squares), $10^{-1}$ (circles), $10^2$ (triangles), and the Gaussian distribution (line).

# 4 The two-community model

At this point, we have completely examined the dynamics of the noisy voter model and the case of only contrarians. Then, we can start analysing the more general model where both contrarians and non-contrarian agents are present.

## 4.1 Theoretical results

As we did in the previous section, we focus now on the derivation of the first moments of the probability function $P(n_1, n_2, t)$ for having $n_1$optimistic noncontrarians and $n_2$ optimistic contrarians at time $t$. By operating with the master equation (12), we have for the first moments

$$\begin{cases} \frac{d}{dt}\langle n_1\rangle = aN_1 - \left(\frac{N_2}{N}h + 2a\right)\langle n_1\rangle + \frac{h}{N}N_1\langle n_2\rangle \\ \frac{d}{dt}\langle n_2\rangle = (a+h)\,N_2 - \left[\left(\frac{N+N_2}{N}\right)h + 2a\right]\langle n_2\rangle - \frac{h}{N}N_2\langle n_1\rangle \end{cases} . \tag{21}$$

The general solution of the previous system describes exponential approach to the unique steady-state solution

$$\begin{cases} \langle n_1\rangle_{st} = \frac{N_1}{2} \\ \langle n_2\rangle_{st} = \frac{N_2}{2} \end{cases} . \tag{22}$$

We notice that the result for the first moment of the community of the contrarians is coherent with the one already obtained for the only-contrarians model, in (22). The same considerations on $a$ and $h$ are valid. Likewise, the same behavior and result holds for the non-contarians community.

Now we compute the second moment for the two-community model: actually, being two variables, $n_1$ and $n_2$, measuring the state of the optimistic agents in the two communities, we then have three second moments in this model, namely $\langle n_1^2\rangle$, $\langle n_2^2\rangle$, and $\langle n_1 n_2\rangle$. In order to find them, we proceed as usual. After a long algebra, we have

$$\begin{aligned} \frac{d}{dt}\langle n_1^2\rangle &= aN_1 + \left[2aN_1 + h\left(1+\frac{N_1}{N}\right)\right]\langle n_1\rangle + \frac{h}{N}N_1\langle n_2\rangle \\ &\quad - 2\left[2a + h\left(1+\frac{1-N_1}{N}\right)\right]\langle n_1^2\rangle - 2\frac{h}{N}(1-N_1)\langle n_1 n_2\rangle , \end{aligned} \tag{23}$$

$$\begin{aligned} \frac{d}{dt}\langle n_2^2\rangle &= (a+h)\,N_2 + \left[2aN_2 - h\left(1-\frac{2N-1}{N}N_2\right)\right]\langle n_2\rangle - \frac{h}{N}N_2\langle n_1\rangle \\ &\quad - 2\left[2a + h\left(1-\frac{1-N_2}{N}\right)\right]\langle n_2^2\rangle + 2\frac{h}{N}(1-N_2)\langle n_1 n_2\rangle , \end{aligned} \tag{24}$$

and

$$\begin{aligned} \frac{d}{dt}\langle n_1 n_2\rangle &= aN_1\langle n_2\rangle + (a+h)\,N_2\langle n_1\rangle - \left[4a + h\left(\frac{N_2-N_1}{N}+2\right)\right]\langle n_1 n_2\rangle \\ &\quad + \frac{h}{N}N_1\langle n_2^2\rangle - \frac{h}{N}N_2\langle n_1^2\rangle . \end{aligned} \tag{25}$$

The corresponding steady–state solutions are the following. For the noncontrarians community:

$$\langle n_1^2 \rangle = \frac{A}{B}, \tag{26}$$

where

$$\begin{aligned} A = & \left\{ N_1 \left\{ h^3 N_2(-N_1+N_2)N \right.\right. \\ & - h^2 \left\{ (2a+h)N_1(2+N_1) + \left[4a - h(-2+N_1)\right] N_2 - 2(a+4aN_1+2hN_1)N_2^2 \right\} N \\ & + h(2a+h)\left[-2a(2+N_1+N_1^2-3N_2-5N_1N_2) + h(-2+N_1+3N_2+2N_1N_2)\right] N^2 \\ & \left.\left. + \; 2(2a+h)^2\left[h+2a(1+N_1)\right] N^3 \right\}\right\} \end{aligned} \tag{27}$$

and

$$\begin{aligned} B = & \left\{ 4(2a+h)N\left[h^2(-2+N_1+N_1^2+N_2-2N_1N_2+N_2^2) \right.\right. \\ & \left.\left. - \; 3h(2a+h)(N_1-N_2)N + 2(2a+h)^2N^2\right]\right\} \end{aligned} \tag{28}$$

As regards the second moment involving only $n_2$, we get

$$\langle n_2^2 \rangle = -\frac{C}{B}, \tag{29}$$

where

$$\begin{aligned} C = & \left\{ N_2 \left\{ h^3 N_1(-N_1^2+N_2^2) - h^2 \left\{ 2N_1\left[2a+h-(a+h)N_1\right] \right.\right.\right. \\ & + \left.\left[4a - 4(2a+h)N_1^2 + h(2+N_1)\right] N_2 + (2a+h)N_2^2 \right\} N \\ & + h(2a+h)\left\{ 2a\left[2+N_2+N_2^2-N_1(3+5N_2)\right] \right. \\ & \left. + \; h\left[2+N_2(3+2N_2)-N_1(3+8N_2)\right]\right\} N^2 \\ & \left.\left. + \; 2(2a+h)^2\left[2a+h+2(a+h)N_2\right] N^3 \right\}\right\} \end{aligned} \tag{30}$$

Finally, as regards the last second moment, involving both $n_1$ and $n_2$, we obtain

$$\langle n_1 n_2 \rangle = \frac{D}{B}, \tag{31}$$

where

$$\begin{aligned} D = & \left\{ N_1N_2 \left\{ h^3(N_1-N_2)N - h^2\left[8aN_1N_2 + h(N_1-N_2+4N_1N_2)\right] N \right.\right. \\ & \left.\left. - \; 2h(2a+h)\left[h(N_1-2N_2)+3a(N_1-N_2)\right] N^2 + 2(2a+h)^3N^3 \right\}\right\} \end{aligned} \tag{32}$$

From the previous expressions we can obtain the first and second moments of the magnetizations by means of the following definitions:

$$\begin{cases} x_1 = \frac{2n_1}{N} - 1 \\ x_2 = \frac{2n_2}{N} - 1 \end{cases}, \tag{33}$$

and also the global magentization using

$$x = \frac{2n}{N} - 1. \tag{34}$$

## 4.2 Numerical results

Here we compare the main theoretical results of the previous section with numerical simulations. The total number of agents is fixed to $N = 200$, while the number of noncontrarians $N_1$, and contrarians $N_2 = N - N_1$, as well as the ratio $a/h$ are variables. We have again used the first reaction method for solving the master equation.

As is apparent from Fig. 5, there is again a very good agreement between theory and simulations for the second moment, since the theoretical results are exact. The second moment of the magnetization of the noncontrarians community is always a decreasing function of the noise to herding ratio $a/h$, where the rate of the increase depends on the number of contrarians in the system. Conversely, the second moment of the magnetization of the contrarians community is an increasing function of the noise to anti-herding ratio $a/h$, for large values of $N_2$, while it is a decreasing or even a nonmonotonic function for smaller values of $N_2$. Finally, the second moment $\langle x_1 x_2 \rangle$ shows that the two communities keep uncorrelated except for smaller fraction of contrarians. In this latter case, the contrarians try to prevent the system from being in the bimodal phase by taken the opinion of the noncontarians (the majority).

In Fig. 6 we also show the probability distribution of the magnetization measured in simulations for different values of $a/h$, in the various cases. Again, since the system always keeps in the unimodal phase, and the fluctuation of the magnetization is very small, the scaled distribution are very good fixed by a Gaussian distribution.

As the main result of this section we see that the system is always in the unimodal phase, for any nonzero value of the number of contrarians. Moreover, both communities are also in the unimodal phase. This can be seen numerically and analytically. Namely, on the one hand, for all the values of the parameters considered, the numerical simulations show that both communities have a second moment of the magnetization smaller than $1/3$ (the value of the uniform phase) and a probability distribution with negative convexity. On the other hand, a careful theoretical analysis of the second moment of the magnetization also confirms that it is always smaller than $1/3$, for both communities and the complete system.

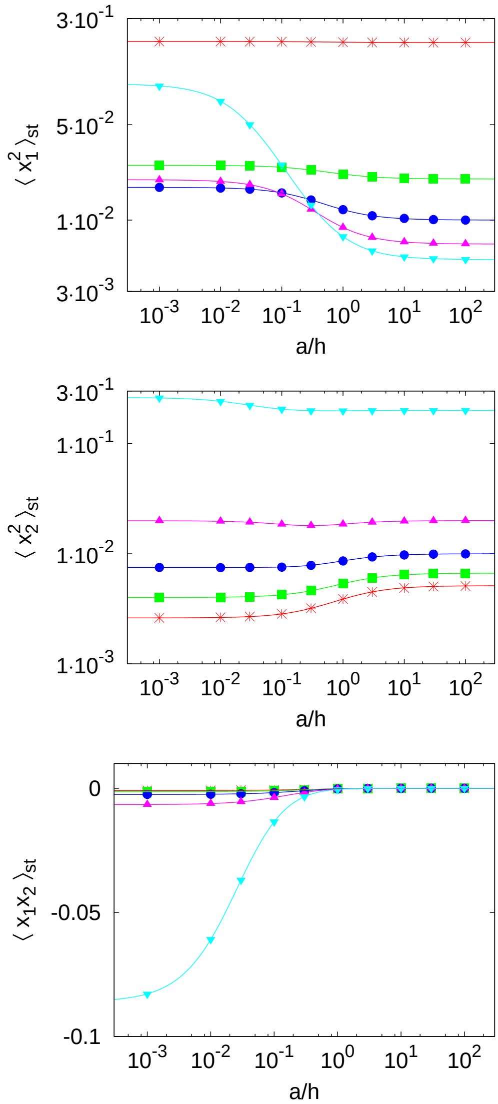


Figure 5: Numerical simulations for a system of $N = 200$ agents, with $N_1$noncontrarians and $N - N_1$ contrarians. The symbols are the simulation results for $N_1 = 5$ (aterisks), 50 (squares), 100 (circles), 150 (traingles), 195 (inverse triangles), and the lines are the theoretical results.

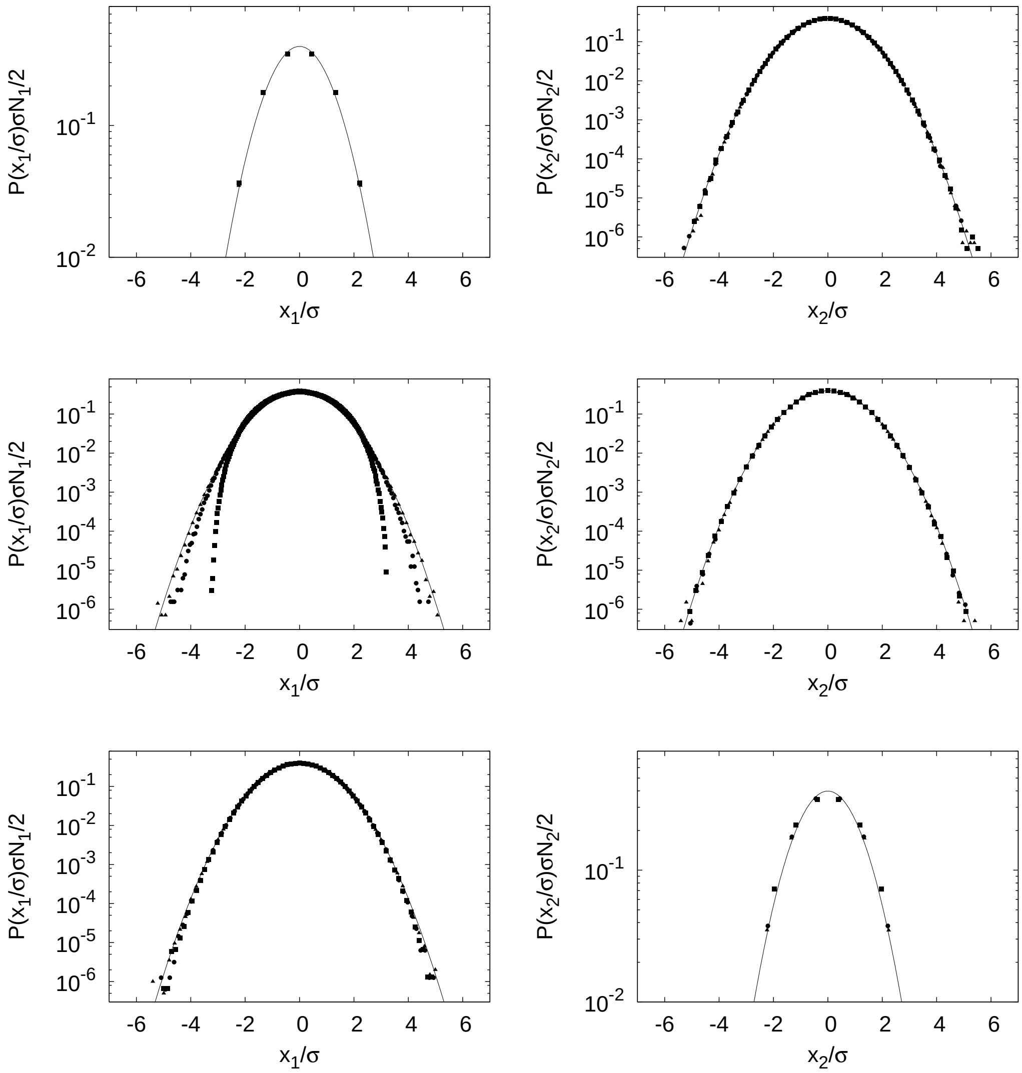


Figure 6: Numerical simulations for a system of $N = 200$ agents, with $N_1$noncontrarians and $N - N_1$ contrarians. The symbols are the simularion results for $N_1 = 5$ (aterisks), 50 (squares), 100 (circles), 150 (traingles), 195 (inverse triangles), and the lines are the theoretical results.

# 5 Conclusions and future work

Among the many different models used to study social systems, using an agent-based approach turns out to be very powerful, since agent-based models provide a natural way of incorporating and testing the mechanism acting at the level of one or few individuals that lead to global and emergent behaviours. A recent, although prominent example of the latter models is the noisy voter model, a variation of the voter model that allows us to study the dynamics of opinions by using two fundamental mechanisms: copy and intrinsic change of opinions. Here, we have proposed a variant version of the noisy voter model, where also contrarian agents are included in the system, that is by copying opposite opinions. In particular, we have studied the possible social collective emergent phenomena in two situations:

1) The case where only contrarians are present (also compared with the classical noisy voter model).

2) The case where both contrarians and non-contrarians are included.

In both cases we have first described the model analytically, then we have compared the theoretical results obtained with the numerical ones.

The case 1) proves to have important differences with the model of only noncontrarians. For any values of the noise to herding parameters, the system always keep on the unimodal state, contrary to what happens with the noisy voter model. While for both cases the system reaches a steady state with a zero magnetization $n$, the second moment of the distribution for the all-contrarians case turns out to be an increasing function of $a/h$. Moreover, since for $a \gg h$ both models coincide due to the prevalence of the noise contribution to the transitions, the mechanism of copying the opposite opinions, or antiherding mechanism, is more effecient upon driving the system towards a situation of co-existence of opinions. The latter has been confirmed theoretically and by means of numerical simulation, where a perfect agreement has been found. The probability distribution for the global magnetization is very accurately approximated by a Gaussian distribution for all the values of the parameters considered.

For case 2), both the theory and the numerical simulations show that the system always reaches the unimodal phase, for any initial condition and regardless of the value of the parameters of the system (provided the number of contrarians is nonzero). This means that the inclusion of contrarians in a system of noisy voter tends to strengthen the effect of the noise in preventing the consensus from being reached, or more precisely on preventing the system from being in the bimodal phase. Moreover, both communities of voters are in the unimodal phase. The theoretical calculation for case 2) is much more involved if compared to the one-community cases, but the situation is much richer, which is reflected, for instance, in the nonmonotonic behaviour of the second moment of the magnetization of the contrarians as a function of the noisy to herding ratio and as a function of the fraction of noncontrarians. The latter prevents us from having a full and intuitive picture on the mutual influence of different agents, in general.

Therefore, we can state with certainty that the major result of the present work is that, in a noisy voter model with contrarian agents, the system is always in the unimodal phase: the inclusion of the contrarians tend to imply the lack of consensus in the system, reinforcing the noise effect in the model. Thus the system appears divided between optimistic and pessimistic voters at the steady-state.

The work can be extended along different and complementary directions. Regarding the completeness of the present work, an explanation of the goodness of the Gaussian distribution as a solution of the master equation for most values of the parameters is lacking. One possibility along this line is to try to reduce the master equation to the Fokker-Planck one. As regards the generalization of the model, an interesting aspect to consider is a weaker influence of the contrarians, by linking them to only a part of the population of noncontrarians. This way we expect the system to reach the bimodal phase.

# 6 Appendix

In this short appendix we describe the numerical algorithm methodology we have used for the simulation part: the first reaction method, as used for the kind of system we had to simulate. Due to the difficulties for the analytical treatment of master equations, it is common to numerically simulate the underlying stochastic process. Here we introduce the basic principles of such a procedure.

Let us consider the simple example of a two-state system with constant rates, with $X$ and $Y$ as the possible states. Then, the rate for the jumps from $X$ to $Y$ is $\omega_{1\to 2}$, while the rate for the jumps from $Y$ to $X$ is $\omega_{2\to 1}$: the two rates do not have any relation between them.

The stochastic process is composed of a series of jumps from one of the two states to the other. Our numerical simulation is aimed at generating of these trajectories: each one jumps from $X$ to $Y$ at a specific time.

Let us assume that, at the initial time $t_0$, we are in the state $X$. The first jump of the particle to $Y$ will be at time $t_1$, which can be proven that follows the distribution

$$f^{1st}\left(2, t_1 | 1, t_0\right) = \omega_{1\to 2} \exp\left[-\omega_{1\to 2}\left(t_1 - t_0\right)\right].$$

Thus, $t_1 - t_0$ follows an exponential distribution. In order to generate values of a random variable distributed according to an exponential distribution, we can generate a $\hat{\mathbf{U}}\left(0,1\right)$ random number $u_0$ uniformly distributed in the interval $(0,1)$ and use $t_1 - t_0 = -\frac{\ln u_0}{\omega_{1\to 2}}$ , or

$$t_1 = t_0 - \frac{\ln u_0}{\omega_{1\to 2}}.$$

At this point the particle is in state $Y$. Then, the next step requires that we determine the time $t_2$ of the next jump to state $X$. The time $t_2$ of the first jump follows the distribution

$$f^{1st}\left(1, t_2 | 2, t_1\right) = \omega_{2\to 1} \exp\left[-\omega_{2\to 1}\left(t_2 - t_1\right)\right].$$

Thus, we can generate $t_2$ by drawing a $\hat{\mathbf{U}}\left(0,1\right)$ random number $u_1$ from a uniform distribution in $(0,1)$. In this way we get

$$t_2 = t_1 - \frac{\ln u_1}{\omega_{2\to 1}}.$$

At this point the particle is again in state $X$ and we can use the transition rate $\omega_{1\to 2}$ to find the time $t_3$ of the next jump, so as to set

$$t_3 = t_2 - \frac{\ln u_0}{\omega_{1\to 2}}.$$

Therefore, the simulation of a stochastic trajectory according to the rules of the alternating rates of the process can be performed.

In our case, we implemented the method in the codes, both for the only contrarians model and for the two-community one: the system starts from and initial configuration, where all the parameters are set (in particular $a$ and $h$, while the moments are set to 0) and the next time for an agent to change its state is computed: at each step the rates and the moments of the system are modified according to the present state of the system and the next time step is calculated. The computations are performed for different trajectories generated.

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