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ON AGGREGATION AND TRANSFORMATION OF GENERALIZED METRICS AND ITS APPLICATIONS

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Doctor by the Universitat de les Illes Balears

Dr. Oscar Valero Sierra, Profesor Titular de Universidad del Departamento de Ciencias Matemáticas e Informática de la Universitat de les Illes Balears

Declara:

Que la tesis doctoral titulada *"On Aggregation and Transformation of Generalized Metrics and its Applications"*, presentada por Juan José Miñana Prats para la obtención del título de doctor, ha sido dirigida bajo mi supervisión.

Y para que así conste firmo el presente documento.

Fdo. Oscar Valero Sierra

Palma, julio de 2019

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Abstract: Español

En muchos problemas, que aparecen de forma natural en diferentes campos de la ciencia, la noción de métrica juega un papel esencial. En ellos, dicha noción se utiliza habitualmente para medir la disimilitud entre puntos u objetos. Sin embargo, en algunos casos la definición de métrica es demasiado restrictiva. Esto ha motivado la introducción de diferentes generalizaciones del concepto de métrica, entre los cuales se incluyen las casi-métricas y las métricas parciales. En cambio, algunos problemas involucran en su propia naturaleza cierta incertidumbre. En dichos casos, la teoría Fuzzy es más apropiada para el tratamiento del problema. Este hecho ha promovido algunas adaptaciones al contexto Fuzzy de la noción de métrica y el estudio de estas. Entre otras, podemos encontrar las nociones de operador de indistinguibilidad o el de métrica difusa.

Un tema de interés, relacionado con las generalizaciones de métrica y sus adaptaciones al entorno Fuzzy, es el estudio de aquellas funciones que transforman una familia de métricas generalizadas, o una sola, en una nueva métrica generalizada. En esta tesis abordamos algunos tópicos relacionados con dichas funciones. Además, proporcionamos algunas aplicaciones relacionadas con algunos de los resultados teóricos obtenidos en la tesis. A continuación se detallan las principales aportaciones:

Damos una nueva caracterización de aquellas funciones que transforman cualquier métrica parcial en una nueva métrica parcial. Además, se estudia qué condiciones deben cumplir dichas funciones para preservar algunas propiedades topológicas.

Caracterizamos las funciones que agregan casi-métricas, que son aquellas

funciones que transforman cualquier familia de casi-métricas definidas sobre un mismo conjunto X en una nueva casi-métrica definida sobre X. Por otro lado, se demuestran algunas propiedades de dichas funciones y se presentan, de modo argumentado, dos posibles campos de aplicación de los resultados obtenidos.

Se demuestra una caracterización de aquellas funciones que generan una métrica parcial a partir de una casi-métrica, y viceversa. Además se estudia qué propiedades deben cumplir dichas funciones para preservar el orden parcial y la topología inducida, tanto por una métrica parcial como por una casi-métrica.

Se caracterizan las funciones que agregan una casi-métrica y una métrica parcial con el fin de obtener una nueva métrica generalizada. En dicha caracterización se demuestra que la métrica generalizada obtenida es una casi-métrica parcial. El resultado obtenido permite desarrollar un marco general para el estudio simultáneo de la semántica de lenguajes y el análisis de complejidad algorítmica.

Se proporcionan dos formas diferentes de construir una métrica difusa a partir de una clásica. Una de ellas mediante funciones que preservan métricas y la otra por medio de generadores aditivos. Se demuestran algunas propiedades de las métricas difusas obtenidas. Además, proporcionamos un método para construir una métrica clásica a partir de una métrica difusa.

Establecemos una relación de dualidad entre métricas difusas y métricas modulares, una generalización de métrica que incluye en su definición un parámetro. Dicha relación motiva la introducción de un nuevo concepto que generaliza tanto a los operadores de indistinguibilidad como a las métricas difusas.

Por último, establecemos una base teórica para generar funciones de respuesta en el problema de asignación de tareas en sistemas Multi-robot. Este se basa en el uso de operadores de indistinguibilidad y preordenes difusos.

Abstract: Català

En molts problemes, que apareixen de forma natural en diferents camps de la ciència, la noció de mètrica juga un paper fonamental. En ells, les mètriques s'utilitzen habitualment per a mesurar la dissimilitud entre objectes o punts. Tot i això, en alguns casos el concepte de mètrica es massa restrictiu, fet que ha motivat la introducció de diferents generalitzacions del concepte mètrica, entre els quals s'inclouen les quasi-mètriques o les mètriques parcials. Per altra banda, hi ha alguns problemes que involucren en la seva pròpia naturalesa una certa incertesa. En aquests casos, la teoria Fuzzy és més adequada per al tractament del problema. Tal fet ha propiciat l'aparició d'algunes adaptacions al context Fuzzy de la noció de mètrica i de l'estudi d'aquestes. Entre altres, podem trobar les nocions d'operador d'indistinguibilitat o del de mètrica difusa.

Un tema d'interès, relacionat amb les diverses generalitzacions de les mètriques i de les seves adaptacions al context Fuzzy, consisteix en l'estudi d'aquelles funcions que transformen una família de mètriques generalitzades, o una tota sola, en una nova mètrica generalitzada. En aquesta tesi s'abordaran alguns temes relacionats amb aquest tipus de funcions. Per altra banda, es proporcionen algunes aplicacions relacionades amb alguns dels resultats teòrics obtinguts en la tesi. A continuació es detallen les aportacions principals:

Donem una nova caracterització d'aquelles funcions que transformen qualsevol mètrica parcial en una nova mètrica parcial. A més, s'estudia les condicions que han de complir aquestes funcions per tal de conservar algunes propietats topològiques.

Es caracteritzen les funcions que agreguen quasi-mètriques, és a dir, aquelles funcions que transformen qualsevol família de quasi-mètriques definides sobre un mateix conjunt X en una nova quasi-mètrica definitiva també sobre X. Per un altre costat, es demostren algunes propietats de dites funcions i s'argumenten dos possibles camps d'aplicació dels resultats obtinguts.

Demostrem una caracterització d'aquelles funcions que generen una mètrica parcial a partir d'una quasi-mètrica, i viceversa. A més, estudiem les propietats que han de complir dites funcions per tal de preservar l'ordre parcial i la topologia induïts, tant per una mètrica parcial com per una quasi-mètrica.

Es caracteritzen les funcions que agreguen una quasi-mètrica i mètrica parcial per tal d'obtenir una nova mètrica generalitzada. Aquesta caracterització demostra que la mètrica generalitzada que s'obté coincideix amb una quasi-mètrica parcial. El resultat obtingut permet desenvolupar un marc general per a l'estudi simultani de la semàntica de llenguatges i l'anàlisi de la complexitat algorítmica.

Es proporcionen dues formes diferents de construir una mètrica difusa a partir d'una de clàssica. Una d'elles per mitjà de les funcions que preserven mètriques i l'altra a partir de generadors additius. A més, es demostren algunes propietats de les mètriques difuses obtingudes. Per altra banda, es proporciona un mètode per a construir una mètrica clàssica a partir d'una mètrica difusa.

Establim una relació de dualitat entre mètriques difuses i mètriques modulars, una generalització de mètrica que inclou en la seva definició un paràmetre. Aquesta relació motiva la introducció d'un nou concepte que generalitza tant els operadors d'indistinguibilitat com les mètriques difuses.

Per últim, s'estableix una base teòrica per generar funcions de resposta en el problema d'assignació de tasques en sistemes Multi-robot. Aquesta es basa en l'ús d'operadors d'indistinguibilitat i de preordres difusos.

Abstract

In many problems of different fields of Science, the notion of metric plays an essential role. Such a notion is commonly used to measure the dissimilarity between points or objects. However, the definition of metric is too restrictive in some problems. It has motivated the introduction of different kind of generalizations of the concept of metric, in which quasi-metrics and partial metrics are included. Moreover, some problems involve in its nature some uncertainty. In such cases, the Fuzzy theory is more appropriate for the treatment of the problem. This fact has promoted some adaptations to the fuzzy context of the notion of metric and their study. Among others, we can find the notions of indistinguishability operator or fuzzy metric.

A topic of interest, related to the generalizations of metric and the adaptation of metric to the fuzzy setting, is the study of those functions that transform a family of generalized metrics, or a single one, into a new generalized metric. In this dissertation we tackle some items related with the aforesaid functions. In addition, we provide some applications related with some of the theoretical results obtained. The main contributions of this dissertation are summarized below:

We prove a new characterization of those functions that transform each partial metric into a new one partial metric. In addition, conditions to preserve some topological properties by the aforementioned functions are studied.

We characterize quasi-metric aggregation functions, which are those functions that transform a family of quasi-metrics defined on the same set X into a new quasi-metric defined on X. Moreover, some properties of such func-

tions are proved and two possible fields of application of the obtained results are presented.

A characterization of those functions that generate a partial metric from a quasi-metric, and vice-versa, are proved Besides, the preservation by means of such functions of the partial order and the topology induced by a quasi-metric or a partial metric are studied.

Functions that merge a quasi-metric and a partial one into a new generalized metric are characterized. In such a characterization it is shown that the generalized metric obtained is a partial quasi-metric. The result obtained allows us to develop a general framework to study, at the same time, denotational semantics and complexity analysis of algorithms.

Two different ways to construct a fuzzy metric from a classical one are provided. One of them by means of metric preserving functions and the other one using additive generators. Some properties of the fuzzy metrics obtained are proved. Furthermore, we give a method to construct a classical metric from a fuzzy one.

We establish a duality relationship between fuzzy metrics and modular metrics, a generalization of metric that include in its definition a parameter. Such a relationship motivates the introduction of a new concept that generalizes both indistinguishability operators and fuzzy metrics.

We establish a theoretical foundation to generate response functions for Multi-robot task allocation problem. It is based on the use of indistinguishability operators and fuzzy preorders.

Contents

Chapter 1

Introduction

1.1 Background of the study

The notion of metric or distance, as it is most commonly named in Computer Science, plays an essential role in many problems arising in different fields of Science. For instance, in Pattern Recognition, Image Analysis, Computational Biology, Decision Making,... (see [2, 14, 47, 68]). In such problems, it is required, in a natural way, to measure the dissimilarity between two objects, and then the concept of metric works as the tool that provides this measurement. Nevertheless, in some cases, the axioms that define a metric are too restrictive to approach the problem under consideration, and so such a concept is not appropriate to solve it. To avoid that inconvenience, we can find in the literature different generalizations of the aforementioned notion, as quasi-metric (or asymmetric distance), partial metric (or non-zero distance), pseudo-metric, etc. Moreover, in some instances, the nature of the addressed problem involves some uncertainty to measure the dissimilarity, so in such a case, it is necessary to handle a notion of measurement framed in the Fuzzy setting. We mean by Fuzzy setting the whole theory that begins in 1965 when L.A. Zadeh introduced in [99] the notion of Fuzzy Set. This concept opened a wide range of research, both from the theoretical point of view and from its applicability to real problems. In this framework, we can find the concept of indistinguishability operator introduced by E. Trillas in [94] (see also [100]), which provides a degree of how indistinguishable are two objects. Also, in the fuzzy realm, we can find some adaptations, to this context, of the different generalizations of metric as, for instance, fuzzy metrics ([50]).

In any case, we can consider all the aforementioned concepts of measurement as generalizations of the notion of metric, although we divide them in two differentiate classes: the first one, which we will refer as *dissimilarity* measurement, which includes the notion of metric and all its generalizations provided by relaxing its axiomatic. The second one, which we call *similarity* measurements, which includes the notion of indistinguishability operator and the concept of fuzzy metric, fuzzy quasi-metric, among the other adaptations to the fuzzy context of the generalizations of metric. The essential difference between these two classes is that the members of the first one can take values into $[0, \infty]$, where 0 indicates that two elements are not distinguishable and greater the value, the more distinguishable are the objects. However, indistinguishable measurements provide a value into $[0, 1]$, where 1 means that two elements are totally indistinguishable and so they have the maximum degree of similarity, while 0 means that the elements are totally distinguishable.

All the concepts included in both families have been deeply studied in the literature separately. Although some authors have treated the problem of studying the relationship among some of them. We focus on two topics related to such research line. The first one has its beginning in 1981, when J. Borsík and J. Doboš [7] studied the problem of characterizing the class of functions that preserve metrics, i.e., they characterized the class of non-negative real functions whose composition with each metric provide a metric. Furthermore, the same authors extended such a study in [8] to the problem of merging a family of distances into a single one. Since then, other authors have continued Borsík and Doboš' work extending their results to those cases in which are considered generalized metrics, as quasi-metric or partial metric among others (see, for instance, [58, 55]). The second topic has been treated in the literature for several authors (see for instance [12, 78] and references therein) and it consists in studying the duality relationship between indistinguishability operators and metrics. Such a relationship consists in obtaining an indistinguishability operator from a classical metric, and vice-versa, by means of a real-valued functions. It can be seen as a transformation, by means of a function, of a classical metric into an indistinguishability operator and vice-versa.

In this chapter, we make an overview on the literature about the study of those functions that preserve, aggregate or transform a family (being able a single distance) of generalized metrics into a different one or another of the same type. Besides, we expose two applications in which generalized metrics play an essential role, as to the analysis of algorithms and to the multi-robot task allocation problem. The remainder of the chapter is divided in four sections. In the first one, we attend some generalizations of metric notion that measure the dissimilarity and some results related to the functions that preserve or aggregate them. The second section is devoted to some transformations of such generalizations. Then, in the third section, we tackle some similarity measurements and their transformations. Finally, in the fourth section we expose some details of the application to the analysis of algorithms and to the multi-robot task allocation problem. Throughout the text, we assume that the reader is familiar with the fundamentals of metric spaces ([14]).

1.1.1 Functions that preserve or aggregate dissimilarity measurements

As was mentioned in the above section, the first results concerning the study of metric preserving functions were due to Borsík and J. Doboš, who, in particular, characterized those functions that merge a family of metrics into a single one. We have collected their results adapting them to the notation that nowadays is commonly used in the literature, in order to unify the notation.

First, recall that a *metric space* is a pair (X, d), where X is a non-empty set and d is non-negative real-valued function on $X \times X$ satisfying, for each $x, y, z \in X$, the following axioms:

(M1) $d(x, y) = 0$ if and only if $x = y$; (Separation)

(M2) $d(x, y) = d(y, x)$; (Symmetry)

(M3) $d(x, z) \leq d(x, y) + d(y, z)$. (Triangle inequality)

If axiom **(M1)** is replaced by the following one

(M0) $d(x, x) = 0$ for each $x \in X$, then the pair (X, d) is said to be a pseudo-metric (such a concept will be useful in Section 1.1.3).

It is a well-known fact that, given a metric space (X, d), the function $d_1 : X \times X \to \mathbb{R}_+$, given by $d_1(x, y) = \frac{d(x,y)}{1+d(x,y)}$ for each $x, y \in X$, is a metric on X. So, one can observe that the function $f : \mathbb{R}_+ \to \mathbb{R}_+$, given by $f(t) = \frac{t}{1+t}$ for each $t \in \mathbb{R}_+$, when it is composed with d provides a new metric on X, for each metric space (X, d). $\mathbb{R}_+$ denotes the interval $[0, \infty[$ along this dissertation. However, from now on, $\mathbb{R}_+$ and $[0, \infty[$ will be used throughout the text in order to follow the notation, in each case, of those previous seminal works taken under consideration.

In [7], it was approached the problem of studying those functions f which satisfy the aforementioned property. To this end, the following concept was introduced in such a paper. A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be *metric preserving* (shortly *mp*-function) if for each metric space (X, d) the function $d_f : X \times X \to \mathbb{R}_+$, given by $d_f(x, y) = f(d(x, y))$ for each $x, y \in X$, is a metric on X.

Two relevant subclasses of non-negative real-valued functions, which will be useful in the problem under consideration, are defined below.

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a function. Then, we will say that

(i) f is *amenable* if $f^{-1}(0) = \{0\}$.

(ii) f is *subadditive* if for each $a, b \in \mathbb{R}_+$ we have that $f(a+b) \leq f(a)+f(b)$.

We will denote by $\mathcal{O}$ the class of all amenable functions.

On the one hand, each *mp*-preserving function is amenable and subadditive. However, in [15], the next example was introduced which yields an amenable and subbaditive function which is not metric preserving.

Example 1.1.1. *Define* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *as follows:*

$$f(x) = \begin{cases} \frac{x}{1+x}, & \text{if } x \in \mathbb{Q} \cap \mathbb{R}_+ (\mathbb{Q} \text{ denotes the set of rational numbers}) \\ 1, & \text{elsewhere}. \end{cases}$$

On the other hand, every amenable, subadditive and monotone function preserves metrics. Nevertheless, there exist *mp*-functions that are not monotone as shows the following instance based on Example 8 in [58].

Example 1.1.2. *Consider the function* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *given by:*

$$f(x) = \begin{cases} 0, & \text{if } x = 0; \\ 2, & \text{if } x \in]0, 1[; \\ 1, & \text{if } x \in [1, \infty[. \end{cases}$$

It is clear that $f(1/2) > f(1)$, but $1/2 < 1$.

With the aim of introducing a characerization of *mp*-functions, we need to recall the notion of triangle triplet. Such a notion was introduced by F. Terpe in [93] and it will be crucial for a subsequent discussion. For each $a, b, c \in \mathbb{R}_+$, the triplet (a, b, c) is called a 1-*dimensional triangle triplet* if

$$a \leq b + c; \quad b \leq a + c \quad \text{and} \quad c \leq a + b.$$

A metric provides a way to construct 1-dimensional triangle triplets. Indeed, if we consider a metric space (X, d) and we take $x, y, z \in X$, then the triangle inequality ensures that (a, b, c) is a 1-dimensional triangle triplet, where $a = d(x, z)$, $b = d(x, y)$ and $c = d(y, z)$.

Now, we are able to introduce the announced characterization of the class of mp-functions such as it was given by Doboš in [15].

Theorem 1.1.3. *Let $f \in \mathcal{O}$. Then the following are equivalent:*

1. *f is an mp-function,*
2. *if (a, b, c) is a 1-dimensional triangle triplet, then $(f(a), f(b), f(c))$ is so,*
3. *if (a, b, c) is a 1-dimensional triangle triplet, then $f(a) \leq f(b) + f(c)$,*
4. *for each $x, y \in \mathbb{R}_+$ we have that $\max\{f(z) : |x - y| \leq z \leq x + y\} \leq f(x) + f(y)$.*

Borsík and Doboš also characterized in [7] those mp-functions for which the topology induced by d_f on X coincides with the topology induced by d on X for each metric space (X, d). Such functions were called strongly metric preserving functions (briefly smp-functions) and they are characterized as follows (see [15]).

Theorem 1.1.4. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a mp-function. Then the following assertions are equivalent:*

1. *f is continuous,*
2. *f is continuous at 0,*
3. *for each $\epsilon > 0$ we can find $x > 0$ such that $f(x) < \epsilon$,*
4. *f is a smp-function.*

In [8], Borsík and Doboš continued the above study. Thus they discussed the case in which a family, instead of a single one, of metric spaces is considered. A motivation of such a work could be the following.

Let $\{(X_i, d_i) : i \in \mathbb{N}\}$ ($\mathbb{N}$ stands for the set of positive integer numbers) be a family of metric spaces. If we denote by X the Cartesian product $\prod_{i\in\mathbb{N}} X_i$, then the function $D : X \times X \to \mathbb{R}_+$, given by

$$D(\mathbf{x}, \mathbf{y}) = \sum_{i\in\mathbb{N}} 2^{-i} \cdot \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)},$$

is a metric on X, which is known in the literature as Fréchet metric on a product of countable family of metric spaces.

In the light of the preceding example, it seems natural to explore those functions which merge a collection (non-necessay finite) of metrics into a single one. This problem was addressed by Borsík, and Doboš in [8]. Although all the following results were given when the considered family is non-finite, we will recall them assuming that the handled family is finite, since we are motivated by the possible applications to real problems in which, obviously, we would consider only the finite case.

Next we will denote by $\mathbb{R}_+^n$ the set of all vectors with n non-negative real components. Moreover, from now on, we will denote by $\mathbf{0}$ the element of $\mathbb{R}_+^n$ given by $\mathbf{0}_i = 0$ for all $i \in \{1, \dots, n\}$. As usual we will consider the set $\mathbb{R}_+^n$ ordered by the point-wise order relation $\preceq$, i. e. $\mathbf{x} \preceq \mathbf{y} \Leftrightarrow x_i \leq y_i$ for all $i \in \{1, \dots, n\}$.

Now, we have adapted some subclasses of 1-dimensional real-valued functions to the n-dimensional context.

Let $F : \mathbb{R}_+^n \to \mathbb{R}_+$. We will say that

(i) F is *amenable* if $F^{-1}(0) = \{\mathbf{0}\}$.

(ii) F is *subadditive* if for each $\mathbf{a}, \mathbf{b} \in \mathbb{R}_+^n$ we have that $F(\mathbf{a}+\mathbf{b}) \leq F(\mathbf{a}) + F(\mathbf{b})$.

(iii) F is *monotone* if for each $\mathbf{a}, \mathbf{b} \in \mathbb{R}_+^n$,with $\mathbf{a} \preceq \mathbf{b}$, we have that $F(\mathbf{a}) \leq F(\mathbf{b})$.

We will denote by $\mathcal{O}^n$ the class of all functions $F : \mathbb{R}_+^n \to \mathbb{R}_+$ which are amenable.

Based on the 1-dimensional case, for $n \in \mathbb{N}$, we will say that a function $F : \mathbb{R}_+^n \to \mathbb{R}_+$ is a *n-metric preserving function* (shortly *nmp*-function) if the function D_F is a metric on the set $X = \prod_{i=1}^{n} X_i$, for every family of n metric spaces $\{(X_1, d_1), \ldots, (X_n, d_n)\}$, where the mapping $D_F : X \times X \to \mathbb{R}_+$ is defined by

$$D_F(\mathbf{x}, \mathbf{y}) = F(d_1(x_1, y_1), \ldots, d_n(x_n, y_n)).$$

The following result, proved in [8], provides sufficient conditions on a function F to be a metric

Proposition 1.1.5. *Let $n \in \mathbb{N}$ and let $F : \mathbb{R}_+^n \to \mathbb{R}_+$ be a monotone, subadditive and amenable function. Then F is an nmp-function.*

One can show, adapting Example 1.1.2 to the n-dimensional case, that there exist *nmp*-functions which are not monotone. Furthermore, this n-dimensional case admits similar characterizations of those given for *mp*-functions in the 1-dimensional case. So, before presenting the characterization of the family of *nmp*-functions, we must adapt the notion of 1-dimensional triangle triplet to the n-dimensional case as follows.

For each $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}_+^n$ we will say that $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is an *n-dimensional triangle triplet* if

$$\mathbf{a} \preceq \mathbf{b} + \mathbf{c}; \quad \mathbf{b} \preceq \mathbf{a} + \mathbf{c} \quad \text{and} \quad \mathbf{c} \preceq \mathbf{a} + \mathbf{b}.$$

Now, we are able to introduce the aforementioned characterization of the class of *nmp*-functions.

Theorem 1.1.6. *Let $n \in \mathbb{N}$ and let $F : \mathbb{R}_+^n \to \mathbb{R}_+$ be a function. Then, F is a nmp-function if and only if it fulfills the following properties:*

(1) $F \in \mathcal{O}^n$.

(2) If $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ *is an n-dimensional triangle triplet, then* $(F(\mathbf{a}), F(\mathbf{b}), F(\mathbf{c}))$ *is a 1-dimensional triangle triplet.*

Later on, in 2000, E. Castiñeira, A. Pradera and E. Trillas slightly modified the problem of merging a finite family of metric spaces into single one. Concretely in [75] (see also [74]), they restricted the study to the case in which all metrics of the collection to be merged are defined on the same non-empty set X. It must be stressed that in the aforementioned papers, the study was made for pseudo-metrics, a more general notion than the metric one. Nevertheless, we have adapted their results to the metric case in order to put them in our context.

With the aim of recalling the result by Castiñeira, Pradera and Trillas, we will need the next notion. We will say that $F : \mathbb{R}_+^n \to \mathbb{R}_+$ is a *metric aggregation function* (shortly *ma*-function) provided that, for any non-empty set X and any collection of metrics $\{d_1, \ldots, d_n\}$ on X, the function $D_F : X \times X \to \mathbb{R}_+$ is a metric, where $D_F(x, y) = F(d_1(x, y), \ldots, d_n(x, y))$ for all $x, y \in X$. We have renamed this concept as *ma*-function because of in this case we are "fusing" n incoming data, which come from the same structure, to a single datum, in such a structure. The preceding fact is in tune with the concept of aggregation function widely studied in the literature (see [28]).

According to [75], Proposition 1.1.5 remains valid when we consider *ma*-functions.

Recently, in [59], the study of the class of *ma*-functions has been continued. In this paper, G. Mayor and O. Valero have refined Proposition 1.1.5 and, in addition, they have provided a characterization of such a class of functions. In both results are involved the following concepts.

Let $n \in \mathbb{N}$ and consider function $F : \mathbb{R}_+^n \to \mathbb{R}_+$. We will say that

(i) F is *positive subadditive* if for each $\mathbf{a}, \mathbf{b} \in]0, \infty[^n$ we have that $F(\mathbf{a} + \mathbf{b}) \leq F(\mathbf{a}) + F(\mathbf{b})$.

(ii) F is *positive monotone* if for each $\mathbf{a}, \mathbf{b} \in]0, \infty[^n$, with $\mathbf{a} \preceq \mathbf{b}$, we have that $F(\mathbf{a}) \leq F(\mathbf{b})$.

Attending to the above introduced notions, in [59] the next proposition was proved.

Proposition 1.1.7. *Let $n \in \mathbb{N}$ and let $F : \mathbb{R}_+^n \to \mathbb{R}_+$ be a function which is positive monotone, positive subadditive and, in addition, it satisfies the following conditions:*

(1) $F(\mathbf{0}) = 0$*,*

(2) If $F(\mathbf{a}) = 0$*, then* $\min\{a_1, \ldots a_n\} = 0$*.*

Then F is a ma-function.

Furthermore, in [59] it has been observed that the positive monotony is not a necessary condition to be a ma-function, such as the following example shows.

Example 1.1.8. *Consider the function $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ given by $F(0,0) = 0$ and*

$$F(a,b) = \begin{cases} 2 & \text{if } \mathit{first}(a,b) \in (0,1) \\ 1 & \text{if } \mathit{first}(a,b) \geq 1 \end{cases},$$

where $(a,b) \neq (0,0)$ and first(a,b) denotes the first value of (a,b) different from 0. It is not hard to check that F is a ma-function but it is not positive monotone.

In order to obtain a characterization of the class of ma-functions similar to that given in Theorem 1.1.6, the following concept was introduced in [59].

Let $n \in \mathbb{N}$ and let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in (0, \infty)^n$. The triplet $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ forms an n-dimensional positive triangle triplet if it forms a n-dimensional triangle triplet.

The aforesaid characterization can be stated as follows ([59]).

Theorem 1.1.9. *Let $n \in \mathbb{N}$. Consider a function $F : \mathbb{R}_+^n \to \mathbb{R}_+$. The following assertions are equivalent.*

(1) F is an ma-function.

(2) F holds the following properties:

(2.1) $F(\mathbf{0}) = 0$,

(2.2) *If* $F(\mathbf{a}) = 0$, *then* $\min\{a_1, \ldots a_n\} = 0$,

(2.3) *If* $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ *is a positive* n*-dimensional triangle triplet, then* $(F(\mathbf{a}), F(\mathbf{b}), F(\mathbf{c}))$ *is a* 1*-dimensional positive triangle triplet.*

We will finish this subsection recalling some examples of ma-functions, which have been included in [59].

Example 1.1.10. *Let* $n \in \mathbb{N}$. *The following functions* $F : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+$ *are* ma*-functions where for all* $\mathbf{a} \in [0, \infty)^n$*:*

(1) $F(\mathbf{a}) = \sum_{i=1}^n w_i a_i$, *where* $w_1, \ldots, w_n \in [0, \infty)$ *with* $max\{w_1, \ldots, w_n\} > 0$. *Of course, this kind of functions includes the class of weighted arithmetic means, and thus the arithmetic mean (see [28]).*

(2) $F(\mathbf{a}) = \max\{w_1 a_1, \ldots, w_n a_n\}$, *where* $w_1, \ldots, w_n \in [0, \infty)$ *are such that* $max\{w_1, \ldots, w_n\} > 0$.

(3) $F(\mathbf{a}) = \sum_{i=1}^n w_i a_{(i)}$ *for all* $w_1, \ldots, w_n \in [0, 1]$ *with* $w_i \geq w_j$ *for* $i < j$ *and* $max\{w_1, \ldots, w_n\} > 0$, *where* $a_{(i)}$ *is the ith largest of the* $a_1, \ldots, a_n$. *This kind of functions includes the OWA operators with decreasing weights (see [28, 78]).*

(4) $F(\mathbf{a}) = (\sum_{i=1}^n (w_i a)_i^p)^{\frac{1}{p}}$ *for given* $p \in [1, \infty[$ *and for all* $w_1, \ldots, w_n \in [0, \infty)$ *such that* $max\{w_1, \ldots, w_n\} > 0$. *This kind of functions includes those root-mean-powers such that* $p \geq 1$ *(see [28]).*

(5) $F(\mathbf{a}) = \min\{c, \sum_{i=1}^n w_i a_i\}$ *for all* $w_1, \ldots, w_n \in [0, \infty)$ *such that* $max\{w_1, \ldots, w_n\} > 0$ *and* $c \in (0, \infty)$.

Preservation and aggregation of generalized metrics

In this subsection we will recall the main results about the study of the functions that preserve and aggregate, respectively, a finite family of generalized metrics into a single one. Concretely, we address the case in which the generalized metrics are quasi-metrics and partial metrics.

The notion of quasi-metric space is a well-known generalization of the concept of metric, in which the symmetry axiom is deleted. Thus we will say that a pair (X, q) is a *quasi-metric space* if X is a non-empty set and q, the quasi-metric, is a non-negative real-valued function on $X \times X$ satisfying for each $x, y, z \in X$ the following:

(QM1) $q(x, y) = q(y, x) = 0$ if and only if $x = y$; (Separation)

(QM2) $q(x, z) \leq q(x, y) + q(y, z)$. (Triangle inequality)

Such a concept has been used in different fields of Computer Science (see, for instance, [22, 80, 82, 83, 88]). In particular, quasi-metrics allow, among other applications, to introduce quantitative techniques of fixed point for the asymptotic complexity analysis of algorithms and for the verification of programs. In this context M.P. Schellekens introduced in [88] a methodology, based on a quasi-metric version of the celebrated Banach Contraction Principle, to determine the complexity class of those algorithms whose execution time satisfies a recurrence equation. Moreover, one can easily observe that there exist several real problems in which considering symmetric distances does not make sense. For instance, if we consider two point in a city plan, taking into account that there exist unidirectional roads, the distance between A and B could be different to the distance between B and A.

As a generalization of metric, and taking into account its applicability to real problems, it is so natural to extend the study started by Borsík and Doboš in metrics to the case of quasi-metrics, which was tackled in [53, 58]. Now, we collect the main results obtained in [58].

In a similar way to the metric case, for $n \in \mathbb{N}$, we will say that a function $F : \mathbb{R}_+^n \to \mathbb{R}_+$ is a *n-quasi-metric preserving function* (shortly *nqmp*-function) if the function Q_F is a quasi-metric on the set $X = \prod_{i=1}^{n} X_i$, for every family of n quasi-metric spaces $\{(X_1, q_1), \ldots, (X_n, q_n)\}$, where the function $Q_F : X \times X \to \mathbb{R}_+$ is defined by $Q_F(\mathbf{x}, \mathbf{y}) = F(q_1(x_1, y_1), \ldots, q_n(x_n, y_n))$.

As in the metric case, amenability is a necessary condition on a function $F : \mathbb{R}_+^n \to \mathbb{R}_+$ to be a *nqmp*-function. Furthermore, subadditivity and monotony are also sufficient conditions to be included in such a class, such as it was shown in the following proposition introduced in [58].

Proposition 1.1.11. *Let $F \in \mathcal{O}^n$. If F is monotone and subadditive, then F is a nqmp-function.*

Moreover, in [58], the next characterization of the class of *nqmp*-functions was given , following the main ideas of Theorem 1.1.6, i.e., by means of triangle triples.

Theorem 1.1.12. *Let $F : \mathbb{R}_+^n \to \mathbb{R}_+$ be a function. Then F is a nqmp-function if and only it satisfies the following properties:*

(1) $F \in \mathcal{O}^n$.

(2) Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}_+^n$. *If* $\mathbf{a} \preceq \mathbf{b} + \mathbf{c}$, *then* $F(\mathbf{a}) \leq F(\mathbf{b}) + F(\mathbf{c})$.

In the light of Theorem 1.1.12, in [58] the fact that that every *nqmp*-function is a *nmp*-function was pointed out. Indeed, every function $F : \mathbb{R}_+^n \to \mathbb{R}_+$ satisfying condition (2) in Theorem 1.1.12 transforms n-dimensional triangle triplets into 1-dimensional triangle triplets. In addition, the reciprocal implication is not true, in general, such as the next example shows.

Example 1.1.13. *Consider the function* $F : \mathbb{R}_+^n \to \mathbb{R}_+$ *given by* $F(\mathbf{0}) = 0$ *and*

$$F(\mathbf{x}) = \begin{cases} 2 & \mathit{first}(\mathbf{x}) \in]0, 1[\\ 1 & \mathit{first}(\mathbf{x}) \geq 1 \end{cases},$$

where $\mathbf{x} \neq \mathbf{0}$ *and* $\mathit{first}(\mathbf{x})$ *denotes the value of the first term of vector* $\mathbf{x}$ *different to* $\mathbf{0}$.

In fact, the inclusion of the class of *nqmp*-functions in the class of *nmp*-functions is strict because of, such as the result below proves, subbaditivity and monotony on functions $F \in \mathcal{O}^n$ are also necessary conditions to be *nqmp*-functions.

Theorem 1.1.14. *Let* $F : \mathbb{R}_+^n \to \mathbb{R}_+$ *be a function. Then the below statements are equivalent:*

(1) F *is subadditive, monotone and* $F \in \mathcal{O}^n$.

(2) F *is a nqmp-function.*

In 1994, S.G. Mathews introduced the notion of partial metric space in order to provide a mathematical framework to model several computational processes that arise in a natural way in denotational semantics for programming languages and in parallel processing (see [56] and [57] and, also, see [45] for additional applications to logic programming).

According to Matthews ([56]), a *partial metric* on a (non-empty) set X is a function $p : X \times X \to \mathbb{R}_+$ such that, for all $x, y, z \in X$, the following axioms are hold:

(P1) $p(x,x) = p(x,y) = p(y,y) \Leftrightarrow x = y;$ (Separation)

(P2) $p(x,x) \leq p(x,y);$ (Monotony)

(P3) $p(x,y) = p(y,x);$ (Symmetry)

(P4) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y).$ (Triangle inequality)

Also, we will say that (X,p) is a partial metric space if X is a non-empty set and p is a partial metric on X. Partial metrics are also known in the literature as non-zero distances.

As in the case of quasi-metric spaces, the applicability of partial metric spaces and, in addition, the growing interest in fusion methods based on the use of numerical aggregation functions, in which the numerical values that are merged come in many cases from measurements among several pieces of information, motivates in a natural way the study about the possibility of extending the merging techniques developed in the metric and quasi-metric case to partial metric framework. Such a topic was addressed in [55] and below, we will collect the main results in this research direction.

Recall that, for $n \in \mathbb{N}$, we will say that $F : \mathbb{R}_+^n \to \mathbb{R}_+$ is a *n-partial metric preserving function* (shortly *npmp*-function) if the function P_F is a partial metric on the set $X = \prod_{i=I}^{n} X_i$, for every family of n partial metric spaces $\{(X_1, p_1, w_1), \ldots, (X_n, p_n, w_n)\}$, where the function $P_F : X \times X \to \mathbb{R}_+$ is defined by $P_F(\mathbf{x}, \mathbf{y}) = F(p_1(x_1, y_1), \ldots, p_n(x_n, y_n))$.

In the light of the preceding concept, the following result provides a partial description of *npmp*-functions. Concretely it provides necessary conditions on a function to be a *npmp*-function.

Proposition 1.1.15. *Let $n \in \mathbb{N}$ and let $F : \mathbb{R}_+^n \to \mathbb{R}_+$ be a function. If F is a npmp-function, then the following statements hold:*

(1) F is monotone.

(2) F is subadditive

(3) If there exists $\mathbf{x} \in \mathbb{R}_+^n$ such that $F(\mathbf{x}) = 0$, then $\mathbf{x} = \mathbf{0}$.

Nevertheless, in [55] it was justified that neither the monotony nor subadditivity are sufficient conditions. Indeed, the following example gives evidence of such a fact.

Example 1.1.16. *Let $n \in \mathbb{N}$ and consider the function $F : \mathbb{R}_+^n \to \mathbb{R}_+$ defined by*

$$F(\mathbf{x}) = \begin{cases} 0 & \textit{if } \mathbf{x} = \mathbf{0} \\ 1 & \textit{otherwise.} \end{cases}$$

F is monotone, subadditive and $F \in \mathcal{O}^n$. However, it is not a npmp-function.

Now, we recall the characterization of *npmp*-functions provided in [55].

Theorem 1.1.17. *Let $n \in \mathbb{N}$ and consider $F : \mathbb{R}_+^n \to \mathbb{R}_+$. Then F is a npmp-function if and only if it satisfies the following two properties for all $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}_+^n$:*

(1) $F(\mathbf{a}) + F(\mathbf{b}) \leq F(\mathbf{c}) + F(\mathbf{d})$ whenever $\mathbf{a} + \mathbf{b} \preceq \mathbf{c} + \mathbf{d}$, $\mathbf{b} \preceq \mathbf{c}$ and $\mathbf{b} \preceq \mathbf{d}$.

(2) If $\mathbf{b} \preceq \mathbf{a}$, $\mathbf{c} \preceq \mathbf{a}$ and $F(\mathbf{a}) = F(\mathbf{b}) = F(\mathbf{c})$, then $\mathbf{a} = \mathbf{b} = \mathbf{c}$.

In the remainder of the section, we discuss the relationship between the classes of *nmp*-functions, *nqmp*-functions and *npmp*-functions.

First, observe that the condition of being amenable is not demanded in the characterization of the class of *npmp*-functions, which was a necessary condition in both characterizations of *nmp*-functions and *nqmp*-functions. The following example, introduced in [55], provides and instance of *npmp*-function, which is neither *nqmp*-function nor a *nmp*-function.

Example 1.1.18. *Let $n = 1$. Define the function $F : \mathbb{R}_+ \to \mathbb{R}_+$ by $F(x) = x + 1$ for all $x \in \mathbb{R}_+$.*

Finally, the following proposition shows that if we demand an extra condition on a *npmp*-function, then such a function is a *nqmp*-function and, thus, a *nmp*-function.

Proposition 1.1.19. *Let $n \in \mathbb{N}$ and let $F : \mathbb{R}_+^n \to \mathbb{R}_+$ be a npmp-function such that $F(\mathbf{0}) = 0$. Then F is a nqmp-function.*

1.1.2 Transformation of dissimilarity measurements

The above section was devoted to compile results related to the problem of merging a finite family of some dissimilarity measuments into a single one. As was observed in such section, the seed of such studies was the work due to J. Borsík and Doboš ([7]), in which it was studied those functions whose composition with every metric provides a metric. Somehow, such functions actually could be considered as a transformation of a metric. Motivated by such an idea, one can consider the problem of studying those functions which transform a generalized metric into another one, although the nature of the input generalized metric can be different to the output one.

According to the preceding argument, we focus our attention on the well-known fact (see [14]) that, given a quasi-metric space (X, q), we always can construct a metric on X from q. Indeed, on the one hand, it is clear that the function $q_{\max} : X \times X \to \mathbb{R}_+$ defined by $q_{\max}(x, y) = \max\{q(x, y), q(y, x)\}$ is a metric. On the other hand, one can verify easily that the function $q_+ : X \times X \to \mathbb{R}_+$ given by $q_+(x, y) = q(x, y) + q(y, x)$ is also a metric. On account of their expressions, the metrics $q_{\max}$ and q_+ are obtained by means of an aggregation of the values $q(x, y)$ and $q(y, x)$. In fact, $q_{\max}(x, y) = F_{\max}(q(x, y), q(y, x))$ and $q_+(x, y) = F_+(q(x, y), q(y, x))$, where $F_{\max}(a, b) = \max\{a, b\}$ and $F_+(a, b) = a + b$ for all $a, b \in \mathbb{R}_+$. Furthermore, note that $F_{\max}$ and F_+ are, by Theorem 1.1.12, $2qmp$-functions. However, both $F_{\max}$ and F_+ can be seen as functions that transform a quasi-metric into a metric.

Taking into account the ideas expressed in the above two paragraphs, it seems natural to wonder whether there exist more ways of generating a metric from a quasi-metric, or in other words, it is natural to ask how are those functions that symmetrize a quasi-metric. Such a problem was tackled in [54], and in this subsection we will collect the main results provided in the aforementioned paper.

On account of [54], a function $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ will be called a *metric generating function* (shortly mg-function) if the function $d_\Phi : X \times X \to \mathbb{R}_+$ is a metric on X, for every quasi-metric space (X, q), where the function d_Φ is defined by

$$d_\Phi(x, y) = \Phi(q(x, y), q(y, x))$$

for all $x, y \in X$.

In [54], it was proved a characterization of the class of mg-functions in which the following notion plays an essential role. Let $a, b, c, d, f, g \in \mathbb{R}_+$. We will say that the triplets (a, b, c) and (d, f, g) are mixed provided that the following inequalities hold:

$$a \leq b + c, \quad b \leq a + f, \quad c \leq g + a,$$

$$d \leq f + g, \quad f \leq d + b, \quad g \leq c + d.$$

The announced characterization is presented below.

Theorem 1.1.20. *Let $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a function. Then the below assertions are equivalent:*

(1) Φ is a mg-function.

(2) Φ holds the following properties:

- *(2.1) $\Phi \in \mathcal{O}$.*
- *(2.2) Φ is symmetric, i.e., $\Phi(a, b) = \Phi(b, a)$ for all $(a, b) \in \mathbb{R}_+^2$.*
- *(2.3) $\Phi(a, d) \leq \Phi(b, g) + \Phi(c, f)$ for all $a, b, c, d, f, g \in \mathbb{R}_+$ such that (a, b, c) and (d, f, g) are mixed triplets.*

Moreover, we have observed that both $F_{\max}$ and F_+ are $nqmp$-functions. Inspired by this fact, J. Martín, G. Mayor and O. Valero tackled the study on the relationship between $nqmp$-functions and metric generating one (see [54]).

The next result shows that demanding an extra condition on the class of $2qmp$-functions we can ensure that they are subclass of mg-functions.

Proposition 1.1.21. *Let $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a function. If Φ is a symmetric $2qmp$-function, then Φ is an mg-function.*

Furthermore, in [54], the following examples of $2qmp$-function, which are mg-functions, was provided .

Example 1.1.22. *It is clear that the below functions $\Phi : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ satisfy statements in Theorem 1.1.12 and, in addition, they are symmetric functions. Thus, by Proposition 1.1.21, all of them are mg-functions:*

(1) $\Phi(a,b) = \begin{cases} 0 & \textit{if } a = b = 0 \\ 1 & \textit{otherwise} \end{cases}$.

(2) $\Phi(a,b) = (w\,(a^p + b^p))^{\frac{1}{p}}$ *for all* $w \in \mathbb{R}_+ \setminus \{0\}$, *where* $p \in [1, \infty[$.

(3) $\Phi(a,b) = w \max\{a,b\}$ *for all* $w \in \mathbb{R}_+ \setminus \{0\}$.

(4) $\Phi(a,b) = w\,(a+b)$ *for all* $w \in \mathbb{R}_+ \setminus \{0\}$.

However, there exist mg-functions which are not a $2qmp$-function, such as the next example shows.

Example 1.1.23. *Let $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ be the function defined by*

$$\Phi(a,b) = \begin{cases} 0 & \textit{if } \max\{a,b\} = 0 \\ 2 & \textit{if } \max\{a,b\} \in]0,1[\\ 1 & \textit{if } \max\{a,b\} \geq 1 \end{cases} .$$

We finish this section with next result that describes the relationship between $2qmp$-functions and mg-functions.

Theorem 1.1.24. *Let $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a mg-function. Then the following assertions are equivalent:*

(1) Φ *is a* $2qmp$*-function.*

(2) Φ *is monotone.*

1.1.3 Similarity measurements and their transformations

In this subsection we compile some results appeared in the literature, in which the relationship between similarity and dissimilarity measurements was studied. Such a study focus on describing those functions which transform a similarity measurement into a dissimilarity one, and vice-versa. So, we will collect some notions of measurements framed in the fuzzy context, as indistinguishability operator or fuzzy metric, in which we are interested in. To this end, we begin recalling the notion of triangular norm, which is crucial in the definition of similarity measurements considered in this dissertation.

Triangular norms

This part of the subsection is devoted to collect some concepts and properties related to the notion of triangular norms, which will be essential in this work. Our main reference for triangular norms is [49].

Recall that a *triangular norm* (briefly, t-norm) is a function $T : [0,1]^2 \to [0,1]$ such that, for all $x, y, z \in [0,1]$, the following four axioms are satisfied:

(T1) $T(x, y) = T(y, x)$; (Commutativity)

(T2) $T(x, T(y, z)) = T(T(x, y), z)$; (Associativity)

(T3) $T(x, y) \geq T(x, z)$, whenever $y \geq z$; (Monotony)

(T4) $T(x,1) = x$. (Boundary Condition)

When T is a continuous function (for the usual topologies in $[0,1]^2$ and in $[0,1]$) we will say that T is a continuous t-norm.

Notice that a t-norm can be considered as a binary operation $*$ on $[0,1]$ which satisfies the aforementioned axioms of commutativity, associativity, monotony and the boundary condition. So, for our convenience, we refer to a t-norm as a binary operation.

An interesting subclass of t-norms, in our subsequent study, are the so called Archimedean. A t-norm $*$ is called *Archimedean* if it satisfies the following condition:

For each $x, y \in]0,1[$ there exists $n \in \mathbb{N}$ such that $x^{(n)} < y$, where $x^{(n)} = x * \cdots * x$ n-times.

In those cases in which $*$ is continuous, then Archimedean t-norms are characterized by the property $x * x < x$ for each $x \in]0,1[$. Two well-known examples of continuous Archimedean t-norms are the usual product, i.e., $x *_P y = x \cdot y$, and the Luckasievicz t-norm, given by $x *_L y = \max\{x+y-1, 0\}$. Of course, in general, continuous t-norms are not Archimedean. Indeed, an example of continuous t-norm which is non-Archimedean is the minimum t-norm, i.e., $x *_M y = \min\{x, y\}$. These t-norms are the most commonly used in Fuzzy Logic. Finally, a well-known example of non-continuous Archimedean t-norm is the so-called Drastic product t-norm, which is given by

$$x *_D y = \begin{cases} 0 & \text{if } x, y \in [0,1[\\ \min\{x, y\} & \text{otherwise} \end{cases}.$$

As indicated above, we focus our interest on Archimedean t-norms, which can be represented by means of a real function called *additive generator*. To introduce this concept we recall before the notion of pseudo-inverse of a real function.

Let $[a, b]$ and $[c, d]$ be two closed subintervals of the extended real line $[-\infty, \infty]$, and let $f : [a, b] \to [c, d]$ be a decreasing function. Then the *pseudo-inverse* $f^{(-1)} : [c, d] \to [a, b]$ of f is defined by

$$f^{(-1)}(y) = \sup\{x \in [a, b] : (f(x) - y)(f(b) - f(a)) < 0\}.$$

Notice that we reserve the term (strictly) monotone for those functions being (strictly) increasing.

In the particular case in which f is a strictly decreasing function, we obtain the next simpler formula

$$f^{(-1)}(y) = \sup\{x \in [a, b] : f(x) > y\}.$$

If in addition, f is left-continuous at 1 the above expression is equivalent to

$$f^{(-1)}(y) = f^{-1}(\min\{f(0), y\}) = \max\{0, f^{-1}(y)\}. \quad (1.1)$$

Using the above concept, in [49], the next theorem was proved.

Theorem 1.1.25. *Let* $f : [0, 1] \to [0, \infty]$ *be a strictly decreasing function with* $f(1) = 0$ *such that*

$$f(x) + f(y) \in Ran(f) \cup [f(0^+), \infty]$$

for all $x, y \in [0, 1]$*. The binary operation* $*$ *on [0,1] is a t-norm, where* $*$ *is defined by*

$$x * y = f^{(-1)}(f(x) + f(y)), \ \textit{for each } x, y \in [0, 1].$$

In the preceding theorem, $f(0^+)$ denotes the right-sided limit as $x \to 0$, i.e., $f(0^+) = \lim_{x\to 0^+} f(x)$. This result motivated the aforementioned concept of additive generator.

In the light of Theorem 1.1.25, we will say that $f_* : [0,1] \to [0,\infty]$ is an *additive generator* of a t-norm $*$, if it is a strictly decreasing function which is also right-continuous at 0 and satisfies $f_*(1) = 0$, such that for $x, y \in [0,1]$ we have

$$f_*(x) + f_*(y) \in Ran(f_*) \cup [f_*(0), \infty],$$
$$x * y = f_*^{(-1)}(f_*(x) + f_*(y)).$$

Each t-norm with an additive generator is Archimedean. The converse of this assertion is not true in general. However, the next theorem shows that for continuous t-norms the converse holds.

Theorem 1.1.26. *A binary operator $*$ on $[0,1]$ is a continuous Archimedean t-norm if and only if there exists a continuous additive generator f_* such that*

$$x * y = f_*^{(-1)}(f_*(x) + f_*(y))$$

for each $x, y \in [0,1]$.

A transformation of Indistinguishability operators

In this subsection, we recall the concept of indistinguishability operator and its relationship with classical metrics.

According to [78, 94] an *indistinguishability operator* on a non-empty set X for a t-norm $*$, is a fuzzy set $E : X \times X \to [0,1]$ which satisfies for each $x, y, z \in X$ the following:

(E1) $E(x,x) = 1$; (Reflexivity)

(E2) $E(x,y) = E(y,x)$; (Symmetry)

(E3) $E(x,z) \geq E(x,y) * E(y,z)$. (Transitivity)

If in addition, E satisfies, for all $x, y \in X$ the following condition:

(E1') $E(x,y) = 1$ implies $x = y$,

then E is said to separate points.

If confusion does not arise, we can call both indistinguishability operators.

Several authors have studied the relationship between indistinguishability operators and metrics (see [12, 25, 46, 49, 67, 78, 95]). In this direction a technique to generate metrics from indistinguishability operators, and vice-versa, has been developed by several authors in the literature. Concretely, an indsitinguishability operator can be provided from a (pseudo-)metric as follows:

Theorem 1.1.27. *Let X be a non-empty set and let $*$ be a t-norm with additive generator $f_* : [0,1] \to [0,\infty]$. If $\diamond$ is a t-norm, then the following assertions are equivalent:*

1) $* \leq \diamond$ (i.e., $x * y \leq x \diamond y$ for all $x, y \in [0,1]$).

2) For any indistinguishability operator E on X for $\diamond$, the function $d^{E,f_*} : X \times X \to [0,\infty]$ defined, for each $x, y \in X$, by

$$d^{E,f_*}(x,y) = f_*(E(x,y)),$$

is a pseudo-metric on X.

3) For any indistinguishability operator E on X for $\diamond$ that separates points, the function $d^{E,f_*} : X \times X \to [0, \infty]$ defined, for each $x, y \in X$, by

$$d^{E,f_*}(x, y) = f_*(E(x, y)),$$

is a metric on X.

In the light of the preceding result, we can see an additive generator of a t-norm as a function that transforms each indistinguishability operator, for a fixed t-norm, into a metric. Hence such a result yields a method to transform a "generalized notion of metric" into a metric. Concretely, it transforms a similarity measurement into a metric.

Conversely, the following theorem provides a technique to construct an indistinguishability operator from a (pseudo-)metric.

Theorem 1.1.28. *Let $*$ be a continuous Archimedean t-norm on X with additive generator $f_* : [0, 1] \to [0, \infty]$. If d is a pseudo-metric on X, then the function $E^{d,f_*} : X \times X \to [0, 1]$ defined, for all $x, y \in X$, by*

$$E^{d,f_*}(x, y) = f_*^{(-1)}(d(x, y)),$$

is an indistinguishability operator for $*$, where $f_*^{(-1)}$ denotes the pseudo-inverse of the additive generator f_*. Moreover, the indistinguishability operator E^{d,f_*} separates points if and only if d is a metric on X.

As before, in this last case, we can affirm that the pseudo-inverse of an additive generator of a continuous Archimedean t-norm is a function that transforms each (pseudo-)metric into an indistinguishable operator.

Fuzzy metrics and their transformations

This section is devoted to collect some results related to a fuzzy version of the notion of (pseudo-)metric, and its relationship between generalized metrics and transformation techniques. Fuzzy metrics were introduced by I. Kramosil and J. Michalek in [50], although nowadays is commonly used as the following reformulation presented by M. Grabiec in [27].

Recall that a KM*-fuzzy metric space* is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times [0, \infty[$ satisfying the following conditions, for all $x, y, z \in X$ and $s, t > 0$:

(KM1) $M(x, y, 0) = 0$;

(KM2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;

(KM3) $M(x, y, t) = M(y, x, t)$;

(KM4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(KM5) $M(x, y, \_) : [0, \infty[\rightarrow [0, 1]$ is left-continuous.

Such a concept was slightly modified later on by A. George and P. Veeramani in [23] as follows, a GV-fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times]0, \infty[$ satisfying $(KM3)$, $(KM4)$ and the following conditions, for all $x, y, z \in X$ and $s, t > 0$:

(GV1) $M(x, y, t) > 0$;

(GV2) $M(x, y, t) = 1$ if and only if $x = y$;

(GV5) $M(x, y, \_) :]0, \infty[\to]0, 1]$ is continuous.

Of course when we work with GV-fuzzy metrics we will refer to the axioms $(KM3)$ and $(KM4)$ as $(GV3)$ and $(GV4)$, respectively.

An immediate consequence of both definitions of fuzzy metric spaces is the next one: if $(X, M, *)$ is a fuzzy metric space and $\diamond$ is a continuous t-norm satisfying $* \geq \diamond$, then $(X, M, \diamond)$ is also a fuzzy metric space.

Taking into account the well-known fact that every GV-fuzzy metric space $(X, M, *)$ can be considered as a KM-fuzzy metric space, simply putting $M(x, y, 0) = 0$ for all $x, y \in X$, GV-fuzzy metric spaces could be considered as subclass of KM-fuzzy metric spaces.

Similar to the classical case, when the axiom $(KM2)$ in the definition of fuzzy metric is replaced by the following weaker one:

(KM2') $M(x, x, t) = 1$ for all $t > 0$.

the triple $(X, M, *)$ is called a *KM-fuzzy pseudo-metric space*.

Replacing axiom $(GV2)$ in the definition of GV-fuzzy metric space by $(KM2')$ we also get the concept of GV-fuzzy pseudo-metric space.

The next notions remain valid for both notions of fuzzy pseudo-metric. So we state them using only the term fuzzy pseudo-metric.

If $(X, M, *)$ is a fuzzy (pseudo-)metric space, we will say that $(M, *)$ is a *fuzzy (pseudo-)metric* on X. Also, if confusion is not possible, we will say that (X, M) is a fuzzy (pseudo-)metric space or M is a fuzzy (pseudo-)metric on X.

Different studies have contributed to the development of a theory of fuzzy metric spaces. Indeed, several works have been devoted to the topological study of fuzzy metrics (see, for instance, [23, 27, 29, 30, 31, 32, 33, 35, 36, 37, 41, 79, 87]). Concretely, it has been proved that each fuzzy metric space $(X, M, *)$ generates a topology τ_M which has as a base the family of open balls given by $\mathcal{B} = \{B_M(x, r, t) : x \in X, r \in]0, 1[, t > 0\}$, where $B_M(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$ for each $x \in X$, $r \in]0, 1[$ and $t > 0$. Moreover, fuzzy metrics have shown to be a significant tool in modeling engineering problems as image filtering (see, for instance [9, 34, 60, 61, 62, 63, 64]).

Now, we recall three interesting classes of fuzzy metrics. They can be defined in both sense of fuzzy metrics presented above.

A (pseudo-)fuzzy metric space $(X, M, *)$ is said to be *principal* (or simply, M is principal) if the family $\{B_M(x, r, t) : r \in]0, 1[\}$ is a local base at $x \in X$, for each $x \in X$ and each $t > 0$.

A (pseudo-)fuzzy metric space $(X, M, *)$ is said to be *strong* (or simply, M is strong) if (in addition) it satisfies the following inequality

(GV4') $M(x, z, t) \geq M(x, y, t) * M(y, z, t)$,

for each $x, y, z \in X$ and each $t > 0$.

A fuzzy (pseudo-)metric space $(X, M, *)$ is said to be *stationary* if M does not depend on $t > 0$, i.e., if the function $M_{x,y} :]0, \infty[\to]0, \infty[$ given by $M_{x,y}(t) = M(x, y, t)$ is constant for each $x, y \in X$. In this case we can write $M(x, y)$ instead of $M(x, y, t)$.

Note that if E is an indistinguishability operator on a non-empty set X, for a continuous t-norm $*$, then the fuzzy set M on $X \times X \times [0, \infty[$ given, for each $x, y \in X$, by

$$M(x,y,t) = \begin{cases} 0, & \text{if } t = 0 \\ E(x,y), & \text{elsewhere,} \end{cases}$$

is a stationary fuzzy pseudo-metric on X for the t-norm $*$. If in addition, E separates points, then M is a fuzzy metric.

The following is a well-known example of GV-fuzzy metric space introduced by George and Veeramani in [23], which we have reformulated as a KM-fuzzy metric space.

Example 1.1.29. *Let (X, d) be a metric space and let M_d be a fuzzy set on $X \times X \times [0, \infty[$ defined by*

$$M_d(x,y,t) = \begin{cases} 0, & \text{if } t = 0, \\ \frac{t}{t+d(x,y)}, & \text{elsewhere,} \end{cases}$$

for each $x, y \in X$.

Following [23], $(X, M_d, \wedge)$ is a fuzzy metric space, where $\wedge$ denotes the minimum t-norm $*_M$. The fuzzy metric M_d is called the standard fuzzy metric induced by d.

In the light of the above example, one can observe that such a fuzzy metric M_d can be seen as a fuzzification of a metric. So, we can see M_d as a transformation of a metric through a function $\Psi_{st} : \mathbb{R}_+ \times \mathbb{R}_+ \to [0, 1]$ defined as follows:

$$\Psi_{st}(a,t) = \begin{cases} 0, & \text{if } t = 0, \\ \frac{t}{t+a}, & \text{elsewhere.} \end{cases}$$

By Example 1.1.29, we have that Ψ_{st} is a function that transforms each metric into a fuzzy metric for each continuous t-norm.

We can find another functions with the same property as, for instance, the following one:

Let $\Psi_e : \mathbb{R}_+ \times \mathbb{R}_+ \to [0,1]$ be the function given by

$$\Psi_e(a,t) = \begin{cases} 0, & \text{if } t = 0, \\ \mathsf{e}^{-\frac{a}{t}}, & \text{elsewhere.} \end{cases}$$

Taking into account that, given a metric space (X,d), the fuzzy set M_e defined on $X \times X \times [0,\infty[$ by

$$M(x,y,t) = \begin{cases} 0, & t = 0; \\ \mathsf{e}^{-\frac{d(x,y)}{t}}, & t > 0; \end{cases}$$

for each $x, y \in X$, is a fuzzy metric on X. Then, Ψ_e is a function that transforms each metric into a fuzzy metric.

The converse problem, i.e., the study of transformations on a fuzzy metric in order to obtain a classical one, has been addressed for different authors too. Some partial results to the aforementioned problem were obtained by V. Radu in [76, 77] and T.L. Hicks in [43]. Recently, in [10] F. Castro-Company et al. generalized the results of Radu and Hicks in the following theorem.

Theorem 1.1.30. *Let $(X, M, *)$ be a KM-fuzzy metric space. Suppose that there exists a function $\alpha : [0,\infty[\to [0,\infty[$ satisfying the following conditions:*

(c1) α is strictly monotone on $[0,1]$;

(c2) $0 < \alpha(t) \le t$ for all $t \in]0,1[$ and $\alpha(t) > 1$ for all $t > 1$;

(c3) $(1-\alpha(t)) * (1-\alpha(s)) \geq 1-\alpha(t+s)$ *for all* $t, s \in [0,1]$.

Then the function $d_\alpha : X \times X \to [0, \infty[$ *defined as*

$$d_\alpha(x,y) = \sup\{t \geq 0 : M(x,y,t) \leq 1-\alpha(t)\},$$

is a metric on X *such that* $d_\alpha(x,y) \leq 1$ *for all* $x, y \in X$.

If, in addition, the function α *is left-continuous on* $]0,1[$*, then*

$$d_\alpha(x,y) < \epsilon \Leftrightarrow M(x,y,t) > 1-\epsilon,$$

for all $\epsilon \in]0,1[$*. Thus the topologies, induced by* $(M,*)$ *and* d_α *coincide on* X*. Moreover,* $(X,M,*)$ *is complete if and only if* (X,d_α) *is complete.*

In the light of the preceding result, let us recall that, according to [23], a sequence $(x_n)_{n\in\mathbb{N}}$ in $(X,M,*)$, where $(X,M,*)$ is a fuzzy pseudo-metric space in any sense, is said to be a Cauchy sequence if for each $\epsilon \in (0,1)$ and each $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1-\epsilon$ for all $n, m \geq n_0$. Moreover, $(X,M,*)$ is called complete if every Cauchy sequence is convergent with respect to τ_M. The preceding two notions remain valid for both type of fuzzy metrics introduced in this text.

1.1.4 Applications

In this dissertation we propose three topics of application of some generalized metrics presented in the previous section. Such topics are denotational semantics of programming languages, computational asymptotic complexity analysis and multi-robot task allocation. This subsection is devoted to detail the aforementioned problems.

Denotational semantics and asymptotic complexity

In Computer Science there are two fields in which generalized metrics have been shown to be useful. Concretely, partial metrics have been applied successfully to denotational semantics and quasi-metrics have been used in asymptotic complexity analysis of algorithms. Let us recall briefly the role of such dissimilarities in the aforementioned fields.

In denotational semantics, one of the aims consists in analysing the correctness of recursive algorithms by means of mathematical models of the programming languages in which the algorithm has been written. Moreover, in many programming languages one can construct recursive algorithms through procedures in such a way that the meaning of such a procedure is expressed in terms of its own meaning. An easy, but illustrative, example is the procedure which computes the factorial function. Indeed, a procedure which computes the factorial of a positive integer number typically uses the following recursive denotational specification:

$$fact(n) = \begin{cases} 1 & \text{if } n = 1 \\ nfact(n-1) & \text{if } n > 1 \end{cases}. \tag{1.2}$$

In order to analyze whether a recursive denotational specification of a procedure is meaningful, it is usual to make use of fixed point mathematical techniques in which the meaning of such recursive denotational specification is obtained as the fixed point of a non-recursive mapping associated to the denotational specification. In the particular case of the factorial function the aforesaid non-recursive mapping ϕ_{fact} will be given as follows:

$$\phi_{fact}(f)(n) = \begin{cases} 1 & \text{if } n = 1 \\ nf(n-1) & \text{if } n > 1 \text{ and } n-1 \in \mathrm{dom} f \end{cases}, \tag{1.3}$$

where ϕ_{fact} is acting over the set of partial functions. Of course, the entire factorial function is given by the unique fixed point of the non-recursive mapping ϕ_{fact}. For a detailed treatment of the set of partial functions and its applications to denotational semantics we refer the reader to [13] and [90].

With the aim of developing quantitative fixed point techniques which will be able to analyze the meaning of recursive denotational specification, S.G. Matthews introduced the notion of partial metric in [56].

According to [56], let us recall that the Baire partial metric space consists in the pair (Σ_∞, p_B), where Σ_∞ is the set of finite and infinite sequences over a non-empty alphabet Σ and the partial metric p_B is given by $p_B(x,y) = 2^{-l(x,y)}$ for all $x, y \in \Sigma_\infty$ with $l(x,y)$ denoting the longest common prefix of the words x and y when it exists and $l(v,w) = 0$ otherwise. Of course the convention that $2^{-\infty} = 0$ is adopted. The success of the Baire partial metric and the Matthews fixed point method in denotational semantics is given by the fact that the natural order between words, the prefix order, is encoded by p_B in the sense that x is a prefix of y if and only if $p_B(x,y) = p_B(x,x)$. Notice that every partial function f can be identified with a word $w^f \in \mathbb{N}_\infty$ such that $w^f = w_1^f w_2^f \ldots w_k^f$ with $\mathrm{dom} f = \{1, \ldots, k\}$ and $w_i^f = f(i)$ for all $i \in \mathrm{dom} f$.

The existence and uniqueness of fixed point of ϕ_{fact} is proved by the so-called Matthews fixed point theorem, where a mapping from a partial metric space (X, p) into itself is said to be a contraction if there exists $c \in [0, 1[$ such that $p(f(x), f(y)) \leq cp(x,y)$ for all $x, y \in X$ and c is said to be the contractive constant of the contraction f, which can be stated as follows:

Theorem 1.1.31. *Let (X, p) be a complete partial metric space and let $f : X \to X$. If f is a contraction from (X, p) into itself, then f has a unique fixed point x_0. Moreover, $p(x_0, x_0) = 0$.*

Let us recall that, according to [56], a partial metric space (X, p) is complete if the associated quasi-metric space (X, q_p) is bicomplete, where $q_p(x, y) = p(x, y) - p(x, x)$ for all $x, y \in X$. We will return to this notion in Section 2.3, where an equivalent definition of completeness will be taken into account.

In the light of the previous result, it can be verified that ϕ_{fact} is a contraction from the complete partial metric space $(\mathbb{N}_\infty, p_B)$ into itself with $\frac{1}{2}$ as contractive constant. The fact that the partial metric space $(\mathbb{N}_\infty, p_B)$ is complete was proved in [66].

Often the running time of computing of the recursive algorithm that performs the computation of the meaning of a recursive denotational specification is discussed in conjunction with the correctness of such a recursive denotational specification. In this direction, M.P. Schellekens introduced the so-called complexity space which allows us to develop quantitative fixed point techniques in order to determine the complexity of recursive algorithms whose running time of computing fulfills a recurrence equation (see [88]).

Going back to the example of the factorial function, it is clear that the running time of computing of an algorithm that computes the factorial of a non-negative integer number, through the recursive denotational specification (1.2), is solution to the following recurrence equation

$$T_{fact}(n) = \begin{cases} c & \text{if } n = 1 \\ T_{fact}(n-1) + c & \text{if } n > 1 \end{cases}, \tag{1.4}$$

where $c \in \mathbb{R}_+$ ($c > 0$) is the time taken by the algorithm to obtain the solution to the problem on the base case.

In contrast to the Matthews approach, Schellekens framework is based on the use of quasi-metrics. Concretely, the complexity space is the pair $(\mathcal{C}, q_{\mathcal{C}})$, where $\mathcal{C} = \{f : \mathbb{N} \to (0, \infty] : \sum_{n=0}^{\infty} 2^{-n} \frac{1}{f(n)} < \infty\}$ and $q_{\mathcal{C}}$ is the quasi-metric on $\mathcal{C}$ given as follows

$$q_{\mathcal{C}}(f, g) = \sum_{n=0}^{\infty} 2^{-n} \max\left(\frac{1}{g(n)} - \frac{1}{f(n)}, 0\right).$$

Obviously we adopt the convention that $\frac{1}{\infty} = 0$.

The success of the Schellekens fixed point method in complexity analysis of algorithms is provided by the fact that the running time of computing of an algorithm can be associated to a function belonging to $\mathcal{C}$. Moreover, given two functions $f, g \in \mathcal{C}$, the numerical value $q_{\mathcal{C}}(f, g)$ (the complexity distance from f to g) can be interpreted as the relative progress made in lowering the complexity by replacing any program P with complexity function f by any program Q with complexity function g. In fact, if $f \neq g$, the condition $q_{\mathcal{C}}(f, g) = 0$ can be understood as f is "at least as efficient" as g on all inputs. Observe that $q_{\mathcal{C}}(f, g) = 0$ implies that $f(n) \leq g(n)$ for all $n \in \mathbb{N}$, and this is key to state an asymptotic bound of the complexity of an algorithm. Furthermore, notice that the asymmetry of the complexity distance $q_{\mathcal{C}}$ is crucial in order to provide information about the increase of complexity whenever a program is replaced by another one. A metric (symmetric) will be able to yield information on the increase but it, however, will not reveal which program is more efficient.

The Schellekens' approach is based on next fixed point theorem. In order to state it, let us recall the next concepts. Following [88], a mapping from a (quasi-)metric space (X, q) into itself is said to be a contraction if there exists $c \in [0,1[$ such that $q(f(x), f(y)) \leq cq(x,y)$ for all $x, y \in X$. As in the partial metric case, the preceding constant c is said to be the contractive constant of the contraction f. Besides, a quasi-metric space (X, d) is said to be bicomplete if the associated metric space (X, d_q) is complete (see [14]), where the metric d_q on X is defined by $d_q(x,y) = \max\{q(x,y), q(y,x)\}$ for all $x, y \in X$.

Theorem 1.1.32. *Let (X, q) be a bicomplete quasi-metric space and let $f : X \to X$. If f is a contraction from (X, q) into itself, then f has a unique fixed point x_0.*

In view of the preceding result, Schellekens' approach provides an upper bound of the solution to the recurrence equation (1.4) making the following statement. Consider the subset $\mathcal{C}_c$ of $\mathcal{C}$ given by $\mathcal{C}_c = \{f \in \mathcal{C} : f(1) = c\}$. Then it is not hard to check that the quasi-metric space $(\mathcal{C}_c, q_{\mathcal{C}}|_{\mathcal{C}_c})$ is bicomplete (see, for instance, [85]). Define the mapping $G_{fact} : \mathcal{C}_c \to \mathcal{C}_c$ by

$$G_{fact}(f)(n) = \begin{cases} c & \text{if } n = 1 \\ f(n-1) + c & \text{if } n \geq 2 \end{cases} \tag{1.5}$$

for all $f \in \mathcal{C}_c$. Clearly $f \in \mathcal{C}_c$ is a solution to the recurrence equation (1.4) if and only if f is a fixed point of the mapping G_{fact}. It can be verified that G_{fact} is a contraction from $(\mathcal{C}_c, q_{\mathcal{C}}|_{\mathcal{C}_c})$ into itself with contractive constant $\frac{1}{2}$. Moreover, if $f_{G_{fact}}$ is the unique fixed point of G_{fact}, then $f_{G_{fact}} \leq g$ provided that there exists $g \in \mathcal{C}_c$ with $q_{\mathcal{C}}|_{\mathcal{C}_c}(G_{fact}(g), g) = 0$ and, this allows us to yield g as an asymptotic upper bound of the running time of computing of the algorithm computing the factorial.

In the light of the exposed facts we have that quantitative fixed point techniques are used in Computer Science in order to discuss the complexity analysis of algorithms and the meaning of recursive denotational specifications for programming languages. Both techniques are independent and they are used separately without any relationship between them. Inspired by the preceding fact it seems natural to consider $(\Sigma_\infty \times \mathcal{C}, \Phi(p_B, q_\mathcal{C}))$, where Φ is any kind of function providing a dissimilarity by means of the aggregation of p_B and $q_\mathcal{C}$, as a first attempt to develop a framework to analyze simultaneously, by means of fixed point methods, the running time of computing of an algorithm that performs a computation using a recursive denotational specification and the meaning of such a specification.

However, the previous proposal presents a handicap. Indeed, it is not clear what kind of generalized metric is the function $\Phi(p_B, q_\mathcal{C})$. It can be easily checked that it is neither a partial metric nor a quasi-metric on $\Sigma_\infty \times \mathcal{C}$.

In Chapter 5, the problem of how to merge a partial metric and a quasi-metric will be studied in depth and the obtained results will be used to develop a framework which remains valid for modeling, at the same time, in denotational semantics and in complexity analysis of algorithms.

Multi-robot task allocation problem

The distribution of a determined number of tasks among a group of agents is a problem intensely studied in different fields, as Economics or Robotics (we refer the reader, for instance, to [24, 97, 98] for a deeper treatment of the topic). It consists in allocating a collection of labours on an amount of agents in the most efficient way, i.e., in such a way that the best agent is selected to perform each one of the labour to be carried out. This problem, commonly referred as task allocation problem, is still an open issue in real environments where the agents have a limited number of resources to obtain the optimal allocation. One of those challenging environments is the formed by two or more autonomous robots that perform cooperatively a common mission, from now on referenced as multi-robot systems.

Among all the methods proposed to address the task allocation problem, this dissertation focuses on swarm methods, which are inspired by insect colonies where an intelligent behaviour emerges from the interaction of very simple skills running on each agent. Concretely, this work focuses on the so-called Response Threshold Method (RTM for short). In these methods each involved agent has associated a task response threshold and a task stimulus. The task stimulus value indicates how much attractive is the task for the agent and its threshold is a parameter of the system. Thus, an agent starts the execution of a task following a probability function, referenced as response function and denoted as P, that depends on both aforementioned values. As the probability of executing a task only depends on the current task, or state, the decision process can be modelled as a probabilistic Markov chain. This classical probabilistic approach presents some well known disadvantages (see [38]), for instance problems with the selection of the probability function (response function) when more than two tasks are considered, asymptotic converge to a system's stable state, and so on. Due to the inconveniences, in [38] it was proposed a new possibilistic theoretical formalism for a RTM. The RTM is implemented considering transitions possibilities (response functions) instead of transitions probabilities (response functions) and possibilistic Markov chains (also known as fuzzy Markov chains) instead the classical probabilistic ones. The theoretical and empirical results demonstrated, among other advantages, that fuzzy Markov chains applied to task allocation problems require a very few number of steps to converge to a stable state.

1.2 Objectives

Inspired by, on the one hand, the intense research activity on generalized metrics and their application (as the references included in this dissertation shown) and, on the other hand, by the usefulness of generalized metrics in several fields of Artificial Intelligence, Engineering and Computer Science, we have focused on continuing the study on preservation, aggregation and transformation of the different metric generalizations detailed in the previous section. Besides, we continue with the development of the applicability of the generalized metrics to three different fields: denotational semantics of programming languages, asymptotic complexity analysis of algorithms and multi-robot task allocation. Below we detail the main objectives tackled in this dissertation.

On the study of functions that preserve generalized metrics, we have focused on the characterization of *npmp*-functions (see Subsection 1.1.1). In this direction, Chapter 2 has been devoted to refine such a characterization for the 1-dimensional case. Besides, the preservation of some topological properties has been treated as completeness and contractiveness. Moreover, the relationship between metric preserving and partial metric preserving functions has been also discussed.

Regarding to aggregation of generalized metrics, we have tackled the problem of characterizing quasi-metric aggregation functions in Chapter 3. Moreover, a few properties of such functions have been discussed and, in addition, a few methods to discard those functions that are useless as quasi-metric aggregation functions have been provided. Furthermore, two possible fields of applicability of the developed theory have been presented.

Concerning transformation of generalized metrics, we have approached different topics. Concretely, we have focused on: obtaining, by means of a transformation function, a quasi-metric from a partial one, and vice-versa; merging, using transformation functions, a quasi-metric and a partial one; generating fuzzy metrics from classical ones, and vice-versa; and, finally, on generating a fuzzy metric from a generalized one that includes in its definition a parameter, the so-called modular metrics. Such topics have been treated in five different chapters. Below we list all of them, detailing the study carried out in each one of them.

Chapter 4 has been devoted to characterize those functions that are able to generate a quasi-metric from a partial metric, and conversely, in such a way that Matthews' relationship between both type of generalized metrics is retrieved as a particular case. Besides, it is studied the preservation of the partial order and the topology induced by a partial metric or a quasi-metric, respectively. Furthermore, we discuss the relationship between the new functions and those families introduced in the literature, i.e., n-metric preserving functions, n-quasi-metric preserving functions, n-partial metric preserving functions and metric generating functions.

In Chapter 5 we have attended the problem of merging a quasi-metric and a partial one by means of a transformation function. Concretely, we have characterized such functions and, in addition, we have discussed the relationship between them and partial metric preserving functions and quasi-metric preserving functions, respectively.

Chapter 6 has been devoted to introduce a method to construct fuzzy metrics from classical ones by means of mp-functions. Moreover, some topological properties of the fuzzy metrics constructed has been studied.

The problem of obtaining a fuzzy metric from a classical one has also been treated in Chapter 7. But in this chapter we has tackled the reciprocal problem too. In fact, it has been provided a duality relationship between fuzzy metrics and classical one based on Theorems 1.1.27 and 1.1.28.

Finally, in Chapter 8 we have approached the study of a duality relationship between fuzzy metrics and modular metrics, a dissimilarity measurement that includes in its definition a parameter. Such a study have motivated the introduction of new concept which generalizes both indistinguishability operators and fuzzy metrics.

Regarding applications, on the one hand, in Chapter 5, a framework which remains valid for modeling, at the same time, in denotational semantics and in complexity analysis of algorithms has been developed. On the other hand, Chapters 9 and 10 have been devoted to address the multi-robot task allocation problem. In particular, given a collection of tasks and robots, we focus our effort on how to select the best robot to execute each tasks by means of the so-called response threshold method and using indistinguishability operators and fuzzy preorders (asymmetric indistinguishability operators) to model response functions.

Chapter 2

On partial metric preserving functions and their characterization

In 1981, J. Borsík and J. Doboš characterized those functions that allow us to transform a metric into another one in such a way that the topology of the metric to be transformed is preserved. Later on, in 1994, S.G. Matthews introduced a new generalized metric notion known as partial metric. In this chapter, motivated in part by the applications of partial metrics, we characterize partial metric preserving functions, i.e., those functions that help to transform a partial metric into another one. In particular we prove that partial metric preserving functions are exactly those that are strictly monotone and concave. Moreover, we prove that the partial metric preserving functions that preserves the topology of the transformed partial metric are exactly those that are continuous. Furthermore, we give a characterization of those partial metric preserving functions which preserve completeness and contractivity. Concretely, we prove that completeness is preserved by those partial metric preserving functions that are non-bounded, and contractivity is kept by those partial metric-functions that satisfy a distinguished functional equation involving contractive constants. The relationship between metric preserving and partial metric preserving functions is also discussed. Finally, appropriate examples are introduced in order to illustrate the exposed theory.

2.1 The new characterization of partial metric preserving functions

In this section we provide a new characterization of partial metric preserving functions (briefly, *pmp*-functions). To this end, we begin making some observations on metric preserving functions (briefly, *mp*-functions) and *pmp*-functions.

The next propositions provide some partial information of *mp*-functions.

Proposition 2.1.1. *Let $f : [0, \infty) \to [0, \infty)$ be an mp-function, then f is amenable.*

Proposition 2.1.2. *Let $f : [0, \infty) \to [0, \infty)$ be an amenable, monotone and subadditive function, then f is an mp-function.*

Proposition 2.1.3. *Let $f : [0, \infty) \to [0, \infty)$ be an amenable and concave function, then f is an mp-function.*

Now, we focus on the characterization of n-partial metric preserving functions given in [55] (see Theorem 1.1.17) when we are restricted to the 1-dimensional case.

Theorem 2.1.4. *Let $f : [0, \infty) \to [0, \infty)$. The below assertions are equivalent:*

(1) f is a pmp-function.

(2) f holds the following properties for all $a, b, c, d \in [0, \infty)$:

(2.1) $f(a)+f(b) \leq f(c)+f(d)$ whenever $a+b \leq c+d$ and $b \leq \min\{c, d\}$.

(2.2) If $\max\{b, c\} \leq a$ and $f(a) = f(b) = f(c)$, then $a = b = c$.

From the preceding result one can derive the following corollary.

Corollary 2.1.5. *Every pmp-function is monotone and subadditive.*

Now we are able to approach the aforementioned aim. First, we prove the following lemmata which will be essential.

Lemma 2.1.6. *Let $f : [0, \infty) \to [0, \infty)$ be a pmp-function. Then f is strictly monotone.*

Proof. Assume for the purpose of contradiction that f is not strictly monotone. Then, there exist $a, b \in [0, \infty)$, with $a < b$, such that $f(a) \geq f(b)$. Consider the partial metric space $([0, \infty), p_m)$, where $p_m(x, y) = \max\{x, y\}$ for each $x, y \in [0, \infty)$. Since f is a *pmp*-function we deduce that $([0, \infty), p_f)$ is a partial metric. Moreover, $p_f(a, a) = f(\max\{a, a\}) = f(a) \geq f(b) = f(\max\{a, b\}) = p_f(a, b)$ and so $p_f(a, a) = p_f(a, b)$. Furthermore, $p_f(b, b) = f(b) = f(\max\{a, b\}) = p_f(a, b)$. Then $p_f(a, a) = p_f(b, b) = p_f(a, b)$ and, thus, $a = b$ which contradicts the fact $a < b$. ■

Observe that Lemma 2.1.6 refines part of the information provided by Corollary 2.1.5 about *pmp*-functions.

Lemma 2.1.7. *Let* $f : [0, \infty) \to [0, \infty)$ *be a strictly monotone function and let* $a, b, c, d \in [0, \infty)$. *If* $\max\{b, c\} \leq a$ *and* $f(a) = f(b) = f(c)$, *then* $a = b = c$.

Proof. Clearly the thesis is derived from the fact that every strictly monotone function is injective.

■

The next result provides a particular method to generate new strictly monotone, concave and *pmp*-functions from old ones.

Lemma 2.1.8. *Let* $f, g : [0, \infty) \to [0, \infty)$ *be two functions such that* $g(a) = f(a) - f(0)$ *for all* $a \in [0, \infty)$. *Then the following assertions are hold:*

(1) If f is strictly monotone and concave, then g is strictly monotone and concave.

(2) If f is a pmp-function, then g is a pmp-function.

Proof. (1). Obviously g is strictly monotone. Next we prove that g is concave too. To this end, let $a, b \in [0, \infty)$ and $\lambda \in]0, 1[$. Then,

$$g(\lambda a + (1 - \lambda)b) = f(\lambda a + (1 - \lambda)b) - f(0) \geq \lambda f(a) + (1 - \lambda)f(b) - f(0) =$$

$$\lambda f(a) - \lambda f(0) + (1 - \lambda)f(b) - (1 - \lambda)f(0) = \lambda g(a) + (1 - \lambda)g(b).$$

It follows that g is concave.

(2). By assertion (1) in Lemma 2.1.8 we have that g is strictly monotone. So, by Lemma 2.1.7, g satisfies condition (2.2) of Theorem 2.1.4. It remains to prove that g satisfies condition (2.1) of Theorem 2.1.4 in order to see that g is a *pmp*-function. With this aim, let $a, b, c, d \in \mathbb{R}_+$ such that $a+b \leq c+d$, $b \leq c$ and $b \leq d$. Then,

$$g(a)+g(b) = f(a)-f(0)+f(b)-f(0) \leq f(c)-f(0)+f(d)-f(0) = g(c)+g(d),$$

since f is a *pmp*-function. Therefore, by Theorem 2.1.4, g is a *pmp*-function.

■

Following similar arguments we can obtain the next lemma whose proof we omit.

Lemma 2.1.9. *Let $\alpha \in [0, \infty)$ and let $f, h : [0, \infty) \to [0, \infty)$ be two functions such that $h(a) = f(a) + \alpha$ for all $a \in [0, \infty)$. If f is strictly monotone and concave, then h is strictly monotone and concave.*

The next result was proved in [15, Theorem 2 in Chapter 1].

Lemma 2.1.10. *Let $f : [0, \infty) \to [0, \infty)$ be an amenable function. Then the following assertions are equivalent:*

(1) f is concave.

(2) If $a, b, c \in [0, d]$ with $d \in (0, \infty)$ and $a + d = b + c$, then $f(a) + f(d) \leq f(b) + f(c)$.

The next result states a relationship between condition (2.1) in Theorem 2.1.4 and condition (2) in Lemma 2.1.10 when strictly monotone functions are under consideration.

Lemma 2.1.11. *Let $f : [0, \infty) \to [0, \infty)$ be a strictly monotone function. Then the following assertions are equivalent:*

(1) If $a, b, c \in [0, d]$ with $d \in (0, \infty)$ and $a + d = b + c$, then $f(a) + f(d) \leq f(b) + f(c)$.

(2) If $a, b, c, d \in \mathbb{R}_+$, then $f(a)+f(b) \leq f(c)+f(d)$ whenever $a+b \leq c+d$, $b \leq c$ and $b \leq d$.

Proof. (1) $\Rightarrow$ (2). Let $a, b, c, d \in \mathbb{R}_+$ such that $a + b \leq c + d$, $b \leq c$ and $b \leq d$. Take $t = c + d - b$. Since $b \leq c$ and $b \leq d$ we have that $t \in [0, \infty)$. However if $t = 0$, then an easy computation shows that $a = b = c = d = 0$ and, thus, condition (2) is hold. So we can assume that $t > 0$. Clearly, $b, c, d \in [0, t]$ and, in addition, $b + t = c + d$. So, by (1), we have that $f(b) + f(t) \leq f(c) + f(d)$. Moreover the facts that $a \leq c + d - b = t$ and f is strictly monotone give that $f(a) \leq f(t)$. Thus we conclude that

$$f(a) + f(b) \leq f(b) + f(t) \leq f(c) + f(d).$$

(2) $\Rightarrow$ (1). Let $d \in (0, \infty)$ and consider $a, b, c \in [0, d]$ such that $a + d = b + c$. Assume that $a > c$. Then $a + d > c + d \geq c + b = a + d$, which is a contradiction. Whence we deduce that $a \leq c$. Similarly one can prove that $a \leq b$. Therefore $a \leq c$ and $a \leq b$. Then $f(a) + f(d) \leq f(b) + f(c)$ as we claimed. ■

In the light of the exposed results we are able to prove the promised new characterization of *pmp*-functions.

Theorem 2.1.12. *Let $f : [0, \infty) \to [0, \infty)$ be a function. Then the following assertions are equivalent:*

(1) f is a pmp-function.

(2) f is strictly monotone and concave.

Proof. Define the function $g : \mathbb{R}_+ \to \mathbb{R}_+$ given by $g(a) = f(a) - f(0)$ for all $a \in [0, \infty)$.

(1) $\Rightarrow$ (2). By assertion (2) of Lemma 2.1.8 we have that g is a *pmp*-function. Lemma 2.1.6 warrantees that g is strictly monotone. Since g is a *pmp*-function g satisfies condition (2.1) in Theorem 2.1.4 and, thus, Lemma 2.1.11 guarantees that, given $d \in (0, \infty)$, $g(a) + g(d) \leq g(b) + g(c)$ whenever $a, b, c \in [0, d]$ with $d \in (0, \infty)$ and $a + d = b + c$. This last condition, by Lemma 2.1.10, is equivalent to the concavity of g, since g is amenable. Now, by Lemma 2.1.9, f is concave and strictly monotone, since $f(a) = g(a) + f(0)$ for all $a \in [0, \infty)$.

(2) $\Rightarrow$ (1). By assertion (1) in Lemma 2.1.8 we have that g is strictly monotone and concave. On the one hand, Lemma 2.1.10 gives that, given $d \in (0, \infty)$, $g(a) + g(d) \leq g(b) + g(c)$ whenever $a, b, c \in [0, d]$ with $d \in (0, \infty)$ and $a + d = b + c$. It follows that, given $d \in (0, \infty)$, $f(a) + f(d) \leq f(b) + f(c)$ whenever $a, b, c \in [0, d]$ with $d \in (0, \infty)$ and $a + d = b + c$. From Lemma 2.1.9, we deduce that f is strictly monotone. Then Lemma 2.1.11 yields that, for each $a, b, c, d \in \mathbb{R}_+$, $f(a) + f(b) \leq f(c) + f(d)$ whenever $a + b \leq c + d$, $b \leq c$ and $b \leq d$. Whence we obtain that f fulfills condition (2.1) in Theorem 2.1.4. Moreover, by Lemma 2.1.7, f satisfies condition (2.2) in Theorem 2.1.4. Therefore, the aforesaid theorem warrantees that f is a *pmp*-function.

■

It must be stressed that, when comparing with Propositions 2.1.2 and

2.1.3, Theorem 2.1.4 shows great differences between *mp*-functions and *pmp*-functions.

Taking into account Theorems 1.1.5 and 2.1.12 we can unify both independent characterizations of *pmp*-functions.

Corollary 2.1.13. *Let $f : [0, \infty) \to [0, \infty)$ be a function. The the following assertions are equivalent:*

(1) f is a pmp-function.

(2) f is strictly monotone and concave.

(3) f holds the following properties:

(3.1) $f(a)+f(b) \leq f(c)+f(b)$ whenever $a+b \leq c+d$ and $b \leq \min\{c, d\}$.

(3.2) If $\max\{b, c\} \leq a$ and $f(a) = f(b) = f(c)$, then $a = b = c$.

In [15, Theorem 1 in Chapter 1], the next result for *mp*-functions was proved.

Proposition 2.1.14. *Let $f : [0, \infty) \to [0, \infty)$ be an amenable function. If f is concave, then f is an mp-function.*

From Proposition 2.1.14 and Theorem 2.1.12 we derive the next interesting relationship between *pmp*-functions and *mp*-functions.

Corollary 2.1.15. *Let $f : [0, \infty) \to [0, \infty)$ be an amenable function. If f pmp-function, then f is an mp-function.*

Notice that the preceding result is consistent with the fact that the same conclusion can be obtained from Corollary 2.1.5 and Proposition 2.1.2. Moreover, the following example shows that the converse of Corollary 2.1.15 is not verified.

Example 2.1.16. *Consider the function $f : [0, \infty) \to [0, \infty)$ defined by*

$$f(a) = \begin{cases} 0 & \text{if } a = 0 \\ 1 & \text{otherwise} \end{cases} .$$

It is clear that f is amenable, monotone, subadditive and concave. Nevertheless, f is not a pmp-function. Indeed, set $a = 3$, $b = 2$ and $c = 1$. Then $b \leq a$ and $c \leq a$ and $f(a) = f(b) = f(c) = 1$ but $a \neq b$. By (2.2) *in Theorem 2.1.4, f is not a pmp-function.*

Finally, it must be pointed out that amenable *pmp*-functions match up with those functions named metric transforms in the sense of L.M. Blumenthal (see [4]).

2.2 Strongly partial metric preserving functions

In this section we focus our attention to discern if *pmp*-functions are able to preserve the topology in the spirit of strongly metric preserving functions, i.e., the topology induced by the transformed partial metric space is equivalent to the topology induced by the partial metric space to be transformed. So the main target of this section is to get a version of Theorem 1.1.4 in the framework of partial metric spaces.

Recall that, each partial metric p on X induces a T_0 topology τ_p on X which has as a base the family of open balls $\{B_p(x; \epsilon) : x \in X, \epsilon > 0\}$, where

$B_p(x;\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$. Taking this fact into account, it can be proved easily that two partial metrics p_1 and p_2 on a set X are topologically equivalent (induce the same topology) if and only if for each $x \in X$ and each $\epsilon \in (0,\infty)$ there exists $\delta \in (0,\infty)$ such that

$$B_{p_1}(x;\delta) \subseteq B_{p_2}(x;\epsilon) \text{ and } B_{p_2}(x;\delta) \subseteq B_{p_1}(x;\epsilon).$$

It seems natural to wonder whether always a *pmp*-function preserves topologies, that is, the topology induced by the transformed partial metric space coincides with the topology induced by the partial metric space to be transformed through the *pmp*-function. Nevertheless, as in the classical metric case, this is not the case such as the next example shows.

Example 2.2.1. *Consider the function $f : [0,\infty) \to [0,\infty)$ given by*

$$f(a) = \begin{cases} 0 & \text{if } a = 0 \\ \frac{1+a}{2+a} & \text{if } a \in (0,\infty) \end{cases}.$$

It not hard to check that f is strictly monotone and concave. So, by Theorem 2.1.12, f is a pmp-function. Moreover, consider the partial metric space $([0,\infty), p_m)$, where $p_m(x,y) = \max\{x,y\}$, for each $x, y \in X$. Furthermore, the partial metric p_{m_f} is given by

$$p_{m_f}(x,y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{1+\max\{x,y\}}{2+\max\{x,y\}} & \text{otherwise} \end{cases}.$$

Obviously $B_{p_{m_f}}(0;\frac{1}{4}) = \{0\}$ and $B_{p_m}(0;\delta) = [0,\delta)$ for each $\delta \in (0,\infty)$. Consequently, the topologies τ_{p_m} and $\tau_{p_{m_f}}$ are not the same. Whence we conclude that the partial metrics p_m and p_{m_f} are not equivalent.

Since *pmp*-functions do not preserve, in general, topologies it makes sense that we introduce the following notion.

Definition 2.2.2. A partial metric preserving function $f : [0, \infty) \to [0, \infty)$ is said to be strongly if for each partial metric space (X, p) the partial metrics p, p_f are topologically equivalent.

Similarly to the classical metric case, the *pmp*-function introduced in Example 2.2.1 fails to be continuous at 0. So apparently the continuity will play a fundamental role so that a *pmp*-function to be strongly. Inspired by this fact we discuss the continuity of a *pmp*-function in the result below.

Lemma 2.2.3. *Let $f : [0, \infty) \to [0, \infty)$ be a pmp-function. The following assertions are equivalent:*

(1) f is continuous.

(2) f is continuous at 0.

Proof. (1) $\Rightarrow$ (2). Clearly if f is continuous, then f is continuous at 0.

(2) $\Rightarrow$ (1). We will distinguish two possible cases:

Case 1. f is amenable. Then, by Corollary 2.1.15, f is an *mp*-function continuous at 0. Then the continuity of f follows from Theorem 1.1.4.

Case 2. f is not amenable. Then, by assertion (2) in Lemma 2.1.8, the function $g : [0, \infty) \to [0, \infty)$, given by $g(a) = f(a) - f(0)$, for all $a \in [0, \infty)$ is a *pmp*-function. Clearly g is amenable and continuous at 0. Then, by Case 1, g is continuous and so, obviously, f is continuous. ■

The next theorem characterizes strongly *pmp*-functions extending Theorem 1.1.4 to the new context under consideration.

Theorem 2.2.4. *Let $f : [0, \infty) \to [0, \infty)$ be a pmp-function. Then the following assertions are equivalent:*

(1) f is strongly.

(2) f is continuous.

(3) f is continuous at 0.

Proof. By Lemma 2.2.3, (2) $\Leftrightarrow$ (3). So it remains to prove (1) $\Leftrightarrow$ (2).

(1) $\Rightarrow$ (2). Consider the partial metric space $(\mathbb{R}_+, d_e)$, where $d_e(x, y) = |x - y|$ for each $x, y \in \mathbb{R}_+$. Since f is strongly we have that the partial metrics d_{ef} and d_e are topologically equivalent. Next we show that f is continuous at 0. To this end, fix $\epsilon \in (0, \infty)$. Then, taking into account that d_e and d_{ef} are topologically equivalent, there exists $\delta \in (0, \infty)$ such that $B_{d_e}(0; \delta) \subseteq B_{d_{ef}}(0; \epsilon)$. Thus, given $y \in B_{d_e}(0; \delta)$ then $y \in B_{d_{ef}}(0; \epsilon)$, i.e., $d_e(0, y) < d_e(0, 0) + \delta$ implies $f(d_e(0, y)) < f(d_e(0, 0)) + \epsilon$. Therefore, for each $y \in [0, \delta[$, we have that $f(y) < f(0) + \epsilon$. Whence f is continuous at 0. Hence f is continuous by Lemma 2.2.3.

(2) $\Rightarrow$ (1). Consider a partial metric space (X, p). Let $x \in X$ and $\epsilon \in (0, \infty)$.

First we show that can find $\delta_1 \in (0, \infty)$ such that $B_p(x; \delta_1) \subseteq B_{p_f}(x; \epsilon)$. With this aim, set $a_0 = p(x, x) \in [0, \infty)$. The continuity of f at a_0 gives the existence of $\delta_1 \in (0, \infty)$ such that for each $b \in (a_0 - \delta_1, a_0 + \delta_1) \cap [0, \infty)$ we have that $|f(a) - f(a_0)| < \epsilon$. Take $y \in B_p(x; \delta_1)$. Then $p(x, y) < p(x, x) + \delta_1$ and so $p(x, y) \in [a_0, a_0 + \delta_1[$. It follows that $|f(p(x, y)) - f(p(x, x))| < \epsilon$. Since f is strictly monotone we have that $f(p(x, y)) < f(p(x, x)) + \epsilon$. Whence $B_p(x; \delta_1) \subseteq B_{p_f}(x; \epsilon)$.

Next we show that there exists $\delta_2 \in (0,\infty)$ such that $B_{p_f}(x;\delta_2) \subseteq B_p(x;\epsilon)$. The strictly monotony and continuity of f provides that $f([0,\infty))$ is the interval either $[f(0),\infty)$ or $[f(0),b]$ with $b \in (0,\infty)$. Moreover, there exists the inverse f^{-1} of f which is continuous. Let $c_0 = f(p(x,x)) \in f([0,\infty))$. By continuity of f^{-1} at c_0 there exists $\delta_2 \in (0,\infty)$ such that for each $c \in]c_0 - \delta_2, c_0 + \delta_2[\cap f([0,\infty))$ we have that $|f^{-1}(c) - f^{-1}(c_0)| < \epsilon$. Take $y \in B_{p_f}(x;\delta_2)$. It follows that $f(p(x,y)) \in]c_0 - \delta_2, c_0 + \delta_2[\cap f([0,\infty))$ and, thus, that $|f^{-1}(f(p(x,y))) - f^{-1}(f(p(x,x)))| < \epsilon$. Whence we deduce that $|p(x,y) - p(x,x)| < \varepsilon$. Hence $B_{p_f}(x;\delta_2) \subseteq B_p(x;\epsilon)$.

Therefore, taking $\delta = \min\{\delta_1,\delta_2\}$, we have that $B_p(x;\delta) \subseteq B_{p_f}(x;\epsilon)$ and $B_{p_f}(x;\delta) \subseteq B_p(x;\epsilon)$. So, we conclude that f is strongly.

■

The next example provides instances of strongly *pmp*-functions.

Example 2.2.5. *Let $\alpha, \beta \in (0,\infty)$. The following functions $f_\alpha : [0,\infty) \to [0,\infty)$ are strongly pmp-functions, where for all $a \in [0,\infty)$ they are defined as follows:*

(1) $f_\alpha(a) = (a+\alpha)^\beta$ *with* $\beta \in (0,1]$.

(2) $f_\alpha(a) = \alpha a + \beta$.

(3) $f_\alpha(a) = \frac{\alpha a}{1+a}$.

(4) $f_\alpha(a) = \frac{1+\alpha a}{2+\alpha a}$.

(5) $f_\alpha(a) = \log_\beta(\alpha + a)$ *with* $\alpha, \beta \in (1,\infty)$.

(6) $f_\alpha(a) = 1 - e^{-\alpha a}$.

Observe that in the preceding example the instances from (1) until (5) are strongly *pmp*-functions that are not *mp*-functions. Moreover, the instance (6) is a strongly *pmp*-function that is, at the same time, strongly metric-preserving. This last fact inspires the next result which discusses the relationship between strongly *pmp*-functions and strongly *mp*-functions.

Corollary 2.2.6. *Let* $f : [0,\infty) \to [0,\infty)$ *be an amenable strongly pmp-function. Then* f *is a strongly mp-function.*

Proof. By Corollary 2.1.15 we have that f is an *mp*-function. Moreover, by Theorem 2.2.4, we obtain that f is continuous. Theorem 1.1.4 gives that f is a strongly *mp*-function. ■

The converse of the preceding result is not satisfied as shows the following example.

Example 2.2.7. *Consider the function* $f : [0,\infty) \to [0,\infty)$ *given by* $f(x) = 3x - 2|x-1| + |x-2|$ *for all* $x \in [0,\infty)$. *According to [15] (see also [7]),* f *is an amenable, monotone, subadditive and continuous function. So, By Theorem 1.1.4, we deduce that* f *is strongly mp-function. However, it is not hard to check that* f *is not concave and, thus by Theorem 2.1.12,* f *is not a pmp-function.*

2.3 Completeness, contractions and partial metric preserving functions

The objective of this section is twofold. On the one hand, we are interested in stating those conditions that a *pmp*-function f must satisfy in order to guarantee that, given a complete partial metric space (X, p), the new induced partial metric space (X, p_f) is, again, complete. So we want to study when a *pmp*-function preserves completeness. On the other hand, we focus our efforts on getting those conditions about *pmp*-functions that help us to induce a new partial metric from an old one in such a way that the contractivity condition of a self-mapping is kept.

2.3.1 Preserving completeness

We begin this subsection recalling that a partial metric space (X, p) is complete if every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ is convergent with respect to τ_p and that, a sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy if there exists $a \in [0, \infty[$ such that $\lim_{n,m} p(x_n, x_m) = a$. Notice that this notion of completeness is equivalent to that given in Subsection 1.1.4 (see [56]).

In order to discuss when *pmp*-functions preserve completeness, the next result will play a crucial role.

Lemma 2.3.1. *Let* $f : [0, \infty) \to [0, \infty)$ *be a strictly monotone continuous function. The following assertions are equivalent:*

(1) f *is surjective on* $[f(0), \infty)$.

(2) f *is non-bounded* ($f([0, \infty)) = [f(0), \infty)$).

Proof. (1) $\Rightarrow$ (2). The strictly monotony and continuity of f provides that $f([0, \infty))$ is an interval. Clearly the sujectivity of f gives that such an interval is $[f(0), \infty)$. Indeed, let $M \in (f(0), \infty)$. Then there exists $a_M \in (0, \infty)$ such that $f(a_M) = M$. Notice that the strictly monotony of f guarantees that $a_M \neq 0$ whenever $M > f(0)$.

(2) $\Rightarrow$ (1). Let $M \in (f(0), \infty)$. Then there exists $a_M \in (0, \infty)$ such that $f(a_M) > M$. Thus we have that $f(0) < M < f(a_M)$. Since f is continuous the Darboux's theorem provides the existence of $b_M \in (0, a_M)$ such that $f(b_M) = M$. Therefore f is surjective on $[f(0), \infty)$. ■

The next result fixes the condition that must be taken under consideration in order to guarantee that a *pmp*-function preserves completeness.

Theorem 2.3.2. *Let $f : [0, \infty) \to [0, \infty)$ be a non-bounded strongly pmp-function and let (X, p) be a partial metric space. The following assertions are equivalent:*

(1) (X, p) is complete.

(2) (X, p_f) is complete.

Proof. (1) $\Rightarrow$ (2). First of all we show that every Cauchy sequence in (X, p_f) is a Cauchy sequence in (X, p). Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (X, p_f). Then there exists $b_0 \in [0, \infty[$ such that $\lim_{n,m\to\infty} p_f(x_n, x_m) = b_0$. Next we prove that there exists $a_0 \in [0, \infty[$ with $f(a_0) = b_0$. Indeed, the fact that $\lim_{n,m\to\infty} p(x_n, x_m) = b_0$ gives that, for each $\epsilon \in (0, \infty)$, we can find $n_0 \in \mathbb{N}$ satisfying $|p_f(x_n, x_m) - b_0| < \epsilon$ for all $n, m \geq n_0$. Thus $b_0 - \epsilon < p_f(x_n, x_m) < b_0 + \epsilon$ for all $n, m \geq n_0$. Whence we deduce that $f(0) \leq f(p(x_n, x_m)) = p_f(x_n, x_m) < b_0 + \epsilon$ for all $\epsilon \in (0, \infty)$. It follows that $f(0) \leq b_0$. By Lemma 2.3.1, there exists $a_0 \in [0, \infty[$ with $f(a_0) = b_0$. Since f is strictly monotone and continuous we have warranted the existence of the inverse f^{-1} of f which is continuous. The continuity of f and the fact that $\lim_{n,m\to\infty} p_f(x_n, x_m) = f(a_0)$ yield that $\lim_{n,m\to\infty} f^{-1}(f(p(x_n, x_m))) = f^{-1}(f(a_0))$. So $\lim_{n,m\to\infty} p(x_n, x_m) = a_0$ and, hence, the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, p).

It remains to prove that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to a point $x \in X$ such that $p_f(x,x) = \lim_{n,m\to\infty} p_f(x_n,x_m) = \lim_{n\to\infty} p_f(x,x_n)$. Since (X,p) is complete we have the existence of x such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m) = \lim_{n\to\infty} p(x,x_n)$. The continuity of f gives immediately the desired conclusion.

(2) $\Rightarrow$ (1). The proof runs following similar arguments to those given for (1) $\Rightarrow$ (2). ■

In view of the preceding result, one can pose the question whether the result remains true when the non-bounded character of the strongly *pmp*-function is deleted from its statement. The next example shows that the answer to the posed question is negative

Example 2.3.3. *Consider the complete partial metric space* $([0,\infty),p_m)$ *and the strongly pmp-function* $f : [0,\infty) \to [0,\infty)$ *given by* $f(a) = \frac{1+a}{2+a}$. *Clearly* f *is bounded because* $f([0,\infty)) = [\frac{1}{2},1)$. *Take the sequence* $(x_n)_{n\in\mathbb{N}}$ *in* $([0,\infty),p_{m_f})$ *with* $x_n = n$ *for all* $n \in \mathbb{N}$, *which is Cauchy in* $([0,\infty),p_{m_f})$. *Indeed,* $\lim_{n,m\to\infty} p_{m_f}(x_n,x_m) = \lim_{n,m\to\infty} p_{m_f}(n,m) = 1$ *because, given* $\epsilon \in (0,\infty)$, *there exists* $n_0 \in \mathbb{N}$ ($n_0 > \frac{1-2\epsilon}{\epsilon}$) *such that for all* $n,m \geq n_0$ *we have that*

$$1 - p_{m_f}(x_n,x_m) = 1 - \frac{1+\max\{n,m\}}{2+\max\{n,m\}} \leq \frac{1}{2+n_0} < \epsilon.$$

However, $(x_n)_{n\in\mathbb{N}}$ *is not convergent with respect to* $\tau_{p_{m_f}}$ *and, thus, we have that* $([0,\infty),p_{m_f})$ *is not complete. Notice that* $1 \notin f([0,\infty))$ *and, thus, by Lemma 2.3.1,* f *is not non-bounded.*

According to [81], a sequence $(x_n)_{n\in\mathbb{N}}$ in a partial metric space (X,p) is said to be 0-convergent to $x \in X$ provided that it converges to x with respect to τ_p and $p(x,x)=0$. Besides, $(x_n)_{n\in\mathbb{N}}$ is called 0-Cauchy whenever $\lim_{n,m\to\infty} p(x_n,x_m)=0$. Furthermore, a partial metric space (X,p) is said to be 0-complete if every 0-Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ 0-converges, with respect to τ_p, to a point $x\in X$. Of course, every complete partial metric space is 0-complete but the converse is not true in general.

Following similar arguments to those given in the proof of Theorem 2.3.2 one can obtain the next surprising result.

Theorem 2.3.4. *Let $f:[0,\infty)\to[0,\infty)$ be an amenable strongly pmp-function. The following assertions hold:*

(1) If (X,p) is a partial metric space then (X,p) is 0-complete if and only if (X,p_f) is 0-complete.

(2) If (X,d) is a metric space then (X,d) is complete if and only if (X,d_f) is complete.

2.3.2 Preserving contractivity

Before starting the study provided in this subsection, recall that given a partial metric space (X,p), we will say that a self-mapping $T:X\to X$ is contractive if there exists $k\in]0,1[$ such that

$$p(T(x),T(y))\leq kp(x,y),\ \text{for each } x,y\in X.$$

To discern what conditions allow *pmp*-functions to keep contractivity of self-mappings, let us introduce the following pertinent notion.

Definition 2.3.5. Let $f : [0, \infty) \to [0, \infty)$ be a *pmp*-function. We will say that f is contraction-preserving provided that, for each partial metric space (X, p), every p-contraction is also a p_f-contraction.

Instance (4) in Example 2.2.5 is an example of *pmp*-function which is not contranction-preserving. Nonetheless, the function $f : [0, \infty) \to [0, \infty)$ defined by $f(a) = \alpha a$ for all $a \in [0, \infty)$ and with $\alpha \in (0, \infty)$ is an example of *pmp*-function which is, in addition, contraction-preserving.

The next result gives a characterization of those *pmp*-functions which preserve contractive mappings.

Theorem 2.3.6. *Let $f : [0, \infty) \to [0, \infty)$ be a pmp-function. The following assertions are equivalent:*

(1) f is contraction-preserving.

(2) For each $k \in]0, 1[$ there exists $c \in]0, 1[$ such that $f(ka) \leq cf(a)$ for all $a \in [0, \infty)$.

Proof. (1) $\Rightarrow$ (2). For the purpose of contradiction, suppose that there exists $k_0 \in]0, 1[$ such that for each $c \in]0, 1[$ we can find $a_c \in [0, \infty[$ satisfying $f(k_0 a_c) > cf(a_c)$. Next we show that f is not contranction-preserving. Indeed, consider the partial metric space $([0, \infty), p_m)$ and define the mapping $T : \mathbb{R}_+ \to \mathbb{R}_+$ by $T(x) = k_0 x$ for all $x \in \mathbb{R}_+$. Then

$$p_m(T(x), T(y)) = \max\{k_0 x, k_0 y\} = k_0 \max\{x, y\} = k_0 p_{\max}(x, y)$$

for all $x, y \in \mathbb{R}_+$ and, hence, T is a $p_{\max}$-contraction. However,

$$p_{m_f}(T(a_c), T(0)) = f(\max\{k_0 a_c, 0\}) >$$

$$> cf(a_c) = cf(\max\{a_c, 0\}) = cp_{m_f}(a_c, 0)$$

and, thus, we have seen that for each $c \in]0,1[$ we can find $x, y \in \mathbb{R}_+$ such that

$$p_{m_f}(T(x), T(y)) > cp_{m_f}(x, y),$$

which contradicts the fact that f is a contraction-pressrving.

(2) $\Rightarrow$ (1). Let (X, p) be a partial metric space and let $T : X \to X$ be a p-contraction. Then there exists $k_0 \in]0,1[$ such that

$$p(T(x), T(y)) \leq k_0 p(x, y)$$

for all $x, y \in X$. Then there exists $c_0 \in]0,1[$ such that $f(ka) \leq c_0 f(a)$ for all $a \in [0,\infty)$. Taking in mind that f is strictly monotone, it follows that

$$p_f(T(x), T(y)) = f(p(T(x), T(y))) \leq f(k_0 p(x, y)) \leq c_0 f(p(x, y)) = c_0 p_f(x, y)$$

for all $x, y \in [0,\infty)$. Therefore, T is a p_f-contraction and so f is contraction-preserving. ■

The previous result allows us to find non-trivial examples of *pmp*-functions which are contraction-preserving.

Example 2.3.7. *Let $\alpha \in (0,\infty)$. Define the function $f : [0,\infty) \to [0,\infty)$ by $f(a) = \sqrt{a^2 + \alpha a}$ for all $a \in [0,\infty)$. It is not hard to check that f is strictly monotone and concave. Then, by Theorem 2.1.12, f is a pmp-function.*

Now, let $k \in]0,1[$ and take $c = \sqrt{k}$. Then we have that

$$f(ka) = \sqrt{k^2a^2 + \alpha ka} \leq \sqrt{c^2a^2 + \alpha c^2 a} = c\sqrt{a^2 + \alpha a} = cf(a)$$

for all $a \in [0,\infty)$. Therefore, by Theorem 2.3.6, f is contraction-preserving.

It is worthy to stress that Example 2.3.7 clarifies that *pmp*-functions being contraction-preserving do not reduce to those that are homogeneous. Recall that, according to [42], a function $f : [0, \infty) \to [0, \infty)$ is homogeneous provided that $f(\alpha a) = \alpha f(a)$ for all $a, \alpha \in [0, \infty)$.

It seems natural to wonder if there exists any relationship between those *pmp*-functions that are contraction-preserving and strongly. The next result makes clear such a question.

Corollary 2.3.8. *Let $f : [0, \infty) \to [0, \infty)$ be a pmp-function. If f is contraction-preserving, then the following assertions hold:*

(1) f is amenable.

(2) f is a strongly pmp-function.

(3) f is a strongly mp-function.

Proof. (1). From Theorem 2.3.6 we deduce that $f(0) = 0$. Then the fact that f is strictly monotone gives that $f(a) = 0 \Leftrightarrow x = 0$.

(2). Next we show that f is continuous at 0. To this end, fix $k \in (0, 1)$. Then there exists $c \in (0, 1)$ such that $f(ka) \leq cf(a)$ for all $a \in [0, \infty)$. It follows that $f(k) \leq cf(1)$ and that $f(k^n) \leq c^n f(1)$ for all $n \in \mathbb{N}$. Suppose for the purpose of contradiction that f is not continuous at 0. By Corollary 2.10 in [7], every *mp*-function g which is discontinuous at 0 satisfies that there exists $\epsilon \in (0, \infty)$ such that $\epsilon \leq g(a)$ for all $a \in [0, \infty)$. By Proposition 2.1.2 we have that f is metric preserving. Then there exists $\epsilon \in (0, \infty)$ such that $\epsilon \leq f(a)$ for all $a \in (0, \infty)$. Moreover, there exists $n_0 \in \mathbb{N}$ with $c^n f(1) < \epsilon$ for all $n \geq n_0$. Whence we obtain that $\epsilon \leq f(k^n) \leq c^n f(1) < \epsilon$ for all $n \geq n_0$, which is a contradiction. Therefore f is continuous at 0. By Theorem 2.2.4 we conclude that f is a strongly *pmp*-function.

(3). Since f is an amenable a strongly *pmp*-function Corollary 2.2.6 guarantees that f is a strongly *mp*-function. ■

The function introduced in Example 2.3.3 is an instance of strongly *pmp*-function which is not contraction-preserving.

In the light of Theorem 2.3.6 and the proof of Corollary 2.3.8 we derive the next result whose easy proof we omit.

Corollary 2.3.9. *Let $f : [0, \infty) \to [0, \infty)$ be a contraction-preserving pmp-function. If (X, p) is a partial metric space and T is a p-contraction with contractive constant k, then there exists $c_k \in (0, 1)$ such that T is a p_f-contraction with contractive constant c_k and, in addition, $f(k) \leq c_k f(1)$.*

In view of the preceding result, it seems natural to ask for those conditions that allow contraction-preserving functions preserve the contractive constant, that is, T is a p_f-contraction with contractive constant k whenever T is a p-contraction with contractive constant k. The next result clarifies this situation.

Corollary 2.3.10. *Let $f : [0, \infty) \to [0, \infty)$ be a contraction-preserving pmp-function. The following assertions are equivalent:*

(1) f preserves the contractive constant of every p-contraction.

(2) $f(ka) = kf(a)$ for all $k \in (0, 1)$ and $a \in [0, \infty)$.

Proof. (1) $\Rightarrow$ (2). Consider the partial metric space $([0, \infty), p_m)$ and

$k \in (0,1)$. Define the mapping $T_k : [0,\infty) \to [0,\infty)$ by $T_k(x) = kx$ for all $x \in [0,\infty)$. It is clear that T is a p_m-contraction with k as contractive constant. Since f preserves the contractive constant we have that T_k is a p_{m_f}-contraction with k as contractive constant. Then

$$f(ka) = f(\max\{ka, 0\}) = p_{m_f}(T_k(a), T_k(0)) \leq kp_{m_f}(a, 0) = kf(a)$$

for all $a \in \mathbb{R}_+$. Moreover, by Theorem 2.1.12, we have that f is concave. Thus $kf(a) \leq f(ka + (1-k)0) = f(ka)$. Whence we conclude that $f(ka) = kf(a)$ for all $k \in (0,1)$ and $a \in [0,\infty)$.

(2) $\Rightarrow$ (1). It is obvious. ■

It is clear that replacing, in Definition 2.3.5, *pmp*-functions and partial metric spaces by *mp*-functions and metric spaces respectively, we obtain a contraction-preserving notion for the classical metric case. From now on, this type of functions will be called metric-contraction-preserving. The next result states a surprising relation between both type of functions.

Theorem 2.3.11. *Let $f : [0,\infty) \to [0,\infty)$ be a pmp-function. The following assertions are equivalent:*

(1) f is contraction-preserving.

(2) f is metric-contraction-preserving.

Proof. (1) $\Rightarrow$ (2). By Corollary 2.3.8 f is a strongly *mp*-function. Then, given a metric space (X, d), d_f is again a metric on X. By Theorem 2.3.6 we have that for each $k \in]0,1[$ there exists $c \in]0,1[$ such that $f(ka) \leq cf(a)$ for all $a \in [0,\infty)$. Next consider a d-contraction $T : X \to X$ with contractive constant k_0. Then, there exist $c_0 \in (0,1)$ such that

$$d_f(T(x),T(y)) = f(d(T(x),T(y))) \leq f(k_0 p(x,y)) \leq$$

$$\leq c_0 f(d(x,y)) = c_0 d_f(x,y)$$

for all $x, y \in [0,\infty)$. Therefore, T is a d_f-contraction and so f is metric-contraction-preserving.

(2) $\Rightarrow$ (1). Assume that f is not contraction-preserving. By Theorem 2.3.6 we have that there exists $k_0 \in]0,1[$ such that for each $c \in]0,1[$ we can find $a_c \in [0,\infty[$ satisfying $f(k_0 a_c) > cf(a_c)$. Consider the partial metric space $([0,\infty), p_m)$ and define the mapping $T : \mathbb{R}_+ \to \mathbb{R}_+$ by $T(x) = k_0 x$ for all $x \in \mathbb{R}_+$. Then it is clear that T is a p_m-contraction. Define the mapping $d_p : [0,\infty) \times [0,\infty) \to \mathbb{R}_+$ by

$$d_{p_m}(x,y) = \begin{cases} 0 & \text{if } x = y \\ p_m(x,y) & \text{if } x \neq y \end{cases} .$$

It is easy to see that d_{p_m} is a metric on $\mathbb{R}_+$ (compare [44]) and that T is a d_{p_m}-contraction. Since f is a metric-contraction-preserving function we have that T is a $d_{p_{m_f}}$-contraction. Hence, there exists $k_1 \in (0,1)$ such that

$$d_{p_{m_f}}(T(x),T(y)) \leq k_1 d_{p_{m_f}}(x,y)$$

for all $x, y \in \mathbb{R}_+$. Concretely, we have that

$$p_{m_f}(T(x),T(y)) \leq k_1 p_{m_f}(x,y)$$

for all $x, y \in \mathbb{R}_+$ with $x \neq y$. It follows that

$$f(k_0 a_{k_1}) = p_{m_f}(T(a_{k_1}),T(0))) \leq k_1 p_{m_f}(a_{k_1},0) = k_1 f(a_{k_1}).$$

Nevertheless $k_1 f(a_{k_1}) < f(k_0 a_{k_1}) \leq k_1 f(a_{k_1})$, which is a contradiction. Consequently f is contraction-preserving. ■

Chapter 3

Characterizing quasi-metric aggregation functions

As stated before, the study of n-metric preserving functions, those functions that merge a family of metric spaces into a single one, was initiated by J. Borsík and J. Doboš in 1981. In particular they obtained a characterization of such functions in terms of triangle triplets. In 2000, E. Castiñeira, A. Pradera and E. Trillas explored in depth the particular case in which each metric of the family of metric spaces to be merge through a function, now called metric aggregation function (briefly, ma-function), is defined on the same set ([74, 75]). They yielded sufficient conditions in order to guarantee that a function is a ma-function and, in addition, they introduced techniques to construct such type of functions. Recently, in 2019, G. Mayor and O. Valero have continued the work in this direction ([59]). Concretely, they have provided a new characterization of ma-functions in terms of positive triangle triplets and they have also discussed techniques to construct them.

Inspired by the fact that, in 2010, Mayor and Valero extended the original work of Borsík and Doboš to the quasi-metric context in such a way that a characterization of those functions that merge a family of quasi-metric spaces into a single one, known as n-quasi-metric preserving functions, was given in terms of (triangle) triplets, in this chapter we focus our attention on exploring the possibility of extending the work of Mayor and Valero about ma-functions to the quasi-metric framework. Thus, we characterize those functions that allow us to combine a family of quasi-metrics, defined all of them on the same set, into a single one and, in addition, we discuss a few of their properties. Moreover, a few methods to discard those functions that are useless as quasi-metric aggregation functions are introduced. Finally, two possible fields where the developed theory can be useful are exposed.

3.1 Quasi-metric aggregation functions and their characterization

As pointed out above we are interested in extending the characterization of metric aggregation functions, given by Theorem 1.1.9, to the context of quasi-metric spaces. Moreover, motivated by the fact that every n-quasi-metric preserving function is a n-metric preserving one, we discuss the relationship between quasi-metric aggregation functions and metric aggregation functions. In addition, the link with n-quasi-metric preserving functions is also explored. All this is illustrated with appropriate examples. To this end, we first introduce the notion of quasi-metric aggregation function following the spirit of [59].

Let $n \in \mathbb{N}$. A function $F : [0, \infty)^n \to [0, \infty)$ will be called a quasi-metric aggregation function (briefly, *qma*-function) if, for each non-empty set X and each family of quasi-metrics $\{q_1, \ldots, q_n\}$ on X, the function $Q_F^n : X \times X \to [0, \infty[$ is a quasi-metric on X, where

$$Q_F^n(x, y) = F(q_1(x, y), \ldots, q_n(x, y))$$

for each $x, y \in X$.

The next example gives a simple, but illustrative, instance of a *qma*-function.

Example 3.1.1. *Let $n \in \mathbb{N}$ and fix a collection of coeficients $\{\alpha_2, \ldots, \alpha_n\}$ such that $\alpha_i \in (0, \infty)$ for all $i = 2, \ldots, n$. Define the function $F_s : [0, \infty)^n \to [0, \infty)$ given, for each $\mathbf{a} \in [0, \infty)^n$, by $F_s(\mathbf{a}) = \sum_{i=2}^n \alpha_i \cdot a_i$. A straightforward computation shows that F_s is a qma-function.*

Observe that every *qma*-function is a *ma*-function. Indeed, let $n \in \mathbb{N}$ and let $F : [0, \infty)^n \to [0, \infty)$ be a *qma*-function. Consider a non-empty set X and a family of metrics $\{d_1, \ldots, d_n\}$ on it. By our assumption, the function Q_F^n is a quasi-metric on X, where $Q_F^n(x, y) = F(d_1(x, y), \ldots, d_n(x, y))$ for each $x, y \in X$. Besides, $Q_F^n(x, y) = Q_F^n(y, x)$ and so Q_F^n is a metric on X, since $d_i(x, y) = d_i(y, x)$ for each $i \in \{1, \ldots, n\}$. Therefore, F is an *ma*-function.

With the aim of getting a characterization of *qma*-functions we introduce the following lemmata which will be crucial later on.

Lemma 3.1.2. *Let $n \in \mathbb{N}$ and let $F : [0, \infty)^n \to [0, \infty)$ be a qma-function. Then, $F(\mathbf{a}) \leq F(\mathbf{b}) + F(\mathbf{c})$ for each $\mathbf{a}, \mathbf{b}, \mathbf{c} \in [0, \infty)^n$ with $\mathbf{a} \preceq \mathbf{b} + \mathbf{c}$.*

Proof. Consider a set $X = \{x, y, z\}$ with all elements different. Next we show that for every $\mathbf{a}, \mathbf{b}, \mathbf{c} \in [0, \infty)^n$ with $\mathbf{a} \preceq \mathbf{b} + \mathbf{c}$, there exists a family of quasi-metrics $\{q_1^{a,b}, \ldots, q_n^{a,b}\}$ such that $q_i^{a,b}$ is defined on X and, in addition, $q_i^{a,b}(x,z) = a_i$, $q_i^{a,b}(z,y) = b_i$ and $q_i^{a,b}(y,z) = c_i$ for all $i \in \{1, \ldots, n\}$. To this end, given $\mathbf{a}, \mathbf{b}, \mathbf{c} \in [0,\infty)^n$, we distinguish two possible cases when each coordinate of $\mathbf{b}$ and $\mathbf{c}$ is considered:

Case 1. $\max\{b_i, c_i\} = 0$. Then we consider the quasi-metric $q_{i,1}^{a,b}$ defined as follows:

$$q_{i,1}^{a,b}(x,y) = q_{i,1}^{a,b}(x,z) = q_{i,1}^{a,b}(y,z) = q_{i,1}^{a,b}(x,x) = q_{i,1}^{a,b}(y,y) = q_{i,1}^{a,b}(z,z) = 0;$$
$$q_{i,1}^{a,b}(y,x) = q_{i,1}^{a,b}(z,x) = q_{i,1}^{a,b}(z,y) = 1.$$

Case 2. $\max\{b_i, c_i\} \neq 0$. Then we consider the quasi-metric $q_{i,2}^{a,b}$ defined as follows:

$$q_{i,2}^{a,b}(x,x) = q_{i,2}^{a,b}(y,y) = q_{i,2}^{a,b}(z,z) = 0;$$
$$q_{i,2}^{a,b}(x,z) = a_i; q_{i,2}^{a,b}(z,x) = \max\{b_i, c_i\};$$
$$q_{i,2}^{a,b}(x,y) = q_{i,2}^{a,b}(z,y) = b_i;$$
$$q_{i,2}^{a,b}(y,z) = q_{i,2}^{a,b}(y,x) = c_i.$$

Since F is a *qma*-function, then the function $Q_F^n : X \times X \to [0,\infty)$ given, for each $u, v \in X$, by

$$Q_F^n(u,v) = F(q_1^{a,b}(u,v), \ldots, q_n^{a,b}(u,v))$$

is a quasi-metric on X, where

$$q_i^{a,b}(u,v) = \begin{cases} q_{i,1}^{a,b}(u,v) & \text{if } \max\{a_i, b_i\} = 0 \\ q_{i,2}^{a,b}(u,v) & \text{if } \max\{a_i, b_i\} \neq 0 \end{cases}, \text{ for each } i \in \{1, \ldots, n\}.$$

It follows that

$$F(\mathbf{a}) = Q_F^n(x,z) \leq Q_F^n(x,y) + Q_F^n(y,z) = F(\mathbf{b}) + F(\mathbf{c})$$

as we claimed. ■

The next results follow immediately from the preceding one.

Corollary 3.1.3. *Let $n \in \mathbb{N}$ and let $F : [0,\infty)^n \to [0,\infty)$ be a qma-function. Then the following assertions hold:*

(1) F is subadditive.

(2) F is monotone.

Proof. By Lemma 3.1.2 we have that $F(\mathbf{a}) \leq F(\mathbf{b}) + F(\mathbf{c})$ for each $\mathbf{a}, \mathbf{b}, \mathbf{c} \in [0,\infty)^n$, with $\mathbf{a} \preceq \mathbf{b}+\mathbf{c}$. Taking $\mathbf{a} = \mathbf{b}+\mathbf{c}$ in the preceding inequality we obtain the subadditivity of F and taking $\mathbf{c} = \mathbf{0}$ we deduce the monotony of F. ■

The below property will paly a central role in our subsequent discussion.

Lemma 3.1.4. *Let $n \in \mathbb{N}$ and let $F : [0,\infty[^n \to [0,\infty[$ be a subadditive function. Then the following assertions are equivalent:*

(i) There exists $i_0 \in \{1,\ldots,n\}$ satisfying the following: for each $\mathbf{a} \in [0,\infty)^n$ with $F(\mathbf{a}) = 0$ we have that $a_{i_0} = 0$;

(ii) If $\mathbf{a} \in [0,\infty)^n$ such that $F(\mathbf{a}) = 0$, then $\min\{a_1,\ldots,a_n\} = 0$.

Proof. It is obvious that $(i) \Rightarrow (ii)$. So we only need to show that $(ii) \Rightarrow (i)$. To this end, suppose for the purpose of contradiction that for each $i \in \{1,\ldots,n\}$ there exists $\mathbf{a}^i \in [0,\infty)^n$ such that $F(\mathbf{a}^i) = 0$ but $a^i_i > 0$. Since F is subadditive we obtain that $F(\mathbf{a}^1 + \cdots + \mathbf{a}^n) \leq F(\mathbf{a}^1) + \cdots + F(\mathbf{a}^n) = 0$. Thus, there exists $\mathbf{c} \in [0,\infty)^n$ with $\mathbf{c} = \mathbf{a}^1 \cdots + \mathbf{a}^n$ such that $F(\mathbf{c}) = 0$ and, however, $c_i > 0$ for each $i \in \{1,\ldots,n\}$, which contradicts (ii). ■

In the particular case of *qma*-functions we have the following.

By Lemma 3.1.4 and Corollary 3.1.3 we obtain the following result.

Lemma 3.1.5. *Let $n \in \mathbb{N}$ and let $F : [0, \infty)^n \to [0, \infty)$ be a qma-function. Then the following assertions are equivalent:*

(i) There exists $i_0 \in \{1, \ldots, n\}$ satisfying the following: for each $\mathbf{a} \in [0, \infty)^n$ with $F(\mathbf{a}) = 0$ we have that $a_{i_0} = 0$;

(ii) If $\mathbf{a} \in [0, \infty)^n$ such that $F(\mathbf{a}) = 0$, then $\min\{a_1, \ldots, a_n\} = 0$.

In the light of the previous results, we are able to prove the promised characterization of *qma*-functions.

Theorem 3.1.6. *Let $n \in \mathbb{N}$ and let $F : [0, \infty)^n \to [0, \infty)$ be a function. Then the following assertions are equivalent:*

(1) F is a qma-function;

(2) F satisfies the following conditions:

(2.1) $F(\mathbf{0}) = 0$;

(2.2) If $F(\mathbf{a}) = 0$, then $\min\{a_1, \ldots, a_n\} = 0$;

(2.3) $F(\mathbf{a}) \leq F(\mathbf{b}) + F(\mathbf{c})$ for each $\mathbf{a}, \mathbf{b}, \mathbf{c} \in [0, \infty)^n$ with $\mathbf{a} \preceq \mathbf{b} + \mathbf{c}$.

(3) F satisfies the following conditions:

(3.1) $F(\mathbf{0}) = 0$;

(3.2) If $F(\mathbf{a}) = 0$, then $\min\{a_1, \ldots, a_n\} = 0$;

(3.3) F is monotone and subadditive.

Proof. (1) $\Rightarrow$ (2). By Lemmas 3.1.2 and 3.1.5 we have that F satisfies conditions (2.2) and (2.3). Next we show that $F(\mathbf{0}) = 0$. Indeed, consider the quasi-metric space $([0,\infty), q_u)$, where $q_u(x,y) = \max\{y-x, 0\}$. The fact that F is a qma-function gives that the function Q_F^n is a quasi-metric on X, where $Q_F^n(x,y) = F(q_u(x,y),\ldots,q_u(x,y))$ for all $x,y \in X$. It follows that, fixed $x \in [0,\infty)$, $F(\mathbf{0}) = F(0,\ldots,0) = F(q_u(x,x),\ldots,q_u(x,x)) = Q_F^n(x,x) = 0$.

(2) $\Rightarrow$ (1). Consider a non-empty set X and a family $\{q_1,\ldots,q_n\}$ of quasi-metrics on X. We will show that the function $Q_F^n : X \times X \to [0,\infty)$ is a quasi-metric.

Suppose that, given $x,y \in X$, $Q_F^n(x,y) = Q_F^n(y,x) = 0$. Then,

$$F(q_1(x,y),\ldots,q_n(x,y)) = F(q_1(y,x),\ldots,q_n(y,x)) = 0.$$

Since F satisfies condition (2.3) we deduce that F is subadditive. In addition F fulfills condition (2.3) and, thus, Lemma 3.1.4 guarantees that there exists $i_0 \in \{1,\ldots,n\}$ such that $q_{i_0}(x,y) = q_{i_0}(y,x) = 0$. Thus $x = y$, since q_{i_0} is a quasi-metric on X. Besides, $Q_F^n(x,x) = 0$ for each $x \in X$, since F satisfies (2.1). So Q_F^n satisfies condition $(q1)$ required for quasi-metrics.

Next we prove that Q_F^n satisfies condition $(q2)$ for quasi-metrics. With this aim consider $x,y,z \in X$ and take $\mathbf{a},\mathbf{b},\mathbf{c} \in [0,\infty)^n$ such that

$$\begin{aligned}\mathbf{a} &= (q_1(x,z),\ldots,q_n(x,z)),\\ \mathbf{b} &= (q_1(x,y),\ldots,q_n(x,y)),\\ \mathbf{c} &= (q_1(y,z),\ldots,q_n(y,z)).\end{aligned}$$

The fact that q_i is a quasi-metric on X for each $i \in \{1,\ldots,n\}$ provides that $\mathbf{a} \preceq \mathbf{b}+\mathbf{c}$. Then condition (2.3) yields that

$$Q_F^n(x,z) = F(\mathbf{a}) \leq F(\mathbf{b}) + F(\mathbf{c}) = Q_F^n(x,y) + Q_F^n(y,z).$$

Therefore, Q_F^n is a quasi-metric on X and, thus, F is a *qma*-function.

(2) $\Leftrightarrow$ (3). It is enough to take into account that F is monotone and subadditive if and only if F satisfies condition (2.3). ■

Theorem 3.1.6 allows us to provide a few examples of *qma*-functions.

Example 3.1.7. *Let $n \in \mathbb{N}$. The following functions $F : [0, \infty)^n \to [0, \infty)$ are qma-functions where for all $\mathbf{a}, \mathbf{w} \in [0, \infty)^n$:*

(1) $F(\mathbf{a}) = \sum_{i=1}^n w_i a_i$ *with* $\max\{w_1, \ldots, w_n\} > 0$. *Notice that weighted arithmetic means, and thus the arithmetic mean, belong to this class of functions (see [28]).*

(2) $F(\mathbf{a}) = \max\{w_1 a_1, \ldots, w_n a_n\}$ *with* $\max\{w_1, \ldots, w_n\} > 0$.

(3) $F(\mathbf{a}) = \sum_{i=1}^n w_i a_{(i)}$ *with $w_i \geq w_j$ for $i < j$ and $max\{w_1, \ldots, w_n\} > 0$, where $a_{(i)}$ is the ith largest of the $a_1, \ldots, a_n$. Notice that OWA operators with decreasing weights belong to this class of functions (see [28, 78]).*

(4) $F(\mathbf{a}) = (\sum_{i=1}^n (w_i a)_i^p)^{\frac{1}{p}}$ *for all $p \in [1, \infty[$ with $max\{w_1, \ldots, w_n\} > 0$. Notice that root-mean-powers with $p \geq 1$ belong to this class of functions (see [28]).*

(5) $F(\mathbf{a}) = \min\{c, \sum_{i=1}^n w_i a_i\}$ *with* $\max\{w_1, \ldots, w_n\} > 0$ *and* $c \in (0, \infty)$.

(6) $F(\mathbf{a}) = \begin{cases} 0 & \text{if } \min\{a_1, \ldots, a_n\} = 0 \\ c & \text{otherwise} \end{cases}$ *with* $c \in (0, \infty)$.

It is worth mentioning that Aumann functions are instances of *qma*-functions. Let us recall that, according to [72], an Aumann function is a monotone and subadditive function $F : [0, \infty[^n \to [0, \infty[$ such that

$$F(a, 0, \dots, 0) = F(0, a, \dots, 0) = F(0, 0, \dots, 0, a) = a$$

for all $a \in [0, \infty)$. Clearly, by Theorem 1.1.9, positive Aumann functions are ma-functions. Let us recall that the concept of positive Aumann function was introduce in [59] by replacing the monotony and the subadditivity by positive monotony and positive subadditivity, respectivaly, in the definition of Aumann function. We can conclude that positive Aumann functions are not qma-functions. Indeed, it is easy to verify that the function provided in Example 3.1.10 below is a positive Aumann function which is not a qma-function.

In the light of Theorems 1.1.12, 1.1.14 and 3.1.6 we immediately obtain that every n-quasi-metric preserving function is a qma-function. Nevertheless, the converse is not true. Certainly the function F_s introduced in Example 3.1.1 satisfies all assumptions in the statement of Theorem 3.1.6 and, hence, it is a qma-function. However, $F_s(1, 0, \dots, 0) = 0$ and so F_s is not amenable. Consequently, Theorem 1.1.12 provides that F is not an n-quasi-metric preserving function.

We have pointed out before that each qma-function is a ma-function. The next example shows that the converse of such affirmation is not true, in general.

Example 3.1.8. *Consider the function $F : [0, \infty)^2 \to [0, \infty)$ given by*

$$F(x, y) = \begin{cases} 0 & \text{if } x = 0, y \in [0, 1[\\ y & \text{if } x = 0, y \in [1, \infty) \\ 0 & \text{if } x \in [0, 1[, y = 0 \\ 0 & \text{if } x \in [1, \infty), y = 0 \\ x + y & \text{if } x, y \in]0, \infty) \end{cases}.$$

It is clear that F verifies all conditions in statement of Theorem 1.1.9 and, thus, that F is an ma-function. Nevertheless, F does not satisfy condition (2.3) *in Theorem 3.1.6. Indeed,* $(1,0) \preceq (\frac{1}{2},0)+(\frac{1}{2},0)$ *but*

$$F(1,0) = 1 > 0 = F\left(\frac{1}{2},0\right) + \left(\frac{1}{2},0\right).$$

So, F is not a qma-function.

The following example shows another instance of an ma-function, which is not a qma-function.

Example 3.1.9. *Let* $F : [0,\infty[\to [0,\infty)$ *be the function given by* $F(0,0) = 0$ *and*

$$F(a,b) = \begin{cases} 2 & \text{if } first(a,b) \in]0,1[\\ 1 & \text{if } first(a,b) \in [1,\infty[\end{cases},$$

where $(a,b) \neq (0,0)$ *and* $first(a,b)$ *denotes the first value of* (a,b) *different from* 0*. According to [59], F is an ma-function. Clearly F is not positive monotone (and so it is not monotone) because* $(\frac{1}{2},\frac{1}{2}) \preceq (1,1)$ *but* $F(\frac{1}{2},\frac{1}{2}) = 2 > F(1,1) = 1$*. Thus, F does not satisfy condition* (3.2) *in Theorem 3.1.6 and, hence, F is not a qma-function.*

In view of the characterization of qma-functions provided by Theorem 3.1.6, one can wonder whether new equivalent conditions to those given in the aforementioned result are obtained when either the monotony is weakened to positive monotony or subadditivity is weakened to positive subadditivity and, in addition, the remaining conditions continue the same. Nonetheless, the answer to the posed question is negative such as the next examples show.

Example 3.1.10. *Consider* $F : [0,\infty[^2 \to [0,\infty)$ *given by*

$$F(x,y) = \begin{cases} y & \text{if } x = 0, y \in [0,\infty) \\ x & \text{if } x \in [0,\infty[, y = 0 \\ \frac{x+y}{2} & \text{if } x,y \in]0,\infty) \end{cases}.$$

It is not hard to check that F is positive monotone, subadditive and, in addition, it satisfies conditions (2.1) *and* (2.2) *in Theorem 3.1.6. However, F is not monotone, since $(0,7) \preceq (1,7)$ but $F(0,7) = 7 > 4 = F(1,7)$. Whence, by Theorem 3.1.6, we deduce that F is not a qma-function.*

Example 3.1.11. *Let $F : [0,\infty)^3 \to [0,\infty)$ be the function given by*

$$F(x,y,z) = \begin{cases} (x+y)^2 & \text{if } x,y \in \left[0,\frac{1}{2}\right[, z=0 \\ (x+z)^2 & \text{if } x,z \in \left[0,\frac{1}{2}\right[, y=0 \\ (y+z)^2 & \text{if } y,z \in \left[0,\frac{1}{2}\right[, x=0 \\ x+y+z & \text{otherwise} \end{cases}.$$

One can easily verify that F is monotone, positive subadditive and, in addition, it satisfies conditions (2.1) *and* (2.2) *in Theorem 3.1.6. However, F is not subadditive, since if we take $\mathbf{a} = \mathbf{b} = \left(\frac{1}{4},\frac{1}{4},0\right)$ we obtain*

$$F(\mathbf{a}+\mathbf{b}) = F\left(\frac{1}{2},\frac{1}{2},0\right) = \frac{1}{2}+\frac{1}{2} = 1 > \frac{1}{2} = \left(\frac{1}{4}+\frac{1}{4}\right)^2+\left(\frac{1}{4}+\frac{1}{4}\right)^2 = F(\mathbf{a})+F(\mathbf{b}).$$

Thus, F is not an ma-aggregation function by condition (3.3) *in Theorem 3.1.6.*

Taking into account the characterization, provided by Theorem 1.1.6, of n-metric preserving functions, it seems natural to discuss the relationship between this kind of functions and the *qma*-functions. In this direction, Example 3.1.9 provides a 2-metric preserving function (compare Example 8 in [58] and Example 10 in [59]) which is not a *qma*-function. Moreover, Example 3.1.1 gives an instance of *qma*-function that it not amenable and, hence, it is not metric preserving.

On account of [14], given a quasi-metric space (X, q), a quasi-metric q^{-1} and a metric d_q can be induced on X from q as follows: $q^{-1}(x, y) = q(y, x)$ and $d_q(x, y) = \max\{q(x, y), q(y, x)\}$ for all $x, y \in X$. Notice that, given a family of quasi-metrics $\{q_1, \ldots, q_n\}$ on X, one can get, on the one hand, the quasi-metric induced by aggregation of $\{q_1, \ldots, q_n\}$ and the quasi-metric induced by aggregation of $\{q_1^{-1}, \ldots, q_n^{-1}\}$. Moreover, every quasi-metric aggregation function is always a metric aggregation function and, thus, one can try to discern what is the relationship between the metric induced by the aggregation of the family $\{d_{q_1}, \ldots, d_{q_n}\}$ and the metric induced on X by the quasi-metric obtained via aggregation of $\{q_1, \ldots, q_n\}$. The next result clarify the posed questions.

Proposition 3.1.12. *Let $n \in \mathbb{N}$ and let $F : [0, \infty)^n \to [0, \infty)$ be an quasi-metric aggregation function. Then the following assertions hold for all $x, y \in X$:*

(1) $(Q_F^n)^{-1}(x, y) = F(q_1^{-1}(x, y), \ldots, q_n^{-1}(x, y))$.

(2) $d_{Q_F^n}(x, y) \leq F(d_{q_1}(x, y), \ldots, d_{q_n}(x, y)) \leq 2d_{Q_F^n}(x, y)$.

Proof. (1). Let $x, y \in X$. Then

$$(Q_F^n)^{-1}(x, y) = Q_F^n(y, x) =$$

$$F(q_1(y, x), \ldots, q_n(y, x)) = F(q_1^{-1}(x, y), \ldots, q_n^{-1}(x, y))$$

(1). Let $x, y \in X$. On the one hand, the monotony of F gives that

$$d_{Q_F^n}(x,y) = \max\{F(q_1(x,y),\ldots,q_n(x,y)), F(q_1^{-1}(x,y),\ldots,q_n^{-1}(x,y))\} \leq$$

$$F(d_{q_1}(x,y),\ldots,d_{q_n}(x,y)).$$

On the other hand, the subadditivity of F yields that

$$\begin{array}{l} F(d_{q_1}(x,y),\ldots,d_{q_n}(x,y)) \leq \\ F(q_1(x,y),\ldots,q_n(x,y)) + F(q_1^{-1}(x,y),\ldots,q_n^{-1}(x,y)), \end{array}$$

since $d_{q_i}(x,y) \leq q_i(x,y) + q_i^{-1}(x,y)$ for all $i \in \{1,\ldots,n\}$. Moreover, we have that

$$F(q_1(x,y),\ldots,q_n(x,y)) + F(q_1^{-1}(x,y),\ldots,q_n^{-1}(x,y)) \leq 2d_{Q_F^n}(x,y).$$

■

We end this section discussing another question that arises in a natural way. When we consider a quasi-metric space (X,q) and the family of quasi-metrics $\{q_1,\ldots,q_n\}$ on a non-empty set X such that $q_i = q$ for all $i \in \{1,\ldots,n\}$, then it seems interesting to know what is the relationship between the quasi-metric generated by aggregation of $\{q_1,\ldots,q_n\}$ and q. In order to give an answer to such a question, let us recall two appropriate notions following [28]. A function $F : [0,\infty)^n \to [0,\infty)$ has $p \in [0,\infty)$ as an idempotent element provided that $F(p,\ldots,p) = p$. As usual, we will say that F is idempotent if each element of $[0,\infty)$ is an idempotent element of F. Moreover, $F : [0,\infty)^n \to [0,\infty)$ is called internal if $\min\{a_1,\ldots,a_n\} \leq F(\mathbf{a}) \leq \max\{a_1,\ldots,a_n\}$ for all $\mathbf{a} \in [0,\infty)^n$.

The next theorem clarifies the issue under discussion.

Theorem 3.1.13. *Let $n \in \mathbb{N}$ and let $F : [0, \infty)^n \to [0, \infty)$ be a qma-function. Then the following assertions are equivalent:*

(1) F is internal.

(2) F is idempotent.

(3) $q(x, y) = F(q(x, y), \ldots, q(x, y))$ for every quasi-metric space (X, q) and for all $x, y \in X$.

(4) $d(x, y) = F(d(x, y), \ldots, d(x, y))$ for every metric space (X, d) and for all $x, y \in X$.

Proof. (1) $\Leftrightarrow$ (2)Theorem 3.1.6 gives that every quasi-metric aggregation function is monotone. Proposition 2.63 in [28] warranties that every monotone function is idempotent if and only if it is internal.

(2) $\Rightarrow$ (3) Let (X, q) be a quasi-metric space and $x, y \in X$. Obviously if F is idempotent, then $q(x, y) = F(q(x, y), \ldots, q(x, y))$.

(3) $\Rightarrow$ (4). Let (X, d) be a metric space and $x, y \in X$. Since every metric d is a quasi-metric we deduce that $d(x, y) = F(d(x, y), \ldots, d(x, y))$.

(4) $\Rightarrow$ (2). It is clear that every quasi-metric aggregation function is an metric aggregation function. According to [59, Theorem 19], a metric aggregation function such that $d(x, y) = F(d(x, y), \ldots, d(x, y))$ for every metric space (X, d) and for all $x, y \in X$ is indempotent. ■

3.2 Discarding functions as quasi-metric aggregation functions

In this section we explore a little more about *qma*-functions in such a way that a few useful methods to discard those functions that are useless as *qma*-functions are presented.

Next we discuss whether a *qma*-function can have absorbent elements. To this end, recall that, given $n \in \mathbb{N}$, a function $F : [0,\infty)^n \to [0,\infty)$ has $u \in [0,\infty)$ as an absorbent (or annihilator) element in its ith coordinate provided that

$$F(a_1, \ldots, a_{i-1}, u, a_{i+1}, \ldots, a_n) = u$$

for each $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in [0,\infty)$ (see [28]). The next results provide information about the matter under consideration when the function has idempotent elements.

Proposition 3.2.1. *Let $n \in \mathbb{N}$ and let $F : [0,\infty)^n \to [0,\infty)$ be a qma-function. Then F has not $u \in [0,\infty)$ as an absorbent element in at least two variables whenever F has an idempotent element $p \in (0,\infty)$ with $p > 2u$.*

Proof. Suppose, without loss of generality, that F has an absorbent element $u \in [0,\infty)$ in the first two variables. By Theorem 3.1.6 we have that F is subadditive. Then we deduce that

$$2u = F(u, p-u, p-u, \ldots, p-u) + F(p-u, u, u, \ldots, u) \geq F(p, \ldots, p) > 2u,$$

which is impossible. ■

As a consequence of the preceding result we obtain the following property.

Corollary 3.2.2. *Let $n \in \mathbb{N}$. If $F : [0,\infty)^n \to [0,\infty)$ is an idempotent qma-function, then F has not $u \in [0,\infty)$ as an absorbent element in at least two variables.*

The preceding results assure that those functions with an absorbent element in at least two variables are not useful to be *qma*-functions when idempotency is enjoyed.

On account of [59, Proposition 20], Proposition 3.2.1 and Corollary 3.2.2 are just true for *ma*-functions whenever $u \in (0,\infty)$. This fact shows, one more time, that *qma*-functions and *ma*-functions enjoy, in general, different properties.

In the particular case in which the quasi-metric aggregation is not idempotent we have that 0 can not become an absorbent element. Indeed, on the one hand, we have that every *qma*-function is always a (subaddtitive) metric aggregation function. On the other hand, in [59, Proposition 23] it was proved that every subadditive *ma*-function has not 0 as an absorbent element. This reasoning allows us to discard all functions having 0 as an absorbent element as quasi-metric aggregation functions. Moreover, in [59, Proposition 22], it was proved that every conjunctive metric aggregation function has 0 as an absorbent element in at least two variables. So we conclude that any *qma*-function cannot be conjunctive. Recall that, following [28], a function $F : [0,\infty)^n \to [0,\infty)$ is conjunctive whenever $F(\mathbf{a}) \leq \min\{a_1, \ldots, a_n\}$ for each $\mathbf{a} \in [0,\infty)^n$. This reasoning allows us to discard all conjunctive functions as *qma*-functions.

Now, we focus our attention on the analysis of the existence of neutral elements of *qma*-functions. According to [28], given $n \in \mathbb{N}$, a function $F : [0,\infty)^n \to [0,\infty)$ has $e \in [0,\infty)$ as a neutral element if $F(\mathbf{a_i e}) = a_i$ for each

$a_i \in [0, \infty[$ and each $i \in \{1, \ldots, n\}$, where $\mathbf{a_i e}$ denotes the element of $[0, \infty[^n$ that consist of a_i in the ith coordinate and e in the rest of coordinates. In [59, Proposition 26], it was proved that every subadditive metric aggregation function has not any neutral element in $(0, \infty)$. Whence we deduce that any *qma*-function has not neutral elements in $(0, \infty)$. Hence we can discard as *qma*-function all functions with neutral elements in $(0, \infty)$.

Notice that the unique possible neutral element of a quasi-metric aggregation function is 0. However, the class of those *qma*-functions with 0 as neutral element matches up with Aumann functions. Furthermore, in [59] it was proved that every subadditive and monotone *ma*-function F with 0 as neutral element is disjunctive, where, following [28], a function $F : [0, \infty)^n \to [0, \infty)$ is said to be disjunctive provided that $\max\{a_1, \ldots, a_n\} \leq F(\mathbf{a})$ for all $\mathbf{a} \in [0, \infty)^n$. Moreover, the aforementioned *ma*-functions are always majorized by a disjunctive function and, in addition, they always majorize a conjunctive function. In particular, $\frac{1}{n} \sum_{i=1}^{n} a_i \leq F(\mathbf{a}) \leq \sum_{i=1}^{n} a_i$ for all $\mathbf{a} \in [0, \infty)^n$. Consequently, every *qma*-function with 0 as neutral element enjoys all the previously indicated properties. So those functions with 0 as neutral element that do not satisfy any of the previously listed properties must be rejected as *qma*-function.

Finally, taking into account the exposed facts we have the next surprising result.

Proposition 3.2.3. *Let $n \in \mathbb{N}$ and let $F : [0, \infty)^n \to [0, \infty)$ be a qma-function with 0 as a neutral element. Then, F fulfills the following inequality for all $\mathbf{a}, \mathbf{b} \in [0, \infty)^n$:*

$$\max\{F(\mathbf{a}) - F(\mathbf{b}), 0\} \leq \sum_{i=1}^{n} \max\{a_i - b_i, 0\}. \tag{3.1}$$

Proof. Let $\mathbf{a}, \mathbf{b} \in [0, \infty)^n$ and take $\mathbf{c} \in [0, \infty)^n$ with $c_i = \max\{a_i - b_i, 0\}$ for each $i \in \{1, \ldots, n\}$. Then $\mathbf{a} \leq \mathbf{b} + \mathbf{c}$. Thus, by Theorem 3.1.6, we have that

$$F(\mathbf{a}) - F(\mathbf{b}) \leq F(\mathbf{c}) = F(\max\{a_1 - b_1, 0\}, \ldots, \max\{a_n - b_n, 0\}).$$

Since $F(\max\{a_1 - b_1, 0\}, \ldots, \max\{a_n - b_n, 0\}) \leq \sum_{i=1}^{n} \max\{a_i - b_i, 0\}$ we deduce that $F(\mathbf{a}) - F(\mathbf{b}) \leq \sum_{i=1}^{n} \max\{a_i - b_i, 0\}$. The last inequality implies the next one $\max\{F(a) - F(b), 0\} \leq \sum_{i=1}^{n} \max\{a_i - b_i, 0\}$. ■

Notice that inequality (3.1) is a quasi-metric Lipschitz condition with constant 1, since $q_l(a, b) = \max\{a - b, 0\}$ is a quasi-metric on $[0, \infty)$.

In the light of Proposition 3.2.3 those functions with 0 as a neutral element which do not satisfy inequality (3.1) cannot be selected as *qma*-function.

From Proposition 3.2.3 we get the following:

Corollary 3.2.4. *Let $n \in \mathbb{N}$ and let $F : [0, \infty)^n \to [0, \infty)$ be a qma-function with 0 as a neutral element. Then, F fulfills the following inequality for all $\mathbf{a}, \mathbf{b} \in [0, \infty)^n$:*

$$|F(\mathbf{a}) - F(\mathbf{b})| \leq \sum_{i=1}^{n} |a_i - b_i| \,. \tag{3.2}$$

Proof. By Proposition 3.2.3 we have that

$$\max\{F(\mathbf{a}) - F(\mathbf{b}), 0\} \leq \sum_{i=1}^{n} \max\{a_i - b_i, 0\}$$

and

$$\max\{F(\mathbf{a}) - F(\mathbf{b}), 0\} \leq \sum_{i=1}^{n} \max\{b_i - a_i, 0\}.$$

It follows that

$$|F(\mathbf{a}) - F(\mathbf{b})| = \max\{\max\{F(\mathbf{a}) - F(\mathbf{b}), 0\}, \max\{F(\mathbf{a}) - F(\mathbf{b}), 0\}\} \leq$$

$$\max\{\sum_{i=1}^{n} \max\{a_i - b_i, 0\}, \sum_{i=1}^{n} \max\{b_i - a_i, 0\}\} \leq$$

$$\sum_{i=1}^{n} |a_i - b_i| .$$

■

3.3 Two possible fields for applications

We end the paper describing two scenarios, that arise in applied fields, where the exposed theory could be helpful. Concretely we illustrate that in many cases the distances used in Asymptotic Complexity Analysis of algorithms and in Location Analysis can be constructed as an aggregation of quasi-metrics defined all of them on the same non-empty subset. So the use of quasi-metric aggregation functions could be useful to construct a mathematical framework under which many of the specific cases exposed in the literature can be unified under the same general framework and, in addition, the new approach could allows us to discern what method of quasi-metric aggregation is the most appropriate in each problem under study.

3.3.1 Asymptotic Complexity Analysis

In 2003, L.M. García-Raffi et al. introduced the theory of polynomial complexity spaces with the aim of developing a general mathematical framework suitable for asymptotic complexity analysis of algorithms ([21]). Let us recall that, fixed a polynomial $P(n)$ such that $P(n) > 0$ for all $n \in \mathbb{N}$, the polynomial complexity space is the quasi-metric space $(\mathcal{C}_{P(n)}, d_{\mathcal{C}_{P(n)}})$, where

$$\mathcal{C}_{P(n)} = \{f : \mathbb{N} \to [0, +\infty) : \sum_{n=1}^{+\infty} 2^{-P(n)} f(n) < +\infty\},$$

and the quasi-metric $d_{\mathcal{C}_{P(n)}}$ is given by

$$d_{\mathcal{C}_{P(n)}}(f, g) = \sum_{n=0}^{+\infty} 2^{-P(n)} \max\{g(n) - f(n), 0\}.$$

The utility of the polynomial complexity space in asymptotic complexity analysis of algorithms is based on the fact that the numerical value $d_{\mathcal{C}_{P(n)}}(f, g)$ can be understood as a measure of the progress made in lowering of complexity when an algorithm with running time represented by g is replaced by another one with running time represented by f. Indeed,

$$d_{\mathcal{C}_{P(n)}}(f, g) = 0 \Leftrightarrow g(n) \leq f(n) \text{ for all } n \in \mathbb{N}$$

and, hence, the running time of computing represented by g is more "efficient" than the algorithm whose running time of computing is represented by f on inputs size when $f \neq g$. Thus when the running time of computing of an algorithm, represented by g, is not known with precision, the fact that $d_{\mathcal{C}_{P(n)}}(f, g) = 0$ guarantees that f provides an asymptotic upper bound of g and, thus, that the algorithm under consideration will take at most $f(n)$ time, when the size of the input is n, to solve the problem for which it has been designed. Observe that the asymmetry of $d_{\mathcal{C}_{P(n)}}$ is crucial in order to get f as an asymptotic upper bound of g. Indeed, a pseudo-metric will be able to provide information about the efficiency but it would be useful to state which algorithm is more efficient of both.

The applicability of the complexity space $\mathcal{C}_{P(n)}$ to the asymptotic analysis of algorithms has been illustrated providing new techniques, by means of fixed point techniques, to specify asymptotic upper bounds for those algorithms whose running time of computing satisfies a recurrence equation. We refer the reader to [19, 52, 83, 84, 86] for a detailed treatment of the topic. In [20], a variant of the polynomial complexity space was introduced in order to provide a mathematical framework to perform an appropriate description of the running time of computing of exponential time algorithms and, thus, to develop suitable fixed point techniques to provide asymptotic upper bounds. In this case the new complexity space was called supremum polynomial complexity space and it was given as the quasi-metric space $(\mathcal{C}_{P(n)}, d_{\infty,\mathcal{C}_{P(n)}})$, where

$$d_{\infty,\mathcal{C}_{P(n)}}(f,g) = \sup_{n\in\mathbb{N}} \left\{ 2^{-P(n)} \max\{g(n) - f(n), 0\} \right\}.$$

According to [22], the asymptotic behaviour of the running time of algorithms can be discussed through finite approximations $d_{\mathcal{C}_{m,P(n)}}(f,g)$ of the numerical value $d_{\mathcal{C}_{P(n)}}(f,g)$, where

$$d_{\mathcal{C}_{m,P(n)}}(f,g) = \sum_{i=1}^{m} 2^{-P(n)} \max\{g(n) - f(n), 0\}$$

for any $m \in \mathbb{N}$. Observe that $d_{\mathcal{C}_{m,P(n)}}$ is a quasi-metric on $\mathcal{C}_{m,P(n)}$, where $\mathcal{C}_{m,P(n)}$ denotes the set all functions belonging to $\mathcal{C}_{P(n)}$ whose domain is restricted to $\{1, \ldots, m\}$. Moreover, notice that

$$d_{\mathcal{C}_{P(n)}}(f,g) = 0 \Leftrightarrow d_{\mathcal{C}_{m,P(n)}}(f,g) = 0 \text{ for all } m \in \mathbb{N}.$$

Therefore, the fixed point techniques developed to specify asymptotic upper bounds for those algorithms whose running time of computing satisfies a recurrence equation can be rewritten in terms of non-asymptotic criteria involving fixed point arguments based on the use of the quasi-metrics $d_{\mathcal{C}_{m,P(n)}}$. Of course, the same happens for the case of $d_{\infty,\mathcal{C}_{P(n)}}$, where now the finite approximations are yielded by the quasi-metrics $d_{\max,m,\mathcal{C}_{m,P(n)}}$ given by

$$d_{\mathcal{C}_{\max,m,P(n)}}(f,g) = \max_{0\leq i\leq m} \left\{2^{-P(n)} \max\{g(n) - f(n), 0\}\right\}.$$

It is clear that the quasi-metrics $d_{\mathcal{C}_{m,P(n)}}$ and $d_{\mathcal{C}_{\max,m,P(n)}}$ can be obtained by aggregation, through the *qma*-functions given by (1) and (2) in Example 3.1.7, of the family of quasi-metrics $\{q_1, \ldots q_m\}$ on $[0,\infty)$, where $q_i = q_u$ and $w_i = 2^{-P(i)}$ for all $i \in \{1,\ldots,m\}$.

In the light of the preceding fact, it seems natural to incorporate quasi-metric aggregation functions in the asymptotic complexity analysis of algorithms via developing general polynomial complexity spaces in such a way that the exposed mathematical frameworks can be retrieved as particular case and, in addition, with the aim of, on the one hand, exploring what quasi-metric aggregation function is the most appropriate for developing measures of the progress made in lowering of complexity and, on the other hand, developing non-asymptotic criteria fixed point methods best adapted to each family of recurrences that (may) arise in the study of algorithms.

3.3.2 Location Analysis

A location problem consists in looking for a new facilities of a company to provide a service for a set of customers. So this is a relevant topic in Logistics because of location of facilities and allocation of customers to the facilities provide constraints in the distribution process and its cost and efficiency. Thus the main problem in Location Analysis is a Decision Making problem in which the company wants to decide how to place new facilities, taking into account the customers allocation, in such a way that the facilities are placed in an optimum way, i.e., reducing the cost or maximizing the customer satisfaction. For a fuller treatment of the topic we refer the reader to [91].

Two typical problems in Location Analysis are the Weber problem (or minsum problem) and the Rawls problem (or minmax problem). In the first one, the target is to place n ($n \in \mathbb{N}$) facilities in n locations minimizing the global cost, which is usually described in terms of time, money, number of trips, etc. Of course, the demand of the customer is associated to each facility location and, thus, each location contributes to the objective in a different way or with a different weight. Examples where this kind of problem arises in a natural way are those where the facility to be located is a distribution center or a center for energy production. The target for the second problem is again to place n facilities but this time minimizing the maximum cost. Typical examples of this problem are those where the facility to be placed is an emergency service like fire or police station and an ambulance service.

From a mathematical viewpoint, the exposed problems can be stated as follows:

We have to choose a location for a facility among a collection of them X (the set of facilities X can be discrete or continuous) in such a way that the selection of a facility is influenced by the cost of the interaction between the facility and a collection of destinations (that represent the customers) normally finite $A = \{a_1, \dots, a_n\}$. The aim is to determine the location that minimizes the global cost interaction C. Normally, the cost is measured as a function of the distance between the facility and the destination. Hence, given a location $x \in X$, the interaction cost between x and destination i is provided as a function $c_i(x, a_i) = c_i(d(x, a_i))$. This kind of cost functions are known as transportation cost functions. According to [16, 69, 70, 71, 92] in many real problems the transportation cost depends on quasi-metrics. This is the case when we consider problems where there are involved one-way paths, rush-hour traffic, navigation in presence of wind, fuel cost, time travel, etc. Observe that the global transportation cost can be understood as a global distance from the facility to all destinations.

A typical cost is proportional to the distance, that is $c_i(x, a_i) = w_i q(x, a_i)$ for all $i \in \{1, \ldots, n\}$, and the global cost is obtained by means of aggregation of each individual transportation cost. Thus the problem under consideration is reduced to the following optimization problem:

$$Min_{x \in X} C(x),$$

where

$C(x) = \sum_{i=1}^{n} w_i q(x, a_i)$ for the Weber problem,

$C(x) = \max\{w_1 q(x, a_1), \ldots, w_n q(x, a_n)\}$ for the Rawls problem.

Notice that in the expression of the global transportation costs can be used, at the same time, different quasi-metrics depending on the nature of the cost under consideration and, thus, one would obtain:

$$Min_{x \in X} C(x),$$

where

$C(x) = \sum_{i=1}^{n} w_i q_i(x, a_i)$ for the Weber problem,

$$C(x) = \max\{w_1 q_1(x, a_1), \ldots, w_n q_n(x, a_n)\} \text{ for the Rawls problem.}$$

In [65], it has been pointed out that the both preceding problems are a particular case of a more general one where the aggregation of costs is made by means of an OWA operators. Thus a unified framework based on OWAs is presented and a deep discussion about how solve such problems is carried out in [65].

In the light of the exposed facts, it appears natural to consider *qma*-functions in Location Analysis with the aim of finding out what they can contribute to develop best adapted global transportation costs functions and new optimization criteria.

Chapter 4

On Matthews' relationship between quasi-metrics and partial metrics: an aggregation perspective

As explained before, J. Borsík and J. Doboš studied the problem of how to merge a family of metric spaces into a single one through a function. They called such functions metric preserving and provided a characterization of them in terms of the so-called triangle triplets. Since then, different papers have extended their study to the case of generalized metric spaces. Concretely, in 2010, G. Mayor and O. Valero provided two characterizations of those functions, called n-quasi-metric preserving functions, that allows us to merge a collection of quasi-metric spaces into a new one ([59]). In 2012, S. Massanet and O. Valero gave a characterization of the functions, called

n-partial metric preserving function, that are useful for merging a collection of partial metric spaces into single one as final output ([55]).

Inspired by the preceding work, in 2013, J. Martín, G. Mayor and O. Valero addressed the problem of constructing metrics from quasi-metrics, in a general way, using a class of functions that they called metric generating functions ([54]). In particular, they solved the posed problem providing a characterization of such functions and, thus, all ways under which a metric can be generated by a quasi-metric from an aggregation viewpoint. Following this idea, we propose the same problem in the framework of partial metric spaces. So, we characterize those functions that are able to generate a quasi-metric from a partial metric, and conversely, in such a way that Matthews' relationship between both type of generalized metrics is retrieved as a particular case. Moreover, we study if both, the partial order and the topology induced by a partial metric or a quasi-metric, respectively, are preserved by the new method in the spirit of Matthews. Furthermore, we discuss the relationship between the new functions and those aforesaid families introduced in the literature, i.e., n-metric preserving functions, n-quasi-metric preserving functions, n-partial metric preserving functions and metric generating functions.

4.1 A general method for generating quasi-metrics from partial metrics

This section is devoted to provide a general method to generate a quasi-metric from a partial metric in such a way that the technique introduced by Matthews in [56] can be retrieved as a particular case. Recall that, given a partial metric p on a non-empty set X, then a quasi-metric q_p can be induced on X by $q_p(x,y) = p(x,y) - p(x,x)$ for each $x, y \in X$. Moreover, a technique for the construction of a partial metric from a quasi-metric was also given in [56]. In order to introduce such a technique let us recall that a *weighted quasi-metric space* is a tern (X, q, w_q), where q is a quasi-metric on X and w_q is a function $w_q : X \to \mathbb{R}_+$ satisfying, for each $x, y \in X$, that

$$q(x,y) + w_q(x) = q(y,x) + w_q(y).$$

The mapping w_q is known as the weight function associated to the quasi-metric q. Thus, given a weighted quasi-metric space (X, q, w_q), a partial metric p_{q,w_q} on X can be defined, for each $x, y \in X$, by

$$p_{q,w_q}(x,y) = q(x,y) + w_q(x).$$

To get our proposed aim, $\mathbf{D}$ will denote the subset of $\mathbb{R}^2_+$ given by $\mathbf{D} = \{(a,b) \in \mathbb{R}^2_+ : a \geq b\}$.

The next notion will be crucial in order to get the solution to the posed problem from the aggregation perspective.

Definition 4.1.1. We will say that a function $\Phi : \mathbf{D} \to \mathbb{R}_+$ is a quasi-metric generating function (briefly, a *qmg*-function) if for each partial metric space (X, p) the function $q_{\Phi,p} : X \times X \to \mathbb{R}_+$ is a quasi-metric on X, where $q_{\Phi,p}(x,y) = \Phi(p(x,y), p(x,x))$ for each $x, y \in X$.

The next example shows that the Matthews' technique is a particular case of the exposed approach.

Example 4.1.2. *Let $\Phi_- : \mathbf{D} \to \mathbb{R}_+$ given by $\Phi_-(a,b) = a - b$. Then, Φ_- is a qmg-function. Indeed, given a partial metric space (X, p) we have that $q_{\Phi_-,p}(x,y) = p(x,y) - p(x,x)$ for each $x, y \in X$, which is the well-known weighted quasi-metric q_p induced by the partial metric p.*

The next example provides an alternative way of generating a quasi-metric from a partial metric which is based on the use of *qmg*-functions.

Example 4.1.3. *Let* $\Phi_{-,\frac{1}{2}} : \mathbf{D} \to \mathbb{R}_+$ *given by*

$$\Phi_{-,\frac{1}{2}}(x,y) = \begin{cases} 0, & \text{if } x = y = 0 \\ x - \frac{y}{2}, & \text{otherwise} \end{cases}.$$

Then, $\Phi_{-,\frac{1}{2}}$ *is a qmg-function. Indeed, given a partial metric space* (X,p)*, it is not hard to check that* $q_{\Phi_{-,\frac{1}{2}},p}$ *is a quasi-metric on* X *with* $q_{\Phi_{-,\frac{1}{2}},p}(x,y) = p(x,y) - \frac{p(x,x)}{2}$ *for each* $x,y \in X$ *with* $x \neq y$*, and* $q_{\Phi_{-,\frac{1}{2}},p}(x,x) = 0$ *for each* $x \in X$*.*

Proposition 4.1.7, below, also yields a way of building quasi-metrics from partial metrics which differs from the Matthews technique.

The next concept will be play a central role in order to characterize those functions $\Phi : \mathbf{D} \to \mathbb{R}_+$ which are *qmg*-functions.

Definition 4.1.4. We will say that $(x_1, x_2, x_3) \in \mathbb{R}^3_+$ is a quadrangular triplet on $(y_1, y_2, y_3) \in \mathbb{R}^3_+$ if the following conditions are satisfied:

(i) $x_1 \geq \max\{y_1, y_3\}$, with $x_1 > y_1$ or $x_1 > y_3$, and $x_1 + y_2 \leq x_2 + x_3$;

(ii) $x_2 \geq \max\{y_2, y_1\}$, with $x_2 > y_2$ or $x_2 > y_1$, and $x_2 + y_3 \leq x_3 + x_1$;

(iii) $x_3 \geq \max\{y_3, y_2\}$, with $x_3 > y_3$ or $x_3 > y_2$, and $x_3 + y_1 \leq x_1 + x_2$.

It is not hard to check that $(2, 1, 2)$ is a quadrangular triplet on $(0, 1, 2)$. Notice that $(0, 1, 2)$ is not a quadrangular triplet on $(2, 1, 2)$.

The next theorem provides a characterization of qmg-functions by means of quadrangular triplets and, thus, a general method to generate a quasi-metric from a partial metric.

Theorem 4.1.5. *Let $\Phi : \mathbf{D} \to \mathbb{R}_+$ be a function. Then the following assertions are equivalent:*

(1) *Φ is a qmg-function.*

(2) *Φ satisfies:*

(i) $\Phi^{-1}(0) = \{(x, y) \in \mathbf{D} : x = y\}$;

(ii) $\Phi(x_1, y_1) \leq \Phi(x_2, y_1) + \Phi(x_3, y_2)$, *whenever* $(x_1, x_2, x_3) \in \mathbb{R}^3_+$ *is a quadrangular triplet on* $(y_1, y_2, y_3) \in \mathbb{R}^3_+$.

Proof. (1) $\Rightarrow$ (2). Let $\Phi : \mathbf{D} \to \mathbb{R}_+$ be a qmg-function.

Next we show that Φ satisfies condition (i). Suppose that $\Phi(x, y) = 0$ for some $x, y \in \mathbf{D}$. Consider $(\mathbb{R}, p_y)$ the partial metric space where $p_y(a, b) = |a - b| + y$ for each $a, b \in \mathbb{R}$, where $\mathbb{R}$ stands for the real number set. Taking into account that Φ is a qmg-function, then $q_{\Phi,p_y} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ is a quasi metric on $\mathbb{R}$, where $q_{\Phi,p_y}(a, b) = \Phi(p_y(a, b), p_y(a, a))$ for each $a, b \in \mathbb{R}$.

Since $x \geq y$ we have that $p_y(y, x) = p_y(x, y) = |x - y| + y = x$. In addition, $p_y(x, x) = |x - x| + y = y$ and $p_y(y, y) = |y - y| + y = y$.

Attending to the above observations and taking into account our assumptions, we have that $q_{\Phi,p_y}(x, y) = \Phi(p_y(x, y), p_y(x, x)) = \Phi(x, y) = 0$. Moreover, $q_{\Phi,p_y}(y, x) = \Phi(p_y(y, x), p_y(y, y)) = \Phi(x, y) = 0$. Thus, $(QM1)$ implies $x = y$.

Next we show that Φ satisfies condition (ii). To this end, suppose that $(x_1, x_2, x_3) \in \mathbb{R}_+^3$ is a quadrangular triplet on $(y_1, y_2, y_3) \in \mathbb{R}_+^3$.

In the following we construct a partial metric space in order to show that $\Phi(x_1, y_1) \leq \Phi(x_2, y_1) + \Phi(x_3, y_2)$.

Let $X = \{a, b, c\}$ be a set of three points. We define $p : X \times X \to \mathbb{R}_+$ as follows:

$$p(a, c) = p(c, a) = x_1 \text{ and } p(a, a) = y_1;$$
$$p(a, b) = p(b, a) = x_2 \text{ and } p(b, b) = y_2;$$
$$p(b, c) = p(c, b) = x_3 \text{ and } p(c, c) = y_3.$$

It is not hard to check that (X, p) is a partial metric space, since (x_1, x_2, x_3) is a quadrangular triplet on (y_1, y_2, y_3).

By our hypothesis, $q_{\Phi,p} : X \times X \to \mathbb{R}_+$ is a quasi-metric, where $q_{\Phi,p}(u, v) = \Phi(p(u, v), p(u, u))$ for each $u, v \in X$. Then

$$q_{\Phi,p}(a, c) \leq q_{\Phi,p}(a, b) + q_{\Phi,p}(b, c),$$

which is equivalent to

$$\Phi(p(a, c), p(a, a)) \leq \Phi(p(a, b), p(a, a)) + \Phi(p(b, c), p(b, b)).$$

Therefore, by definition of p, we have that

$$\Phi(x_1, y_1) \leq \Phi(x_2, y_1) + \Phi(x_3, y_2).$$

(2) $\Rightarrow$ (1). Assume that $\Phi : \mathbf{D} \to \mathbb{R}_+$ is a function satisfying conditions (i) and (ii). Let (X, p) be a partial metric space, we will show that $q_{\Phi,p}$ is a quasi-metric on X, where $q_{\Phi,p}(x, y) = \Phi(p(x, y), p(x, x))$ for each $x, y \in X$.

First we suppose that $q_{\Phi,p}(x,y) = 0 = q_{\Phi,p}(y,x)$ for some $x, y \in X$. Then,

$$\Phi(p(x,y), p(x,x)) = q_{\Phi,p}(x,y) = 0,$$

and

$$\Phi(p(y,x), p(y,y)) = q_{\Phi,p}(y,x) = 0.$$

Condition (i) implies that $p(x,y) = p(x,x)$ and $p(y,x) = p(y,y)$. Taking into account that p is a partial metric on X we have that $p(x,y) = p(x,x) = p(y,y)$, and so $x = y$. Since Φ satisfies (i) we deduce that $q_{\Phi,p}(x,y) = 0 = q_{\Phi,p}(y,x)$ provided that $x = y$. Thus $q_{\Phi,p}$ satisfies axiom (QM1) of quasi-metrics.

It remains to prove that $q_{\Phi,p}$ fulfils the triangle inequality, i.e., axiom $(QM2)$ of quasi-metrics. With this aim, let $x, y, z \in X$. We sill show that $q_{\Phi,p}(x,z) \leq q_{\Phi,p}(x,y) + q_{\Phi,p}(y,z)$. Observe that the cases $x = y$, $y = z$ or $x = z$ are obvious. So, we assume that $x \neq y$, $x \neq z$ and $y \neq z$. In such a case we obtain:

$$p(x,z) > p(x,x) \text{ or } p(x,z) > p(z,z) \text{ and } p(x,z) \geq \max\{p(x,x), p(z,z)\};$$

$$p(x,y) > p(x,x) \text{ or } p(x,y) > p(y,y) \text{ and } p(x,y) \geq \max\{p(x,x), p(y,y)\};$$

$$p(y,z) > p(y,y) \text{ or } p(y,z) > p(z,z) \text{ and } p(y,z) \geq \max\{p(y,y), p(z,z)\}.$$

Moreover, by axiom $(P4)$ of partial metrics, we have that

$$p(x,z) + p(y,y) \leq p(x,y) + p(y,z);$$

$$p(x,y) + p(z,z) \leq p(x,z) + p(z,y);$$

$$p(y,z) + p(x,x) \leq p(y,x) + p(x,z).$$

Then, $(p(x,z), p(x,y), p(y,z)) \in \mathbb{R}^3_+$ is quadrangular triplet on $(p(x,x), p(y,y), p(z,z)) \in \mathbb{R}^3_+$. Thus, by condition (ii), we have that

$$\Phi(p(x,z), p(x,x)) \leq \Phi(p(x,y), p(x,x)) + \Phi(p(y,z), p(y,y)),$$

and so

$$q_{\Phi,p}(x,z) \leq q_{\Phi,p}(x,y) + q_{\Phi,p}(y,z).$$

■

According to [26], given a quasi-metric sapce (X,q), then q induces a partial order $\preceq_q$ on X given by $x \preceq_q y \Leftrightarrow q(x,y) = 0$. In [56], Matthews showed that given a partial metric space (X,p), then p also induces a partial order $\preceq_p$ on X given by $x \preceq_p y \Leftrightarrow p(x,y) = p(x,x)$. Moreover, in the same reference, it was proved that $\preceq_{q_p} = \preceq_p$.

In the light of the preceding facts, it seems natural to discuss whether, given a *qmg*-function $\Phi : \mathbf{D} \to \mathbb{R}_+$ and a partial metric space (X,p), the partial orders $\preceq_{q_{\Phi,p}}$ and $\preceq_p$ are exactly the same on X, i.e., whether a *qmg*-function preserves the order induced by the partial metric that it transforms. The next result gives a positive answer to the questions under consideration.

Proposition 4.1.6. *Let $\Phi : \mathbf{D} \to \mathbb{R}_+$ be a qmg-function and let (X,p) be a partial metric space. Then $\preceq_{q_{\Phi,p}} = \preceq_p$.*

Proof. Let $x, y \in X$ .On the one hand, we have that $x \preceq_p y \Leftrightarrow p(x,x) = p(x,y)$. On the other hand, we have that $x \preceq_{q_{\Phi,p}} y \Leftrightarrow q_{\Phi,p}(x,y) = 0$. Theorem 4.1.5 guarantees that $\Phi^{-1}(0) = \{(x,y) \in \mathbf{D} : x = y\}$ and, thus, that $x \preceq_p y \Leftrightarrow x \preceq_{q_{\Phi,p}} y$ as claimed. ■

Following [26], each quasi-metric q on X induces a T_0 topology $\tau(q)$ on X which has as a base the family of open balls $\{B_q(x;\epsilon) : x \in X, \epsilon > 0\}$, where $B_q(x;\epsilon) = \{y \in X : q(x,y) < \epsilon\}$. Moreover, according to [56], each partial metric p on X induces a T_0 topology $\tau(p)$ on X which has as a base the family of open balls $\{B_p(x;\epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x;\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$. As in the case of the partial order, Matthews proved that the topology induced by a partial metric p and by the associated quasi-metric q_p coincide, i.e., $\tau(p) = \tau(q_p)$.

Similarly to the partial order case, it seems natural to ask if the same situation happens in our context, i.e., if a *qmg*-function Φ preserves the topology induced by the partial metric p which it transforms and, thus, $\tau(p) = \tau(q_{\Phi,p})$. Nevertheless, the behaviour of *qmg*-functions regarding the preservation of the topology is slightly different. In fact, the answer to the question posed is negative such as Example 4.1.8 reveals.

The next proposition will be crucial to show, by means of Example 4.1.8, that the topology induced by a partial metric p on a set X does not coincide, in general, with the topology induced by the generated quasi-metric $q_{\Phi,p}$.

Proposition 4.1.7. *Let $\Phi_2 : \mathbf{D} \to \mathbb{R}_+$ be the function given by $\Phi(0,0) = 0$ and $\Phi_2(x,y) = \frac{x-y}{x}$ for each $(x,y) \in \mathbf{D}\backslash\{(0,0)\}$. Then Φ_2 is a qmg-function.*

Proof. First of all, we observe that $\Phi_2(x,y) \geq 0$, for each $(x,y) \in \mathbf{D}$. Now, we show that Φ satisfies conditions (i) and (ii) in the statement of Theorem 4.1.5.

Clearly, by definition, $\Phi(0,0) = 0$. Next suppose that $\Phi_2(x,y) = 0$ for some $(x,y) \in \mathbf{D} \setminus \{(0,0)\}$. Then $\frac{x-y}{x} = 0$. The last equality is held if and only if $x = y$. Thus the aforesaid condition (i) is satisfied by Φ.

In order to prove that Φ fulfils condition (ii), assume that $(x_1, x_2, x_3) \in \mathbb{R}^3_+$ is a quadrangular triplet on $(y_1, y_2, y_3) \in \mathbb{R}^3_+$. It remains to show that $\Phi_2(x_1, y_1) \leq \Phi_2(x_2, y_1) + \Phi_2(x_3, y_2)$. First, observe that, by definition of quadrangular triplet $(x_1, x_2, x_3) \neq (0,0,0)$. So, $\Phi_2(x_1, y_1) = \frac{x_1-y_1}{x_1}$, $\Phi_2(x_2, y_1) = \frac{x_2-y_1}{x_2}$ and $\Phi_2(x_3, y_2) = \frac{x_3-y_2}{x_3}$. Now, we distinguish two possible cases:

Case 1. $x_1 \geq \max\{x_2, x_3\}$. On the one hand, we have that

$$\Phi_2(x_1, y_1) = \frac{x_1 - y_1}{x_1} \leq \frac{x_1 - y_1}{\max\{x_2, x_3\}}$$

and

$$\Phi_2(x_2, y_1) + \Phi_2(x_3, y_2) = \frac{x_2 - y_1}{x_2} + \frac{x_3 - y_2}{x_3} \geq \frac{x_2 - y_1}{\max\{x_2, x_3\}} + \frac{x_3 - y_2}{\max\{x_2, x_3\}}.$$

On the other hand $x_1 - y_1 \leq x_2 - y_1 + x_3 - y_2$, since (x_1, x_2, x_3) is a quadrangular triplet on (y_1, y_2, y_3). It follows that

$$\frac{x_1 - y_1}{\max\{x_2, x_3\}} \leq \frac{x_2 - y_1}{\max\{x_2, x_3\}} + \frac{x_3 - y_2}{\max\{x_2, x_3\}}$$

and, hence, that

$$\Phi_2(x_1, y_1) \leq \Phi_2(x_2, y_1) + \Phi(x_3, y_2).$$

Case 2. $x_1 < \max\{x_2, x_3\}$. Put $x_1' = \max\{x_2, x_3\}$. It is a routine to check that (x_1', x_2, x_3) is a quadrangular triplet on (y_1, y_2, y_3), since $x_1' > x_1 \geq \max\{y_1, y_3\}$ and $x_1' + y_2 \leq x_2 + x_3$. Therefore, by Case 1, we deduce that $\Phi_2(x_1', y_1) \leq \Phi_2(x_2, y_1) + \Phi_2(x_3, y_2)$. Moreover, we observe that, for each $a, b, \alpha \in \mathbb{R}_+$ with $a \leq b$, we have that $\frac{a}{b} \leq \frac{a+\alpha}{b+\alpha}$. Letting $\alpha = \max\{x_2, x_3\} - x_1$ and $x_1' = \max\{x_2, x_3\}$ we obtain that

$$\Phi_2(x_1, y_1) = \frac{x_1 - y_1}{x_1} \leq \frac{x_1 - y_1 + \alpha}{x_1 + \alpha} = \frac{\max\{x_2, x_3\} - y_1}{\max\{x_2, x_3\}} = \Phi_2(x_1', y_1).$$

Thus, $\Phi_2(x_1, y_1) \leq \Phi_2(x_2, y_1) + \Phi_2(x_3, y_2)$.

Consequently Φ fulfils condition (ii) in Theorem 4.1.5. Hence, we deduce that Φ is a *qmg*-function. ■

As announced before, the next example shows that, in general, a *qmg*-function Φ does not preserve the topology induced by the partial metric p which it transforms and, thus, that $\tau(p) \neq \tau(q_{\Phi,p})$ in general.

Example 4.1.8. *Let* $(\mathbb{R}_+, p_m)$ *be the partial metric space such that* $p_m(x, y) = \max\{x, y\}$ *for each* $x, y \in \mathbb{R}_+$. *It is not hard to verify that, for each* $x \in \mathbb{R}_+$ *and* $\epsilon > 0$, *the open ball centred at* x *with radius* ϵ *is given by* $B_{p_m}(x; \epsilon) = [0, x + \epsilon[$.

It is clear that the quasi-metric generated by means of the function Φ_2 *(introduced in Proposition 4.1.7) from* p_m *is given by* $q_{\Phi_2,p_m}(0, 0) = 0$ *and* $q_{\Phi_2,p_m}(x, y) = \frac{\max\{x,y\}-x}{\max\{x,y\}}$ *for each* $(x, y) \in \mathbf{D} \setminus \{(0, 0)\}$. *Then* $q_{\Phi_2,p_m}(0, y) = 1$, *for each* $y \in]0, \infty[$ *and, hence,* $B_{q_{\Phi_2,p_m}}(0; \epsilon) = \{0\}$ *for each* $\epsilon \in]0, 1[$. *Hence,* $B_{q_{\Phi_2,p_m}}(0; \epsilon) \notin \tau(p_m)$ *and, therefore,* $\tau(p_m) \neq \tau(q_{\Phi_2,p_m})$.

On account of the above example, we focus our effort on seeking conditions on *qmg*-functions Φ in order to ensure that, for each partial metric space (X, p), $\tau(p) = \tau(q_{\Phi,p})$. To this end, we introduce the next concept.

Definition 4.1.9. Let $\Phi : \mathbf{D} \to \mathbb{R}_+$ be a *qmg*-function. We will say that Φ is a strongly quasi-metric generating function (*sqmg*-function for short) if for each partial metric space (X, p) we have that $\tau(p) = \tau(q_{\Phi,p})$.

It must be stressed that the name of strongly quasi-metric generating function has been inspired by strongly metric preserving functions introduced in [15].

In the light of the preceding definition, an instance of *sqmg*-functions is given by the function Φ_- introduced in Examples 4.1.2.

The next result will be essential for getting a characterization of those *qmg*-functions that are *sqmg*-functions.

Lemma 4.1.10. *Let* $\Phi : \mathbf{D} \to \mathbb{R}_+$ *be a qmg-function. Then,* Φ *is monotone in the first component, i.e.,* $\Phi(x, z) \geq \Phi(y, z)$ *whenever* $x \geq y$.

Proof. Let $(x, z), (y, z) \in D$ with $x \geq y$. It is clear that if $x = y$, then $\Phi(x, z) = \Phi(y, z)$. So we assume that $x > y$. Moreover, we can consider that $y > 0$ because otherwise we have that $y = z = 0$ and, thus, that $\Phi(x, z) \geq \Phi(y, z) = \Phi(0, 0) = 0$. Consider the terns $(y, x, x - y), (z, x - y, 0) \in \mathbb{R}_+^3$. It is not hard to see that $(y, x, x - y)$ is a quadrangular triplet on $(z, x - y, 0)$. Therefore, by Theorem 4.1.5, we have that

$$\Phi(y, z) \leq \Phi(x, z) + \Phi(x - y, x - y) = \Phi(x, z).$$

Thus, Φ is monotone in the first component.

■

Although *qmg*-functions do not preserve the topology of the partial metric that they transform, we have always that the topology induced by the quasi-metric $q_{\Phi,p}$ generated from a *qmg*-function Φ is finer than the topology induced by the partial metric p from which it is constructed. Let us recall a topology τ_1 is said to be finer than a topology τ_2 provided that each open set in τ_2 it is so in τ_1 (see, for instance, [1]). From now on, the fact that a topology τ_1 is finer that a topology τ_2 will be denoted by $\tau_2 \subseteq \tau_1$.

Theorem 4.1.11. *Let $\Phi : \mathbf{D} \to \mathbb{R}_+$ be a qmg-function and let (X, p) be a partial metric space. Then, $\tau(p) \subseteq \tau(q_{\Phi,p})$.*

Proof. Consider $x \in X$ and the real number $\epsilon > 0$. Put $\delta = \Phi(p(x, x) + \epsilon, p(x, x))$. Then $\delta > 0$. Next we prove that $B_{q_{\Phi,p}}(x; \delta) \subseteq B_p(x; \epsilon)$. Indeed, let $y \in B_{q_{\Phi,p}}(x; \delta)$. Then $\Phi(p(x, y), p(x, x)) < \delta = \Phi(p(x, x) + \epsilon, p(x, x))$. Assume for the purpose of contradiction that $p(x, y) - p(x, x) \geq \epsilon$. By Lemma 4.1.10 we obtain that $\Phi(p(x, x) + \epsilon, p(x, x)) \leq \Phi(p(x, y), p(x, x)) < \Phi(p(x, x) + \epsilon, p(x, x))$ which is not possible. It follows that $p(x, y) - p(x, x) < \epsilon$ and, hence, that $y \in B_p(x; \epsilon)$. Therefore, $\tau(p) \subseteq \tau(q_{\Phi,p})$ as claimed. ■

The next theorem characterizes *sqmg*-functions.

Theorem 4.1.12. *Let* $\Phi : \mathbf{D} \to \mathbb{R}_+$ *be a qmg-function. Then the following assertions are equivalent:*

(1) Φ *is a sqmg-function.*

(2) For each $a \in \mathbb{R}_+$*, the function* $\Phi_a : \mathbb{R}_+ \to \mathbb{R}_+$ *is continuous at* 0*, where* $\Phi_a(x) = \Phi(x + a, a)$ *for each* $x \in \mathbb{R}_+$*.*

Proof. (1) $\Rightarrow$ (2). Suppose for the purpose of contradiction that there exists $a_0 \in \mathbb{R}_+$ such that Φ_{a_0} is not continuous at 0. Next we see that Φ is not a *sqmg*-function.

Since Φ_{a_0} is not continuous at 0, then there exist $\epsilon_0 > 0$ such that for each $\delta > 0$ we can find $x_\delta \in [0, \delta[$ satisfying $\Phi_{a_0}(x_\delta) \geq \epsilon_0$ (observe that $\Phi_{a_0}(0) = 0$).

Consider the partial metric space $(\mathbb{R}_+, p_m)$ introduced in Example 4.1.8 and take $x = a_0$. Clearly, for each $\delta > 0$ we have that $y_\delta = x_\delta + a_0 \notin B_{q_{\Phi,p_m}}(x; \epsilon_0)$, since

$$q_{\Phi,p_m}(x, y_\delta) = \Phi(p_m(x, y_\delta), p_m\{x, x\}) = \Phi(x_\delta + a_0, a_0) = \Phi_{a_0}(x_\delta) \geq \epsilon_0.$$

It follows that $\tau(p) \neq \tau(q_{\Phi,p})$ because $B_{q_{\Phi,p_m}}(x; \epsilon_0) \in \tau(q_{\Phi,p})$ and, in addition, $B_{q_{\Phi,p_m}}(x; \epsilon_0) \notin \tau(p)$. Indeed, for each $\delta > 0$, we have that $p_m(x, y_\delta) - p_m(x, x) = x_\delta + a_0 - a_0 = x_\delta < \delta$ and, thus, $y_\delta \in B_{p_m}(x; \delta)$ but $y_\delta \notin B_{q_{\Phi,p_m}}(x; \epsilon_0)$. Consequently, Φ is not a *sqmg*-function which is a contradiction.

(2) $\Rightarrow$ (1). By Theorem 4.1.11 we have, for each partial metric space (X, p), that $\tau(p) \subseteq \tau(q_{\Phi,p})$. It remains to prove that $\tau(q_{\Phi,p}) \subseteq \tau(p)$. With

this aim we show that, for each $x \in X$, given $\epsilon > 0$ there exists $\delta > 0$ such that $B_p(x;\delta) \subseteq B_{q_{\Phi,p}}(x;\epsilon)$. Indeed, by the continuity of $\Phi_{p(x,x)}$ at 0, there exists $\delta > 0$ such that, for each $\alpha \in [0,\delta[$, we have that $\Phi_{p(x,x)}(\alpha) < \epsilon$. It follows that $B_p(x;\delta) \subseteq B_{q_{\Phi,p}}(x;\epsilon)$. Hence if $y \in B_p(x;\delta)$, then $p(x,y) - p(x,x) < \delta$ and so $p(x,y) - p(x,x) \in [0,\delta[$. Whence, $q_{\Phi,p}(x,y) = \Phi(p(x,y),p(x,x)) = \Phi_{p(x,x)}(p(x,y) - p(x,x)) < \epsilon$ and, therefore, $y \in B_{q_{\Phi,p}}(x;\epsilon)$. This ends the proof. ■

As a consequence of the previous theorem, we can show that the function $\Phi_{-,\frac{1}{2}}$ introduced in Example 4.1.3 also constitutes an instance of *sqmg*-function. Indeed, on the one hand,

$$\Phi_{\left(-,\frac{1}{2}\right)_0}(x) = \begin{cases} 0, & \text{if } x = 0 \\ \frac{x}{2}, & \text{otherwise} \end{cases},$$

and, on the other hand, for each $a \in]0,\infty[$, $\Phi_{\left(-,\frac{1}{2}\right)_a}(x) = x + \frac{a}{2}$. Then, for each $a \in \mathbb{R}_+$, we have that $\Phi_{\left(-,\frac{1}{2}\right)_a}$ is continuous at 0 and so, by Theorem 4.1.12, we conclude that $\Phi_{-,\frac{1}{2}}$ is a *sqmg*-function. However, the function Φ_2, given in Proposition 4.1.7, is not a *sqmg*-function because Φ_{2_0} is not continuous at 0.

We finish this section exploring the relationship between *qmg*-functions, n-(quasi-)metric preserving functions, n-partial metric preserving functions and metric generating functions, whenever all of them are defined on $\mathbf{D}$.

Recall that Theorems 1.1.6 and 1.1.12 states that 2-metric preserving functions and 2-quasi-metric preserving functions belong to $\mathcal{O}$. Proposition 4.1.7 shows that there are *qmg*-functions that are neither 2-metric preserving functions nor 2-quasi-metric preservings functions because $\Phi_2 \notin \mathcal{O}$. By Theorem 1.1.20 we know that metric generating functions also belong to $\mathcal{O}$, and thus, Proposition 4.1.7 gives an instance, namely Φ_2, of *qmg*-functions which does not belong to $\mathcal{O}$ and, in addition, it is not a 2-metric generating function. By Corollary 1.1.15, 2-partial metric preserving functions are monotonemonotone and, therefore, qmg-functions are not, in general, 2-partial metric preserving functions. Indeed, the aforesaid mapping Φ_2 is not monotone, since $(2,1) \preceq (2,2)$ and $\frac{1}{2} = \Phi_2(2,1) \not\leq \Phi_2(2,2) = 0$.

Reciprocally we analyze if 2-(quasi-)metric preserving functions, 2-partial metric preserving functions and metric generating functions are qmg-functions.

The next example shows that there are 2-metric and 2-quasi-metric preserving functions that are not qmg-functions.

Example 4.1.13. *Consider the function* $\Phi_{0,1} : \mathbf{D} \to \mathbb{R}_+$ *defined by*

$$\Phi_{0,1}(a,b) = \begin{cases} 0 & \text{if } a = b = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Since $\Phi_{0,1}$ *is monotone, subadditive and* $\Phi_{0,1} \in \mathcal{O}$ *we have, by Theorems 1.1.6 and 1.1.12, that it is a 2-metric and 2-quasi-metric preserving function. However,* $\Phi_{0,1}(1,1) = 1$ *and thus, by Theorem 4.1.5, it is not a qmg-function.*

Next we show that there are 2-partial metric preserving functions that are not qmg-functions.

Example 4.1.14. *Consider the function* $\Phi_{+1} : \mathbf{D} \to \mathbb{R}_+$ *defined by* $\Phi_{+1}(a,b) = a+b+1$ *for all* $(a,b) \in \mathbf{D}$*. It is clear that the function* Φ_{+1} *fulfills conditions in the statement of Theorem 1.1.17 and, thus, it is a 2-partial metric preserving function. Nevertheless,* $\Phi_{+1}(1,1) = 3$ *and thus, by Theorem 4.1.5, it is not a qmg-function.*

The next example gives an instance of 2-metric generating function which is not a qmg-function.

Example 4.1.15. *Consider the function* $\Phi_{med} : \mathbf{D} \to \mathbb{R}_+$ *given by* $\Phi_{med}(a,b) = \frac{a+b}{2}$ *for all* a, b. *It is not hard to check that* Φ_{med} *satisfies Theorem 1.1.20 and, hence, that it is a* 2*-metric generating function. However,* $\Phi_{med}(1,1) = \frac{1}{2}$ *and, therefore, by Theorem 4.1.5 it is not a qmg-function.*

Despite the above exposed facts, surprisingly, qmg-functions can be useful to construct (quasi-)metric preserving functions, i.e., one dimensional n-(quasi-)metric preserving functions, such as the next result shows.

Proposition 4.1.16. *Let* $\Phi : \mathbf{D} \to \mathbb{R}_+$ *be a qmg-function. Then, for each* $a \in \mathbb{R}_+$, *the function* $\Phi_a : \mathbb{R}_+ \to \mathbb{R}_+$ *is a quasi-metric preserving function, where* $\Phi_a(x) = \Phi(x+a, a)$ *for each* $x \in \mathbb{R}_+$

Proof. Fix $a \in \mathbb{R}_+$. Since $\Phi : \mathbf{D} \to \mathbb{R}_+$ is a qmg-function we have that $\Phi_a^{-1}(0) = \{0\}$ Lemma 4.1.10 ensures that Φ_a is monotone. So, we only need to show that Φ_a is also subadditive. To this end, consider $x, y \in \mathbb{R}_+$ with $x \neq 0$ and $y \neq 0$ (the case $x = 0$ or $y = 0$ is obvious). It is not hard to check that $(x+y+a, x+a, y+a)$ is a quadrangular triplet on (a,a,a). Then, by Theorem 4.1.5, we deduce that

$$\Phi_a(x+y) = \Phi(x+y+a, a) \leq \Phi(x+a, a) + \Phi(y+a, a) = \Phi_a(x) + \Phi_a(y).$$

Thus, by Theorem 1.1.12, we obtain that Φ_a is a quasi-metric preserving function. ■

Since every quasi-metric preserving function is a metric preserving function (see Theorems 1.1.6 and 1.1.12) we obtain immediately from Proposition 4.1.16 the next consequence.

Corollary 4.1.17. *Let* $\Phi : \mathbf{D} \to \mathbb{R}_+$ *be a qmg-function. Then, for each* $a \in \mathbb{R}_+$, *the function* $\Phi_a : \mathbb{R}_+ \to \mathbb{R}_+$ *is a metric preserving function, where* $\Phi_a(x) = \Phi(x+a, a)$ *for each* $x \in \mathbb{R}_+$

Examples 4.1.2 and 4.1.3 provide instances of *qmg*-functions that, by Proposition 4.1.16 and Corollary 4.1.17, allow us to induce (quasi-)metric preserving functions.

Observe that Proposition 4.1.16 does not allows us to generate, in general, partial metric preserving functions (one dimensional n-partial metric preserving functions). To clarify this assertion it is sufficient that we consider, again, the function Φ_2 given in Proposition 4.1.7. Then $\Phi_{2_0}(x) = 1$ for all $x \in \mathbb{R}_+$. By Theorem 1.1.17 we have that Φ_{2_0} is not a partial metric preserving function because $1 \leq 3$, $2 \leq 3$ and $\Phi_{2_0}(3) = \Phi_{2_0}(1) = \Phi_{2_0}(2)$.

4.2 A general method for generating partial metrics from quasi-metrics

The aim of this section is to introduce a general method to generate a partial metric from a quasi-metric in such a way that the technique introduced by Matthews can be recovered as a particular case. With this objective, we introduce the next concept which will be essential for tackling the posed problem.

Definition 4.2.1. We will say that a function $\Psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a partial metric generating function (briefly, *pmg*-function) if for each weighted quasi-metric space (X, q, w_q) the function $p_{\Psi,q,w_q} : X \times X \to \mathbb{R}_+$ is a partial metric on X, where $p_{\Psi,q,w_q}(x, y) = \Psi(q(x, y), w_q(x))$ for each $x, y \in X$.

The Matthews technique is a particular case of the exposed approach, as shows the next example.

Example 4.2.2. *Let* $\Psi_+ : \mathbb{R}_+^2 \to \mathbb{R}_+$ *given by* $\Psi_+(a, \alpha) = a + \alpha$ *for each*

$a, \alpha \in \mathbb{R}_+$. Then, Ψ_+ is a pmg-function. Indeed, given a weighted quasi-metric space (X, q, w_q) we have that $p_{\Psi_+,q,w_q}(x, y) = q(x, y) + w_q(x)$ for each $x, y \in X$, which is the well-known partial metric p_{q,w_q} induced by the weighted quasi-metric space q.

The next example provides an alternative way of generating a partial metric from a quasi-metric which is based on the use of partial metric generating functions.

Example 4.2.3. *Define $\Psi_{+1} : \mathbb{R}_+^2 \to \mathbb{R}_+$ by $\Psi_{+1}(a, \alpha) = a + \alpha + 1$. It is not hard to check that Ψ_{+1} is a pmg-function. Indeed, given a weighted quasi-metric space (X, q, w_q), an easy verification shows that p_{Ψ_{+1},q,w_q} is a partial metric on X with $p_{\Psi_{+1},q,w_q}(x, y) = q(x, y) + w_q(x) + 1$ for each $x, y \in X$.*

The so-called upper quasi-metric space will be crucial in order to achieve our target. Although previously introduced, let us recall that the upper quasi-metric space is the weighted quasi-metric space given by the tern $(\mathbb{R}_+, q_u, w_{q_u})$ such that $q_u(x, y) = \max\{y - x, 0\}$ for each $x, y \in X$ and $w_{q_u}(x) = x$ for each $x \in \mathbb{R}_+$.

The next result will be crucial in order to yield a characterization of *pmg*-functions later on.

Lemma 4.2.4. *Let $\Psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a pmg-function and $\alpha \in \mathbb{R}_+$. Then Ψ is monotone in the first component.*

Proof. Fix $\alpha \in \mathbb{R}_+$. Since Ψ is a *pmg*-function we have that, given $(\mathbb{R}_+, q_u, w_{q_u})$, the function $p_{\Psi,q_u,w_{q_u}}$ is a partial metric on $\mathbb{R}_+^2$. Let $a, b \in \mathbb{R}_+$ and assume that $a \leq b$. Then, $q_u(\alpha, a+\alpha) = a$, $q_u(\alpha, \alpha) = 0$, $q_u(\alpha, b+\alpha) = b$ and $q_u(b + \alpha, a + \alpha) = 0$. So, on the one hand, we have that

$$\Psi(a,\alpha)+\Psi(0,b+\alpha)=$$

$$\Psi(q_u(\alpha,a+\alpha),w_{q_u}(\alpha))+\Psi(q_u(b+\alpha,b+\alpha),w_{q_u}(b+\alpha))=$$

$$p_{\Psi,q_u,w_{q_u}}(\alpha,a+\alpha)+p_{\Psi,q_u,w_{q_u}}(b+\alpha,b+\alpha).$$

On the other hand, we have that

$$\Psi(b,\alpha)+\Psi(0,b+\alpha)=$$

$$\Psi(q_u(\alpha,b+\alpha),w_{q_u}(\alpha))+\Psi(q_u(b+\alpha,a+\alpha),w_{q_u}(b+\alpha))=$$

$$p_{\Psi,q_u,w_{q_u}}(\alpha,b+\alpha)+p_{\Psi,q_u,w_{q_u}}(b+\alpha,a+\alpha).$$

Now, since $p_{\Psi,q_u,w_{q_u}}$ is a partial metric on $\mathbb{R}_+$ we have that

$$p_{\Psi,q_u,w_{q_u}}(\alpha,a+\alpha)+p_{\Psi,q_u,w_{q_u}}(b+\alpha,b+\alpha)\leq p_{\Psi,q_u,w_{q_u}}(\alpha,b+\alpha)+p_{\Psi,q_u,w_{q_u}}(b+\alpha,a+\alpha).$$

It follows that

$$\Psi(a,\alpha)+\Psi(0,b+\alpha)\leq\Psi(b,\alpha)+\Psi(0,b+\alpha).$$

This last inequality implies $\Psi(a,\alpha)\leq\Psi(b,\alpha)$, as we claimed. ■

The next theorem provides a characterization of the class of pmg-functions and, thus, a general method to generate a partial metric from a (weighted) quasi-metric.

Theorem 4.2.5. *Let $\Psi:\mathbb{R}_+^2\to\mathbb{R}_+$ be a function. The the following assertions are equivalent:*

(1) Ψ *is a pmg-function.*

(2) Ψ *satisfies, for each* $a, b, c, \alpha, \beta \in \mathbb{R}_+$, *the following conditions:*

(i) $\Psi(a, \alpha) = \Psi(a + \alpha - \beta, \beta)$, *whenever* $a + \alpha \geq \beta$*;*

(ii) $\Psi(c, \alpha) + \Psi(0, \beta) \leq \Psi(a, \alpha) + \Psi(b, \beta)$, *whenever* $c \leq a + b$ *and* $\beta \leq a + \alpha$*;*

(iii) If $\Psi(0, \alpha) = \Psi(a, \alpha) = \Psi(0, \beta)$ *and* $a + \alpha \geq \beta$, *then* $a = 0$ *and* $\alpha = \beta$.

Proof. (1) $\Rightarrow$ (2). Let $a, b, c, \alpha, \beta \in \mathbb{R}_+$.

In order to prove (i), consider that $a+\alpha \geq \beta$. Notice that the case $a = 0$ and the case $\alpha = \beta$ are obvious. So, we suppose that $a > 0$ or $\alpha \neq \beta$.

Consider the set $X = \{x, y, z\}$ and define the function q on $X \times X$ as follows:

$$q(x, y) = a; \quad q(y, x) = a + \alpha - \beta; \quad q(x, z) = 2a + \alpha - \beta;$$

$$q(z, x) = 2a + 2\alpha - \beta; \quad q(y, z) = a + \alpha - \beta; \quad q(z, y) = a + \alpha;$$

$$q(x, x) = q(y, y) = q(z, z) = 0.$$

It is not hard to verify that q is a quasi-metric on X. Besides, if we define the function w_q on X given by

$$w_q(x) = \alpha; \quad w_q(y) = \beta \quad \text{and} \quad w_Q(z) = 0,$$

we have that

$$q(x, y) + w_q(x) = a + \alpha = a + \alpha - \beta + \beta = q(y, x) + w_q(y);$$

$$q(x, z) + w_q(x) = 2a + \alpha - \beta + \alpha = 2a + 2\alpha - \beta = q(z, x) + w_q(z)$$

and

$$q(y,z) + w_q(y) = a + \alpha - \beta + \beta = a + \alpha = q(z,y) + w_q(z).$$

Therefore, (X, q, w_q) is a weighted quasi-metric space.

Now, by hypothesis, p_{Ψ,q,w_q} is a partial metric on X, where $p_{\Psi,q,w_q}(u,v) = \Psi(q(u,v), w_q(u))$ for each $u, v \in X$. Then, $p_{\Psi,q,w_q}(u,v) = p_{\Psi,q,w_q}(v,u)$, for each $u, v \in X$. Hence

$$\Psi(a,\alpha) = \Psi(q(x,y), w_q(x)) =$$

$$p_{\Psi,q,w_q}(x,y) = p_{\Psi,q,w_q}(y,x) =$$

$$\Psi(q(y,x), w_q(y)) = \Psi(a + \alpha - \beta, \beta).$$

Next we show that Φ fulfils (ii). To this end, let $a, b, c, \alpha, \beta \in \mathbb{R}_+$ with $c \leq a + b$ and $\beta \leq a + \alpha$. Lemma 4.2.4 ensures that the inequality under consideration is hold for the case $a = 0$ and $\alpha = \beta$, and for the case $b = \beta = 0$. So, we suppose that $a > 0$ or $\alpha \neq \beta$, and $b > 0$ or $\beta > 0$. First, we will prove that the next inequality is fulfilled

$$\Psi(a + b, \alpha) + \Psi(0, \beta) \leq \Psi(a, \alpha) + \Psi(b, \beta).$$

With this purpose we consider the set $X = \{x, y, z\}$. Define the function q on $X \times X$ by:

$$q(x,y) = a; \quad q(y,x) = a + \alpha - \beta; \quad q(y,z) = b; \quad q(z,y) = b + \beta;$$

$$q(x,z) = a + b; \quad q(z,x) = a + b + \alpha;$$

and

$$q(x,x) = q(y,y) = q(z,z) = 0.$$

Then one can verify that q is a quasi-metric on X. Even more, if we define $w_q(x) = \alpha$, $w_q(y) = \beta$ and $w_q(z) = 0$, then

$$q(x, y) + w_q(x) = a + \alpha = a + \alpha - \beta + \beta = q(y, x) + w_q(y);$$

$$q(x, z) + w_q(x) = a + b + \alpha = q(z, x) + w_q(z)$$

and

$$q(y, z) + w_q(y) = b + \beta = q(z, y) + w_q(z).$$

Therefore, (X, q, w_q) is a weighted quasi-metric space.

Since Ψ is a *pmg*-function we have that p_{Ψ,q,w_q} is a partial metric on X, where $p_{\Psi,q,w_q}(u, v) = \Psi(q(u, v), w_q(u))$ for each $u, v \in X$. From this fact we deduce, on the one hand, that

$$\Psi(a + b, \alpha) + \Psi(0, \beta) = \Psi(q(x, z), w_q(x)) + \Psi(q(y, y), w_q(y)) =$$

$$p_{\Psi,q,w_q}(x, z) + p_{\Psi,q,w_q}(y, y),$$

and, on the other hand, that

$$\Psi(a, \alpha) + \Psi(b, \beta) = \Psi(q(x, y), w_q(x)) + \Psi(q(y, z), w_q(y)) = p_{\Psi,q}(x, y) + p_{\Psi,q}(y, z).$$

Since $p_{\Psi,q}(x, z) + p_{\Psi,q}(y, y) \leq p_{\Psi,q}(x, y) + p_{\Psi,q}(y, z)$, we obtain that

$$\Psi(a + b, \alpha) + \Psi(0, \beta) \leq \Psi(a, \alpha) + \Psi(b, \beta).$$

Thus, by Lemma 4.2.4, we conclude that

$$\Psi(c, \alpha) + \Psi(0, \beta) \leq \Psi(a + b, \alpha) + \Psi(0, \beta) \leq \Psi(a, \alpha) + \Psi(b, \beta).$$

It remains to prove condition (iii). Suppose that $\Psi(0, \alpha) = \Psi(a, \alpha) = \Psi(0, \beta)$ and $a + \alpha \geq \beta$.

Consider the weighted quasi-metric space $(\mathbb{R}_+, q_u, w_{q_u})$. Then, $p_{\Psi,q_u,w_{q_u}}$ is a partial metric on $\mathbb{R}_+$, where $p_{\Psi,q_u,w_{q_u}}(x,y) = \Psi(q_u(x,y), w_{q_u}(x))$ for each $x, y \in \mathbb{R}_+$.

First, we will see that $\alpha = \beta$. Suppose that $\alpha \geq \beta$ (the proof for the case $\beta \geq \alpha$ runs following similar arguments). In such a case,

$$p_{\Psi,q_u,w_{q_u}}(\alpha,\alpha) = \Psi(q_u(\alpha,\alpha), w_{q_u}(\alpha)) = \Psi(0,\alpha);$$

$$p_{\Psi,q_u,w_{q_u}}(\beta,\beta) = \Psi(q_u(\beta,\beta), w_{q_u}(\beta)) = \Psi(0,\beta);$$

$$p_{\Psi,q_u}(\alpha,\beta) = \Psi(q_u(\alpha,\beta), w_{q_u}(\alpha)) = \Psi(\max\{\beta-\alpha,0\},\alpha) = \Psi(0,\alpha).$$

Since $p_{\psi,q_u,w_{q_u}}$ is a partial metric then $p_{\Psi,q_u w_{q_u}}(\alpha,\beta) = p_{\Psi,q_u,w_{q_u}}(\beta,\alpha)$ and, hence,

$$p_{\Psi,q,w_{q_u}}(\alpha,\alpha) = p_{\Psi,q,w_{q_u}}(\alpha,\beta) = p_{\Psi,q,w_{q_u}}(\beta,\beta),$$

and so $\alpha = \beta$.

Now, for the purpose of contradiction, we assume that $a > 0$. Consider the set $X = \{x, y, z\}$ with $x \neq y$. Define the function q on $X \times X$ by:

$$q(x,y) = q(z,y) = q(y,x) = q(y,z) = a; \qquad q(x,z) = q(z,x) = 2a;$$

and

$$q(x,x) = q(y,y) = q(z,z) = 0.$$

One can verify that q is a quasi-metric on X. Moreover, if we define $w_q(x) = w_q(y) = w_q(z) = \alpha$, we have that

$$q(x,y) + w_q(x) = a + \alpha = a + \alpha = q(y,x) + w_q(y);$$

$$q(x,z) + w_q(x) = 2a + \alpha = q(z,x) + w_q(z)$$

and

$$q(y,z) + w_q(y) = a + \alpha = q(z,y) + w_q(z).$$

Therefore, (X, q, w_q) is a weighted quasi-metric space. Then, p_{Ψ,q,w_q} is a partial metric on X, where $p_{\Psi,q,w_q}(u, v) = \Psi(q(u, v), w_q(u))$ for each $u, v \in X$. Furthermore we have

$$p_{\Psi,q,w_q}(x, x) = \Psi(q(x, x), w_q(x)) = \Psi(0, \alpha);$$

$$p_{\Psi,q,w_q}(x, y) = \Psi(q(x, y), w_q(x)) = \Psi(a, \alpha);$$

and

$$p_{\Psi,q,w_q}(y, y) = \Psi(q(y, y), w_q(y)) = \Psi(0, \alpha).$$

Since $\Psi(0, \alpha) = \Psi(a, \alpha)$ we obtain that

$$p_{\Psi,q,w_q}(x, x) = p_{\Psi,q,w_q}(x, y) = p_{\Psi,q,w_q}(y, y)$$

and, thus, that $x = y$, which is a contradiction.

(2) $\Rightarrow$ (1). Let (X, q, w_q) be a weighted quasi-metric space. Define $p_{\Psi,q,w_q}(x, y) = \Psi(q(x, y), w_q(x))$ for each $x, y \in X$. Next we show that p_{Ψ,q,w_q} is a partial metric on X. To this aim, let $x, y, z \in X$.

Suppose that $p_{\Psi,q,w_q}(x, x) = p_{\Psi,q,w_q}(x, y) = p_{\Psi,q,w_q}(y, y)$. By construction of p_{Ψ,q,w_q} we have that

$$\Psi(0, w_q(x)) = \Psi(q(x, x), w_q(x)) = \Psi(q(x, y), w_q(x)) =$$

$$\Psi(q(y, y), w_q(y)) = \Psi(0, w_q(y)) .$$

Besides $q(x, y) + w_q(x) \geq w_q(y)$, since $q(x, y) + w_q(x) = q(y, x) + w_q(y)$. Whence we deduce that $q(x, y) = 0$ and $w_q(x) = w_q(y)$, because of Ψ satisfies (*iii*). Moreover, in such a case we have that $w_q(x) = q(y, x) + w_q(y)$, which implies that $q(y, x) = 0$. Thus, $q(x, y) = q(y, x) = 0$ and so $x = y$.

Obviously, if $x = y$ we have that $p_{\Psi,q,w_q}(x,x) = p_{\Psi,q,w_q}(x,y) = p_{\Psi,q,w_q}(y,y)$. We conclude that p_{Ψ,q,w_q} satisfies axiom $(P1)$ of partial metrics.

The definition of Ψ gives that $p_{\Psi,q,w_q}(x,x) = \Psi(0, w_q(x)) \geq 0$. Moreover, Lemma 4.2.4 guarantees that

$$p_{\Psi,q,w_q}(x,x) = \Psi(0, w_q(x)) \leq \Psi(q(x,y), w_q(x)) = p_{\Psi,q,w_q}(x,y).$$

It follows that p_{Ψ,q,w_q} satisfies axiom $(P2)$ of partial metrics.

Since $q(x,y) + w_q(x) \geq w_q(y)$ and $q(x,y) + w_q(x) = q(y,x) + w_q(y)$ we obtain from condition (i) that

$$p_{\Psi,q,w_q}(x,y) = \Psi(q(x,y), w_q(x)) = \Psi(q(y,x), w_q(y)) = p_{\Psi,q,w_q}(y,x).$$

So p_{Ψ,q,w_q} fulfils axiom $(P3)$ of partial metrics.

Finally we show that p_{Ψ,q,w_q} satisfies axiom $(P4)$ of partial metrics. On the one hand,

$$p_{\Psi,q,w_q}(x,z) + p_{\Psi,q,w_q}(y,y) =$$

$$\Psi(q(x,z), w_q(x)) + \Psi(q(y,y), w_q(y)) =$$

$$\Psi(q(x,z), w_q(x)) + \Psi(0, w_q(y)).$$

On the other hand,

$$p_{\Psi,q,w_q}(x,y) + p_{\Psi,q,w_q}(y,z) = \Psi(q(x,y), w_q(x)) + \Psi(q(y,z), w_q(y)).$$

Since $q(x,z) \leq q(x,y) + q(y,z)$ and $w_q(y) \leq q(x,y) + w_q(x)$, we deduce from condition (ii) that

$$\Psi(q(x,z), w_q(x)) + \Psi(0, w_q(y)) \leq \Psi(q(x,y), w_q(x)) + \Psi(q(y,z), w_q(y)).$$

Hence we conclude that

$$p_{\Psi,q,w_q}(x,z) + p_{\Psi,q,w_q}(y,y) \leq p_{\Psi,q,w_q}(x,y) + p_{\Psi,q,w_q}(y,z).$$

Therefore p_{Ψ,q,w_q} is a partial metric on X, and this ends the proof. ■

An immediate consequence of the above characterization is given by the following result which will be key in our subsequent discussion.

Corollary 4.2.6. *Let $\Psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a pmg-function and let $\alpha, a \in \mathbb{R}_+$. If $\Psi(a,\alpha) = \Psi(0,\alpha)$, then $a = 0$.*

Proof. Suppose that $\Psi(a,\alpha) = \Psi(0,\alpha)$. Set $a + \alpha = \beta$. By condition (i) in the statement of Theorem 4.2.5 we have that $\Psi(a,\alpha) = \Psi(0, a+\alpha)$. Consequently, $\Psi(a,\alpha) = \Psi(0,\alpha) = \Psi(0, a+\alpha)$ and, by condition $(iiii)$ in the aforesaid theorem we deduce that $a = 0$. ■

Similar to the case of quasi-metric generating functions one can explore whether, given a *pmg*-function $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ and a weighted quasi-metric space (X, q, w_q), the partial orders $\preceq_{p_{\Psi,q,w_q}}$ and $\preceq_q$ are exactly the same on X, i.e., whether a *pmg*-function preserves the order induced by the quasi-metric that it transforms. The next result gives a positive answer to the posed inquiry.

Proposition 4.2.7. *Let $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a pmg-function and let (X, q, w_q) a weighted quasi-metric space. Then $\preceq_{p_{\Psi,q,w_q}} = \preceq_q$.*

Proof. Let $x, y \in X$. Suppose that $x \preceq_q y$. Then $q(x,y) = 0$ and, thus, we have that

$$p_{\Psi,q,w_q}(x,y) = \Psi(q(x,y), w_q(x)) = \Psi(0, w_q(x)) = \\ \Psi(q(x,x), w_q(x)) = p_{\Psi,q,w_q}(x,x).$$

Whence we get that $x \preceq_{p_{\Psi,q,w_q}} y$. Next assume that $x \preceq_{p_{\Psi,q,w_q}} y$. Then $p_{\Psi,q,w_q}(x,y) = p_{\Psi,q,w_q}(x,x)$. Hence we have that

$$\Psi(q(x,y), w_q(x)) = \Psi(q(x,x), w_q(x)) = \Psi(0, w_q(x)).$$

Corollary 4.2.6 ensures that $q(x,y) = 0$ and, thus, that $x \preceq_q y$. ■

Similarly to the the *qmg*-functions, it seems natural to wonder if a *pmg*-function Φ preserves the topology induced by the weighted quasi-metric q that it transforms and, hence, $\tau(q) = \tau(p_{\Psi,q,w_q})$. However, the next example shows that this is not the case.

Example 4.2.8. *Define $\Psi_1 : \mathbb{R}_+^2 \to \mathbb{R}_+$ by $\Psi_1(0,0) = 0$ and $\Psi_1(a,\alpha) = a + \alpha + 1$ otherwise. A straightforward computation gives that Ψ_1 is a pmg-function. Consider the weighted quasi-metric $(\mathbb{R}_+, q_u, w_{q_u})$. It is not hard to check that, for each $x \in \mathbb{R}_+$ and $\epsilon > 0$, the open ball centered at x with radius ϵ is given by $B_{q_u}(x;\epsilon) = [0, x+\epsilon[$. However, $B_{p_{\Psi_1,q_u,w_{q_u}}}(0;\epsilon) = \{0\}$ for each $\epsilon \in]0,1[$. Hence, $B_{p_{\Psi_1,q_u,w_{q_u}}}(0;\epsilon) \notin \tau(q_u)$ and, therefore, $\tau(q_u) \neq \tau(p_{\Phi_1,q_u,w_{q_u}})$.*

In the light of the preceding example we focus our effort on characterizing those *pmg*-functions which preserve the topology of the weighted quasi-metric that it transforms. With this aim we introduce the notion below.

Definition 4.2.9. Let $\Psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a *pmg*-function. We will say that Ψ is a strongly partial metric generating function (*spmg*-function for short) if for each weighted quasi-metric space (X, q, w_q) we have that $\tau(q) = \tau(p_{\Psi,q,w_q})$.

It is easily seen that Examples 4.2.2 and 4.2.3 provide instances of *spmg*-functions.

Similar to *qmp*-functions, we have that the topology induced by the partial metric p_{Ψ,q,w_q} generated from a *pmg*-function Ψ is always finer than the topology induced by the weighted quasi-metric q from which it is constructed. The next result states such an affirmation.

Theorem 4.2.10. *Let $\Psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a pmg-function and let (X, q, w_q) be a weighted quasi-metric space. Then, $\tau(q) \subseteq \tau(p_{\Psi,q,w_q})$.*

Proof. Consider $x \in X$ and the real number $\epsilon > 0$. Put $\delta = \Psi(\epsilon, w_q(x)) - \Psi(0, w_q(x))$. Corollary 4.2.6 guarantees that $\delta > 0$. Next we show that $B_{p_{\Psi,q,w_q}}(x; \delta) \subset B_q(x; \epsilon)$. Let $y \in B_{p_{\Psi,q,w_q}}(x; \delta)$. Then, $p_{\Psi,q,w_q}(x, y) < p_{\Psi,q,w_q}(x, x) + \delta$. Then

$$\Psi(q(x, y), w_q(x)) < \Psi(0, w_q(x)) + \delta = \Psi(\epsilon, w_q(x)).$$

Now, by Lemma 4.2.4, we deduce that $q(x, y) < \epsilon$. Therefore, $y \in B_q(x; \epsilon)$.

■

The next theorem characterizes *spmg*-functions.

Theorem 4.2.11. *Let $\Psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a pmg-function. Then the following assertions are equivalent:*

(1) Ψ is a spmg-function.

(2) For each $\alpha \in \mathbb{R}_+$, the function $\Psi_\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous at 0, where $\Psi_\alpha(a) = \Psi(a, \alpha)$ for each $a \in \mathbb{R}_+$.

Proof. (1) $\Rightarrow$ (2). Suppose for the purpose of contradiction that there exists $\alpha_0 \in \mathbb{R}_+$ such Ψ_{α_0} is not continuous at 0. Next we show that Ψ is not a *spmg*-function.

Since Ψ_{α_0} is not continuous at 0, then there exist $\epsilon_0 > 0$ such that for each $\delta > 0$ we can find $a_\delta \in [0, \delta[$ satisfying $\Psi_{\alpha_0}(a_\delta) - \Psi_{\alpha_0}(0) \geq \epsilon_0$ (observe that $\Psi_{\alpha_0}(0) \leq \Psi_{\alpha_0}(a_\delta)$ by Lemma 4.2.4).

Consider the weighted quasi-metric space $(\mathbb{R}_+, q_u, w_{q_u})$. Take $x = \alpha_0$. Then $\tau(q_u) \neq \tau(p_{\Psi,q_u,w_{q_u}})$, since we have that $B_{p_{\Psi,q_u,w_q}}(x;\epsilon_0) \in \tau(p_{\Psi,q_u,w_{q_u}})$ but $B_{p_{\Psi,q_u,w_q}}(x;\epsilon_0) \notin \tau(q_u)$. Indeed, for each $\delta > 0$, put $y_\delta = a_\delta + \alpha_0$. Then $y_\delta \in B_{q_u}(x;\delta)$, since $q_u(x, y_\delta) = \max\{y_\delta - x, 0\} = a_\delta < \delta$. Nevertheless, we have that $y_\delta \notin B_{p_{\Psi_{\alpha_0},q_u,w_{q_u}}}(x;\epsilon_0)$ because

$$p_{\Psi,q_u,w_{q_u}}(x, y_\delta) = \Psi(q_u(x, y_\delta), w_{q_u}(x)) = \Psi(a_\delta, \alpha_0) = \Psi_{\alpha 0}(a_\delta) \geq$$

$$\Psi_{\alpha_0}(0) + \epsilon_0 = \Psi(0, w_{q_u}(\alpha_0)) + \epsilon_0 = p_{\Psi,q_u,w_{q_u}}(x, x) + \epsilon_0.$$

Therefore, Ψ is not a *spmg*-function which is a contradiction.

(2) $\Rightarrow$ (1). Let (X, q, w_q) be a weighted quasi-metric space. Suppose that, for each $\alpha \in \mathbb{R}_+$, the function Ψ_α is continuous at 0. By Theorem 4.2.10 we deduce that $\tau(q) \subseteq \tau(p_{\Psi,q,w_q})$. So we just need to show that $\tau(p_{\Psi,q,w_q}) \subseteq \tau(q)$. Next we prove that, given $x \in X$ we have that, for each real number $\epsilon > 0$ there exist a real number $\delta > 0$ such that $B_q(x;\delta) \subseteq B_{p_{\Psi,q,w_q}}(x;\epsilon)$. Since $\Psi_{w_q(x)}$ is continuous at 0, there exists $\delta > 0$ such that, for each $a \in [0, \delta[$ we have that $\Psi_{w_q(x)}(a) - \Psi_{w_q(x)}(0) < \epsilon$. In such a case, $B_q(x;\delta) \subseteq B_{p_{\Psi,q,w_q}}(x;\epsilon)$. Indeed, if $y \in B_q(x;\delta)$, then $q(x, y) < \delta$ and, in addition, $p_{\Psi,q,w_q}(x, y) - p_{\Psi,q,w_q}(x, x) = \Psi_{w_q(x)}(q(x, y)) - \Psi_{w_q(x)}(0) < \epsilon$.

Therefore $y \in B_{p_{\Psi,q,w_q}}(x;\epsilon)$. ■

Note that the *pgm*-function Ψ_1 introduced in Example 4.2.8 fulfils that the function Ψ_0 is not continuous at 0, which agrees with the fact that Ψ_1 is not a *spgm*-function.

We conclude this section discussing the relationship between *pmg*-functions, 2-(quasi-)metric preserving functions, 2-partial metric preserving functions and 2-metric generating functions.

Example 4.2.3 shows that there are *pmg*-functions that are neither 2-(quasi-)metric preserving functions nor metric generating functions because the function Ψ_{+1} introduced in Example 4.2.3 fulfils that $\Psi_{+1} \notin \mathcal{O}$. Let us recall that Theorems 1.1.6, 1.1.12 and 1.1.20 state that 2-(quasi-)metric preserving functions and metric generating functions belong to $\mathcal{O}$.

Nevertheless, the next result shows that every *pmg*-function is always a 2-partial metric preserving function.

Proposition 4.2.12. *Let $\Psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a pmg-function. Then Ψ is a 2-partial metric preserving function.*

Proof. We prove that Ψ satisfies (1) and (2) in Theorem 1.1.17. To this end, assume that $(x_1,x_2),(y_1,y_2),(z_1,z_2),(w_1,w_2) \in \mathbb{R}_+^2$, with $(x_1,x_2)+(y_1,y_2) \preceq (z_1,z_2)+(w_1,w_2)$, $(y_1,y_2) \preceq (z_1,z_2)$ and $(y_1,y_2) \preceq (w_1,w_2)$. Next we show that

$$\Psi(x_1,x_2) + \Psi(y_1,y_2) \leq \Psi(z_1,z_2) + \Psi(w_1,w_2).$$

Set $c = x_1+x_2$, $a = z_1+z_2$, $b = w_1+w_2-y_1-y_2$, $\alpha = 0$ and $\beta = y_1+y_2$. Then, $c,a,b,\alpha,\beta \in \mathbb{R}_+$, $c \leq a+b$ and $a+\alpha \geq \beta$. So, by (*ii*) in Theorem

4.2.5, we have that

$$\Psi(x_1+x_2,0)+\Psi(0,y_1+y_2)\leq \Psi(z_1+z_2,0)+\Psi(w_1+w_2-y_1-y_2,y_1+y_2).$$

Besides, by (i) in Theorem 4.2.5, we have that $\Psi(x_1+x_2,0)=\Psi(x_1,x_2)$, $\Psi(0,y_1+y_2)=\Psi(y_1,y_2)$, $\Psi(z_1+z_2,0)=\Psi(z_1,z_2)$ and $\Psi(w_1,w_2)=\Psi(w_1+w_2-y_1-y_2,y_1+y_2)$. Therefore we obtain that $\Psi(x_1,x_2)+\Psi(y_1,y_2)\leq \Psi(z_1,z_2)+\Psi(w_1,w_2)$ and, thus, that (1) in Theorem 1.1.17 is satisfied.

Next we show that Ψ satisfies (2) in Theorem 1.1.17. Take the elements $(x_1,x_2),(y_1,y_2),(z_1,z_2)\in\mathbb{R}^2_+$, with $(x_1,x_2)\preceq(z_1,z_2)$ and $(y_1,y_2)\preceq(z_1,z_2)$, such that $\Psi(x_1,x_2)=\Psi(y_1,y_2)=\Psi(z_1,z_2)$. We claim that $(x_1,x_2)=(y_1,y_2)=(z_1,z_2)$.

Consider $a=z_1+z_2-x_1-x_2$, $\alpha=x_1+x_2$ and $\beta=y_1+y_2$. Then $a,\alpha,\beta\in\mathbb{R}_+$ and $a+\alpha\geq\beta$. By (i) in Theorem 4.2.5, we have that $\Psi(0,\alpha)=\Psi(0,x_1+x_2)=\Psi(x_1,x_2)$, $\Psi(a,\alpha)=\Psi(z_1+z_2-x_1-x_2,x_1+x_2)=\Psi(z_1,z_2)$ and $\Psi(0,\beta)=\Psi(0,y_1+y_2)=\Psi(y_1,y_2)$. Then $\Psi(0,\alpha)=\Psi(a,\alpha)=\Psi(0,\beta)$. Hence, by (iii) in Theorem 4.2.5, we obtain that $z_1+z_2=x_1+x_2=y_1+y_2$. Moreover one can show easily that $z_1=x_1=y_1$ and $z_2=x_2=y_2$, since $z_1\geq\max\{x_1,y_1\}$ and $z_2\geq\max\{x_2,y_2\}$. Therefore, $(x_1,x_2)=(y_1,y_2)=(z_1,z_2)$ as we claimed. ∎

Conversely we analyze if 2-(quasi-)metric preserving functions, 2-partial metric preserving functions and metric generating functions are *pmg*-functions.

The next example shows that there are 2-metric and 2-quasi-metric preserving functions that are not *qmg*-functions.

Example 4.2.13. *Consider the function* $\Psi_{0,1}:\mathbb{R}^2_+\to\mathbb{R}_+$ *defined by*

$$\Psi_{0,1}(a,b)=\begin{cases}0 & \text{if } a=b=0\\ 1 & \text{otherwise.}\end{cases}$$

Clearly $\Psi_{0,1}$ *is monotone, subadditive and* $\Psi_{0,1} \in \mathcal{O}$. *By Theorems 1.1.6 and 1.1.12 we have that it is a* 2*-metric and* 2*-quasi-metric preserving function. However,* $\Psi_{0,1}$ *is not a pmg-function. Indeed,* $\Psi_{0,1}(0,1) = \Psi_{0,1}(2,1) = \Psi_{0,1}(0,\frac{1}{2}) = 1$ *and, in addition,* $\frac{1}{2} \leq 1+2$ *but* $2 \neq 0$. *So,* $\Psi_{0,1}$ *does not satisfy condition* (iii) *in the statement of Theorem 4.2.5.*

Next we show that there are 2-partial metric preserving functions that are not *pmg*-functions.

Example 4.2.14. *Consider the function* $\Psi_{\frac{1}{2}} : \mathbb{R}_+^2 \to \mathbb{R}_+$ *defined by* $\Psi_{\frac{1}{2}}(a,b) = a + \frac{b}{2}$ *for all* $(a,b) \in \mathbb{R}_+^2$. *It is clear that the function* $\Psi_{\frac{1}{2}}$ *fulfils the conditions in the statement of Theorem 1.1.17 and, thus, it is a* 2*-partial metric preserving function. However,* $\Psi_{\frac{1}{2}}$ *does not satisfies condition* (iii) *in the statement of Theorem 4.2.5 and, hence, it is not a pmg-function. Notice that* $0 \leq 1+1$ *and that* $\Psi_{\frac{1}{2}}(1,1) \neq \Psi_{\frac{1}{2}}(2,0)$.

The next example gives an instance of metric generating function which is not a *pmg*-function.

Example 4.2.15. *Consider the function* $\Psi_{max} : \mathbb{R}_+^2 \to \mathbb{R}_+$ *defined by* $\Psi_{max}(a,b) = \max\{a,b\}$ *for all* $a,b \in \mathbb{R}_+$. *It is not har to check that* Ψ_{max} *satisfies Theorem 1.1.20 and, hence, that it is a metric generating function. Moreover,* $0 \leq 1+1$ *and* $\Psi_{\max}(1,1) \neq \Psi_{\max}(2,0)$. *This* $\Psi_{\max}$ *does not satisfy condition* (iii) *in Theorem 4.2.5 and, therefore, it is not a pmg-function.*

Similar to the (quasi-)metric preserving approach, a method for generating partial metric preserving functions from *pmg*-functions can be obtained such as the next result shows.

Theorem 4.2.16. *Let* $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a pmg-function. Then for each*

$\alpha \in \mathbb{R}_+$, the function $\Psi_\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a partial metric preserving function, where $\Psi_\alpha(a) = \Psi(a, \alpha)$ for each $a \in \mathbb{R}_+$.

Proof. Fix $\alpha \in \mathbb{R}_+$. First we show that Ψ_α satisfies condition (1) in statement of Theorem 1.1.17. Let $x, y, z, w \in \mathbb{R}_+$ such that $x + y \leq z + w$ and $y \leq \min\{z, w\}$. Set $a = z$, $b = w - y$, $c = x$ and $\beta = y + \alpha$. It follows that that $c \leq a + b$ and $\beta \leq a + \alpha$ and, by condition (i) in Theorem 4.2.5, that $\Psi(0, \beta) = \Psi(y, \alpha)$ and $\Psi(b, \beta) = \Psi(w, \alpha)$. By condition (ii) in Theorem 4.2.5 we deduce that

$$\Psi_\alpha(x) + \Psi_\alpha(y) = \Psi(c, \alpha) + \Psi(0, \beta) \leq \Psi(a, \alpha) + \Psi(b, \beta) = \Psi_\alpha(z) + \Psi_\alpha(w).$$

Next we prove that Ψ_α satisfies condition (ii) in statement of Theorem 1.1.17. Let $x, y, z \in \mathbb{R}_+$, with $x \geq \max\{y, z\}$, and suppose that $\Psi_\alpha(x) = \Psi_\alpha(y) = \Psi_\alpha(z)$. Set $a = x - y$, $\alpha' = y + \alpha$ and $\beta = z + \alpha$. In such a case, we have that $\Psi(a, \alpha') = \Psi(x, \alpha)$, $\Psi(0, \alpha') = \Psi(y, \alpha)$ and $\Psi(0, \beta) = \Psi(z, \alpha)$ because of Ψ satisfies condition (i) in Theorem 4.2.5. Then we obtain that $\Psi(a, \alpha') = \Psi(0, \alpha') = \Psi(0, \beta)$ and $a + \alpha' \geq \beta$. So, $a = 0$ and $\alpha' = \beta$ due to Ψ satisfies condition (iii) in Theorem 4.2.5. Hence, $x = y = z$. This ends the proof. ■

As an immediate consequence of Theorem 4.2.16 and Proposition 1.1.19 we obtain a method for generating (quasi-)metric metric preserving functions from *pmg*-functions.

Corollary 4.2.17. *Let $\Psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a pmg-function such that $\psi(0, 0) = 0$. Then for each $\alpha \in \mathbb{R}_+$, the function $\Psi_\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a quasi-metric preserving function and, thus, a metric preserving function, where $\Psi_\alpha(a) = \Psi(a, \alpha)$ for each $a \in \mathbb{R}_+$.*

Notice that Example 4.2.3 gives an instance of *pmg*-functions that allows us to induce, following Theorem 4.2.16, a partial metric preserving function. In addition, Examples 4.2.2 and 4.2.8 yield instances of *pmg*-functions which are able to generate, according to Corollary 4.2.17, partial metric preserving functions that are at the same time (quasi-)metric preserving functions.

Chapter 5

What is the aggregation of a partial metric and a quasi-metric?

Generalized metrics have been shown to be useful in many fields of Computer Science. In particular, partial metrics and quasi-metrics are used to develop quantitative mathematical models in denotational semantics and in asymptotic complexity analysis of algorithms, respectively (see Subsection 1.1.4). The aforesaid models are implemented independently and they are not related. However, it seems natural to consider a unique framework which remains valid for the applications to the both aforesaid fields. A first natural attempt to achieve that target suggests that the quantitative information should be obtained by means of the aggregation of a partial metric and a quasi-metric. Inspired by the preceding fact, we explore the way of merging, by means of a function, the aforementioned generalized metrics into a new one. We show that the induced generalized metric matches up with a partial quasi-metric. Thus, we characterize those functions that allow to generate partial quasi-metrics from the combination of a partial metric and a quasi-metric. Moreover, the relationship between the problem under consideration and the problems of merging partial metrics and quasi-metrics is discussed. Examples that illustrate the obtained results are also given. Finally, an application of the exposed theory to develop a framework which remains valid, at the same time, for modeling in denotational semantics and in complexity analysis of algorithms has been also given.

5.1 Partial quasi-metric generating functions and their characterization

In [51], the notions of partial metric and quasi-metric were generalized. Specifically, H.-P.A. Künzi et al. introduced the concept of partial quasi-metric. Let us recall that a partial quasi-metric space is a pair (X, pq), where X is a non-empty set and $pq : X \times X \to \mathbb{R}_+$ is a function satisfying for all $x, y, z \in X$ the following:

(PQ1) $pq(x,x) \leq pq(x,y)$ and $pq(x,x) \leq pq(y,x)$;

(PQ2) $pq(x,z) + pq(y,y) \leq pq(x,y) + pq(y,z)$;

(PQ3) $x = y \Leftrightarrow pq(x,x) = pq(x,y)$ and $pq(y,y) = pq(y,x)$.

With the aim of providing a solution to the problem stated in the preceding section we introduce the notion of partial quasi-metric generating function. Thus, we will say that a function $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a partial quasi-metric generating function (*pqmg*-function for short) provided that for each partial metric space (X, p) and each quasi-metric space (Y, q), the function $PQ_\Phi : (X \times Y) \times (X \times Y) \to \mathbb{R}_+$ is a partial quasi-metric on $X \times Y$, where $PQ_\Phi((x, y), (u, v)) = \Phi(p(x, u), q(y, v))$ for each $(x, y), (u, v) \in X \times Y$.

Later, Proposition 5.1.4 and Example 5.1.8 will provide non-trivial instances of *pqmg*-functions. Furthermore, an instance of a function that is not a *pqmg*-function, and that arises in a natural way in aggregation operator theory, will be given by Example 5.1.10.

Before, we yield a characterization of *pqmg*-functions. To this end, recall that $\mathbb{R}_+^2$ becomes a partial ordered set when we endow it with the pointwise partial order $\preceq$, i.e., $(a, \alpha) \preceq (b, \beta) \Leftrightarrow a \leq b$ and $\alpha \leq \beta$. Moreover, a function $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is monotone provided that $\Phi(a, \alpha) \leq \Phi(b, \beta)$ whenever $(a, \alpha) \preceq (b, \beta)$.

The following result will be useful in our subsequent work.

Lemma 5.1.1. *If $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a pqmg-function, then it is monotone.*

Proof. Let $a, b, \alpha, \beta \in \mathbb{R}_+$ such that $(a, \alpha) \preceq (b, \beta)$. Consider the partial metric space $(\mathbb{R}_+, p_m)$ and the quasi-metric space $(\mathbb{R}, q_u)$, where $p_m(x, y) = \max\{a, b\}$ for all $a, b \in \mathbb{R}_+$and $q_u(x, y) = \max\{y - x, 0\}$ for all $x, y \in \mathbb{R}$. By our assumption, the function $PQ_\Phi : (\mathbb{R}_+ \times \mathbb{R}) \times (\mathbb{R}_+ \times \mathbb{R}) \to \mathbb{R}_+$ defined by $PQ_\Phi((x, y), (u, v)) = \Phi(p_m(x, u), q_u(y, v))$ is a partial quasi-metric on

$\mathbb{R}_+ \times \mathbb{R}$. Then, for each $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}_+ \times \mathbb{R}$, we have that

$$\begin{aligned}
&\Phi(p_m(x_1, x_3), q_s(y_1, y_3)) + \Phi(p_m(x_2, x_2), q_s(y_2, y_2)) = \\
&PQ_\Phi((x_1, y_1), (x_3, y_3)) + PQ_\Phi((x_2, y_2), (x_2, y_2)) \leq \\
&PQ_\Phi((x_1, y_1), (x_2, y_2)) + PQ_\Phi((x_2, y_2), (x_3, y_3)) = \\
&\Phi(p_m(x_1, x_2), q_s(y_1, y_2)) + \Phi(p_m(x_2, x_3), q_s(y_2, y_3)).
\end{aligned}$$

Taking in the preceding inequalities $x_1 = a$, $x_2 = b$, $x_3 = 0$, $y_1 = 0$, $y_2 = \beta$ and $y_3 = \alpha$ we obtain that $\Phi(a, \alpha) + \Phi(b, 0) \leq \Phi(b, \beta) + \Phi(b, 0)$, since $p_m(x_1, x_3) = a$, $p_m(x_2, x_2) = p_m(x_1, x_2) = p_m(x_2, x_3) = b$, $q_s(y_1, y_3) = \alpha$, $q_s(y_1, y_2) = \beta$, $q_s(y_2, y_2) = 0$ and $q_s(y_2, y_3) = 0$. So $\Phi(a, \alpha) \leq \Phi(b, \beta)$. ■

The next theorem provides a characterization of those functions $\Phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ which are a *pqmg*-functions.

Theorem 5.1.2. *A function $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a pqmg-function if and only if for each $a, b, c, d, \alpha, \beta, \gamma \in \mathbb{R}_+$ the following assertions hold:*

(i) If $b \geq a$, then $\Phi(a, 0) \leq \Phi(b, \alpha)$;

(ii) If $c \geq \max\{a, b\}$, $\Phi(a, 0) = \Phi(c, \alpha)$ and $\Phi(b, 0) = \Phi(c, \beta)$, then $a = b = c$ and $\alpha = \beta = 0$;

(iii) If $b \leq \min\{c, d\}$, $a + b \leq c + d$ and $\alpha \leq \beta + \gamma$, then $\Phi(a, \alpha) + \Phi(b, 0) \leq \Phi(c, \beta) + \Phi(d, \gamma)$.

Proof. Assume that $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a *pqmg*-function. Next we show that Φ satisfies (i), (ii) and (iii).

(i) It is fulfilled by Lemma **??**.

(ii) Let $a, b, c, \alpha, \beta \in \mathbb{R}_+$ such that $c \geq \max\{a, b\}$, $\Phi(a, 0) = \Phi(c, \alpha)$ and $\Phi(b, 0) = \Phi(c, \beta)$. We will show that $a = b = c$ and $\alpha = \beta = 0$.

On the one hand, consider the set $X = \{a, b, c\}$ and we define the function p_X on $X \times X$ as follows: $p_X(a,b) = p_X(b,a) = p_X(a,c) = p_X(c,a) = p_X(b,c) = p_X(c,b) = c$ and, $p_X(a,a) = a$, $p_X(b,b) = b$ and $p_X(c,c) = c$. By our assumption on $a, b, c \in \mathbb{R}_+$, one can verify that p_X is a partial metric on X.

On the other hand, we will distinguish two cases on $\alpha, \beta \in \mathbb{R}_+$:

Case 1. Suppose that $\alpha = \beta = 0$. Then the function $PQ_\Phi : (X \times \mathbb{R}) \times (X \times \mathbb{R}) \to \mathbb{R}_+$ given, for each $(x,y),(u,v) \in X \times \mathbb{R}$, by $PQ_\Phi((x,y),(u,v)) = \Phi(p_X(x,u), d_e(y,v))$ is a partial quasi-metric, where d_e denotes the euclidean metric (note that every metric is a quasi-metric). Moreover,

$$\begin{aligned}
PQ_\Phi((a,0),(a,0)) &= \Phi(p_X(a,a), d_e(0,0)) = \Phi(a,0),\\
PQ_\Phi((a,0),(b,0)) &= \Phi(p_X(a,b), d_e(0,0)) = \Phi(c,0),\\
PQ_\Phi((b,0),(b,0)) &= \Phi(p_X(b,b), d_e(0,0)) = \Phi(b,0),\\
PQ_\Phi((b,0),(a,0)) &= \Phi(p_X(b,a), d_e(0,0)) = \Phi(c,0).
\end{aligned}$$

Since $\Phi(a,0) = \Phi(c,0)$ and $\Phi(b,0) = \Phi(c,0)$ we deduce that

$$\begin{aligned}
PQ_\Phi((a,0),(a,0)) &= PQ_\Phi((a,0),(b,0)) \text{ and}\\
PQ_\Phi((b,0),(b,0)) &= PQ_\Phi((b,0),(a,0)).
\end{aligned}$$

Therefore, by axiom (PQ3), we deduce that $(a,0) = (b,0)$ and so $a = b$. Furthermore, if we repeat the above process using now a and c, we deduce that $a = c$ and so, $a = b = c$ as we claimed.

Case 2. Suppose that either $\alpha \neq 0$ or $\beta \neq 0$. We will show that this case cannot be given. Consider the partial metric space (X, p_X) introduced in Case 1 and the quasi-metric space (Y, q_Y), where $Y = \{1, 2, 3\}$ and the quasi-metric q_Y on Y is given by $q_Y(1,2) = q_Y(2,3) = q_Y(1,3) = \alpha$, $q_Y(2,1) = q_Y(3,2) = q_Y(3,1) = \beta$ and

$q_Y(i,i) = 0$ for all $i \in \{1,2,3\}$. Then the function $PQ_\Phi : (X \times Y) \times (X \times Y) \to \mathbb{R}_+$ given by

$$PQ_\Phi((x,y),(u,v)) = \Phi(p_X(x,u), q_Y(y,v)),$$

for each $(x,u),(y,v) \in X \times Y$, is a partial quasi-metric. Moreover

$$\begin{aligned} PQ_\Phi((a,1),(a,1)) &= \Phi(p_X(a,a), q_Y(1,1)) = \Phi(a,0), \\ PQ_\Phi((a,1),(b,2)) &= \Phi(p_X(a,b), q_Y(1,2)) = \Phi(c,\alpha), \\ PQ_\Phi((b,2),(b,2)) &= \Phi(p_X(b,b), q_Y(2,2)) = \Phi(b,0), \\ PQ_\Phi((b,2),(a,1)) &= \Phi(p_X(b,a), q_Y(2,1)) = \Phi(c,\beta). \end{aligned}$$

Since $\Phi(a,0) = \Phi(c,\alpha)$ and $\Phi(b,0) = \Phi(c,\beta)$ we deduce that

$$\begin{aligned} &PQ_\Phi((a,1),(a,1)) = PQ_\Phi((a,1),(b,2)) \text{ and} \\ &PQ_\Phi((b,2),(b,2)) = PQ_\Phi((b,2),(a,1)). \end{aligned}$$

So, by axiom (PQ3), $(a,1) = (b,2)$ which gives a contradiction. Thus, such a case cannot be given.

Hence, we have shown that $a = b = c$ and $\alpha = \beta = 0$.

(iii) Let $a,b,c,d,\alpha,\beta,\gamma \in \mathbb{R}_+$, with $b \leq \min\{c,d\}$, $a + b \leq c + d$ and $\alpha \leq \beta + \gamma$.

First, we will show that $\Phi(c+d-b,\alpha) + \Phi(b,0) \leq \Phi(c,\beta) + \Phi(d,\gamma,)$. To this end we distinguish two cases:

Case 1. $\alpha = \beta = \gamma = 0$. Set $X = \{x_1, x_2, x_3\}$ with $x_1 = a$, $x_2 = b$ and $x_3 = c$ and, besides, define the partial metric p'_X on X as follows: $p'_X(x_1,x_3) = p'_X(x_3,x_1) = c+d-b$, $p'_X(x_1,x_2) = p'_X(x_2,x_1) = c$, $p'_X(x_2,x_3) = p'_X(x_3,x_2) = d$, $p'_X(x_1,x_1) = p'_X(x_3,x_3) = 0$ and $p'_X(x_2,x_2) = b$. Then the function $PQ_\Phi : (X \times \mathbb{R}) \times (X \times \mathbb{R}) \to \mathbb{R}_+$ defined, for each $(x,y),(u,v) \in X \times \mathbb{R}$, by $PQ_\Phi((x,y),(u,v)) =$

$\Phi(p'_X(x,u), d_e(y,v))$, is a partial quasi-metric. By axiom (PQ2), we have that

$$\Phi(p'_X(x_1,x_3), d_e(0,0)) + \Phi(p'_X(x_2,x_2), d_e(0,0)) =$$
$$PQ_\Phi((x_1,0),(x_3,0)) + PQ_\Phi((x_2,0),(x_2,0)) \leq$$
$$PQ_\Phi((x_1,0),(x_2,0)) + PQ_\Phi((x_2,0),(x_3,0)) =$$
$$\Phi(p'_X(x_1,x_2), d_e(0,0)) + \Phi(p'_X(x_2,x_3), d_e(0,0)).$$

So $\Phi(c+d-b,0)+\Phi(b,0) \leq \Phi(c,0)+\Phi(d,0)$. Thus, in this case, $\Phi(c+d-b,\alpha)+\Phi(b,0) \leq \Phi(c,\beta)+\Phi(d,\gamma)$.

Case 2. The condition $\alpha = \beta = \gamma = 0$ does not hold, i.e., at least one element of $\{\alpha,\beta,\gamma\}$ is different of 0. Notice that $\alpha \leq \beta + \gamma$ and, thus, that either $\beta \neq 0$ or $\gamma \neq 0$. Suppose, without loss of generality, that $\beta \neq 0$.

Consider the quasi-metric space (Y, q'_Y) such that $Y = \{1,2,3\}$ and the quasi-metric q'_Y is defined as follows:

$q'_Y(1,3) = \alpha$, $q'_Y(1,2) = q'_Y(3,1) = q'_Y(3,2) = \beta$, $q'_Y(2,3) = q'_Y(2,1) = \gamma$ and $q'_Y(i,i) = 0$ for each $i \in \{1,2,3\}$.

Then the function $PQ_\Phi : (X \times Y) \times (X \times Y) \to \mathbb{R}_+$ defined by $PQ_\Phi((x,y),(u,v)) = \Phi(p'_X(x,u), q'_Y(y,v))$, is a partial quasi-metric. Here (X, p'_X) is the partial metric introduced in the preceding Case 1. By axiom (PQ2) we have that

$$\Phi(p'_X(x_1,x_3), q'_Y(1,3)) + \Phi(p'_X(x_2,x_2), q'_Y(2,2)) =$$
$$PQ_\Phi((x_1,1),(x_3,3)) + PQ_\Phi((x_2,y_2),(2,2)) \leq$$
$$PQ_\Phi((x_1,1),(x_2,2)) + PQ_\Phi((x_2,2),(x_3,3)) =$$
$$\Phi(p'_X(x_1,x_2), q'_Y(1,2)) + \Phi(p'_X(x_2,x_3), q'_Y(2,3)).$$

Thus, $\Phi(c+d-b,\alpha)+\Phi(b,0) \leq \Phi(c,\beta)+\Phi(d,\gamma)$ in this case too.

Now, taking into account that $a \leq c+d-b$, we have, by Lemma 5.1.1, that $\Phi(a,\alpha) \leq \Phi(c+d-b,\alpha)$ and so $\Phi(a,\alpha)+\Phi(b,0) \leq \Phi(c+d-b,\alpha)+\Phi(b,0) \leq \Phi(c,\beta)+\Phi(d,\gamma)$, as we claimed.

Hence, we have proved that if Φ is a *pqmg*-function then it satisfies (i), (ii) and (iii).

Next we assume that $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ satisfies (i), (ii) and (iii). We will prove that Φ is a *pqmg*-function. To this end, (X, p) be a partial metric space and let (Y, q) be a quasi-metric space. Define the function $PQ_\Phi : (X \times Y) \times (X \times Y) \to \mathbb{R}_+$ by $PQ_\Phi((x, y), (u, v)) = \Phi(p(x, u), q(y, v))$ for each $(x, y), (u, v) \in X \times Y$. We will prove that PQ_Φ is a partial quasi-metric on $X \times Y$.

(PQ1). Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Since p is a partial metric on X, we have that $p(x_1, x_2) = p(x_2, x_1) \geq p(x_1, x_1)$. The fact that Φ fulfills (i) gives $\Phi(p(x_1, x_1), 0) \leq \Phi(p(x_1, x_2), q(y_1, y_2))$. Thus

$$PQ_\Phi((x_1, y_1), (x_1, y_1)) = \Phi(p(x_1, x_1), q(y_1, y_1)) = \Phi(p(x_1, x_1), 0) \leq \Phi(p(x_1, x_2), q(y_1, y_2)) = PQ_\Phi((x_1, y_1), (x_2, y_2)).$$

Similarly we can show that

$$PQ_\Phi((x_1, y_1), (x_1, y_1)) \leq PQ_\Phi((x_2, y_2), (x_1, y_1)),$$

since $\Phi(p(x_1, x_1), 0) \leq \Phi(p(x_1, x_2), q(y_2, y_1))$.

(PQ2). Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$. Then we have that $p(x_2, x_2) \leq \min\{p(x_1, x_2), p(x_2, x_3)\}$, $p(x_1, x_3) + p(x_2, x_2) \leq p(x_1, x_2) + p(x_2, x_3)$ and $q(y_1, y_3) \leq q(y_1, y_2) + q(y_2, y_3)$. Since Φ satisfies (iii) we obtain that

$$\Phi(p(x_1, x_3), q(y_1, y_3)) + \Phi(p(x_2, x_2), 0) \leq \Phi(p(x_1, x_2), q(y_1, y_2)) + \Phi(p(x_2, x_3), q(y_2, y_3)).$$

Therefore,

$$PQ_\Phi((x_1, y_1), (x_3, y_3)) + PQ_\Phi((x_2, y_2), (x_2, y_2)) \leq PQ_\Phi((x_1, y_1), (x_2, y_2)) + PQ_\Phi((x_2, y_2), (x_3, y_3)).$$

(PQ3). Obviously if $(x_1, y_1) = (x_2, y_2)$, then

$$PQ_\Phi((x_1, y_1), (x_1, y_1)) = PQ_\Phi((x_1, y_1), (x_2, y_2))$$

and

$$PQ_\Phi((x_2, y_2), (x_2, y_2)) = PQ_\Phi((x_2, y_2), (x_1, y_1)).$$

Conversely, suppose that

$$PQ_\Phi((x_1, y_1), (x_1, y_1)) = PQ_\Phi((x_1, y_1), (x_2, y_2))$$

and, in addition, that

$$PQ_\Phi((x_2, y_2), (x_2, y_2)) = PQ_\Phi((x_2, y_2), (x_1, y_1))$$

for some $(x_1, y_1), (x_2, y_2) \in X \times Y$. It follows that $\Phi(p(x_1, x_1), 0) = \Phi(p(x_1, x_1), q(y_1, y_1)) = \Phi(p(x_1, x_2), q(y_1, y_2))$ and $\Phi(p(x_2, x_2), 0) = \Phi(p(x_2, x_2), q(y_2, y_2)) = \Phi(p(x_2, x_1), q(y_2, y_1))$. Since p is a partial metric on X we have that $\max\{p(x_1, x_1), p(x_2, x_2)\} \leq p(x_1, x_2) = p(x_2, x_1)$. The fact that Φ satisfies (ii) yields that

$$p(x_1, x_1) = p(x_1, x_2) = p(x_2, x_2) \text{ and } q(y_1, y_2) = q(y_2, y_1) = 0.$$

Whence we conclude that $x_1 = x_2$ and $y_1 = y_2$ and, hence, that $(x_1, y_1) = (x_2, y_2)$.

Therefore PQ_Φ is a partial quasi-metric on $X \times Y$. ■

The following result will be crucial in order to provide examples of *pqmg*-functions.

Lemma 5.1.3. *Let $a, b, c, d \in \mathbb{R}_+$ with $a + b \leq c + d$ and $b \leq \min\{c, d\}$. Then, $a \cdot b \leq c \cdot d$.*

Proof. First, we will show that $(c+d-b)\cdot b \leq c\cdot d$. To this end, we can suppose, with out loss of generality, that $d \geq c$. Then,

$$0 \leq (c-b)^2 \leq (c-b)\cdot(d-b) = c\cdot d - c\cdot b - d\cdot b + b^2 = c\cdot d - (c+d-b)\cdot b.$$

Therefore, $(c+d-b)\cdot b \leq c\cdot d$. It follows that $a\cdot b \leq (c+d-b)\cdot b \leq c\cdot d$, since $0 \leq a \leq c+d-b$. ■

The next result provides instances of *pqmg*-functions.

Proposition 5.1.4. *Let $M, N, L \in \mathbb{R}_+$. Then the following functions $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ are pqmg-functions:*

1) $\Phi(x,y) = Mx + Ny + L$ *for each* $(x,y) \in \mathbb{R}_+^2$*;*

2) $\Phi(x,y) = \sqrt{Mx + Ny + L}$ *for each* $(x,y) \in \mathbb{R}_+^2$*;*

3) $\Phi(x,y) = \log(Mx + Ny + L)$ *for each* $(x,y) \in \mathbb{R}_+^2$ *and* $L \geq 1$*.*

Proof. 1) Consider the function given by $\Phi(x,y) = \sqrt{Mx + Ny + L}$. Next we show that it fulfills (i), (ii) and (iii) in Theorem 5.1.2.

(i) Suppose that $b \geq a$. Then, $\Phi(a,0) = Ma + L \leq Mb + L \leq Mb + N\alpha + L = \Phi(b,\alpha)$.

(ii) Now, suppose that $c \geq \max\{a,b\}$. Then, on the one hand,

$$\Phi(a,0) = Ma + L = Mc + N\alpha + L = \Phi(c,\alpha) \Leftrightarrow a = c \text{ and } \alpha = 0,$$

and, on the other hand,

$$\Phi(b,0) = Mb + L = Mc + N\beta + L = \Phi(c,\beta) \Leftrightarrow b = c \text{ and } \beta = 0.$$

Therefore $\Phi(a,0) = \Phi(c,\alpha)$ and $\Phi(b,0) = \Phi(c,\beta)$ implies $a = b = c$ and $\alpha = \beta = 0$.

(iii) Finally, suppose that $b \leq \min\{c,d\}$, $a+b \leq c+d$ and $\alpha \leq \beta+\gamma$. Then

$$\Phi(a,\alpha)+\Phi(b,0) = Ma+N\alpha+L+Mb+L = M(a+b)+N\alpha+2L$$
$$\leq M(c+d)+N(\beta+\gamma)+2L = Mc+N\beta+L+Md+N\gamma+L =$$
$$\Phi(c,\beta)+\Phi(d,\gamma).$$

2) Consider the function given by $\Phi(x,y) = \sqrt{Mx+Ny+L}$. We will only prove that Φ satisfies condition *(iii)* in Theorem 5.1.2, since the fact that Φ satisfies conditions *(i)* and *(ii)* can be proved following similar reasoning to those given in statement 1) in this result.

Suppose that $b \leq \min\{c,d\}$, $a+b \leq c+d$ and $\alpha \leq \beta+\gamma$. Then,

$$\Phi(a,\alpha)+\Phi(b,0) \leq \Phi(c,\beta)+\Phi(d,\gamma),$$

if and only if

$$\sqrt{Ma+N\alpha+L}+\sqrt{Mb+L} \leq \sqrt{Mc+N\beta+L}+\sqrt{Md+N\gamma+L},$$

which is equivalent to

$$\left(\sqrt{Ma+N\alpha+L}+\sqrt{Mb+L}\right)^2 \leq$$

$$\left(\sqrt{Mc+N\beta+L}+\sqrt{Md+N\gamma+L}\right)^2,$$

since

$$\sqrt{Ma+N\alpha+L}, \sqrt{Mb+L}, \sqrt{Mc+N\beta+L}, \sqrt{Md+N\gamma+L} \in \mathbb{R}_+.$$

Therefore, we need to prove that

$$Ma+N\alpha+L+Mb+L+2\cdot\left(\sqrt{Ma+N\alpha+L}\right)\cdot\left(\sqrt{Mb+L}\right) \leq$$

$$Mc+N\beta+L+Md+N\gamma+L+2\cdot\left(\sqrt{Mc+N\beta+L}\right)\cdot\left(\sqrt{Md+N\gamma+L}\right).$$

We have shown that $Ma+N\alpha+L+Mb+L \leq Mc+N\beta+L+Md+N\gamma+L$ in the proof of 1). So it remains to prove that

$$(Ma+N\alpha+L)\cdot(Mb+L) \leq (Mc+N\beta+L)\cdot(Md+N\gamma+L),$$

or equivalently,

$$M^2ab+MaL+N\alpha Mb+N\alpha L+LMb+L^2 \leq$$

$$\leq M^2cd+McN\gamma+McL+N\beta Md+N^2\beta\gamma+N\beta L+LMd+LN\gamma+L^2.$$

It is clear that

$$MaL+LMb = ML(a+b) \leq ML(c+d) = McL+LMd,$$

$$N\alpha Mb = NMb\alpha \leq NM\min\{c,d\}(\beta+\gamma) \leq McN\gamma+N\beta Md,$$

$$N\alpha L = NL\alpha \leq NL(\beta+\gamma) = N\beta L+LN\gamma.$$

By Lemma 5.1.3, we have that $M^2ab \leq M^2cd$, and so the proof is concluded.

3) Consider the function given by $\Phi(x,y)=\sqrt{Mx+Ny+L}$ with $L \geq 1$. We will only prove that Φ satisfies condition (iii) in Theorem 5.1.2, since the fact that Φ satisfies conditions (i) and (ii) can be proved following similar reasoning to those given in statement 1) in this result.

It is clear that $\Phi(a,\alpha)+\Phi(b,0)=\log((Ma+N\alpha+L)\cdot(Mb+L))$ and $\Phi(c,\beta)+\Phi(d,\gamma)=\log((Mc+N\beta+L)\cdot(Md+N\gamma+L))$. Suppose that $b\leq\min\{c,d\}$, $a+b\leq c+d$ and $\alpha\leq\beta+\gamma$. Then, in statement 2) in this result we have shown that

$$(Ma+N\alpha+L)\cdot(Mb+L) \leq (Mc+N\beta+L)\cdot(Md+N\gamma+L).$$

Since log is a monotone function we have that $\Phi(a,\alpha)+\Phi(b,0) \leq \Phi(c,\beta)+\Phi(d,\gamma)$.

The proof is finished. ■

In the light of Proposition 5.1.4 we immediately obtain the following instances of partial quasi-metrics. The example below is relevant because of the lack of instances of partial quasi-metrics in the literature which limits its applicability.

Example 5.1.5. *Let* (X, p) *be a partial metric space and let* (X, q) *be a quasi-metric space. If* $M, N, L \in \mathbb{R}_+$*, then the following functions are partial quasi-metrics on* $X \times Y$*.*

1. $PQ_\Phi\left((x, y), (u, v)\right) = M \cdot p(x, u) + N \cdot q(y, v) + L.$
2. $PQ_\Phi\left((x, y), (u, v)\right) = \sqrt{M \cdot p(x, u) + N \cdot q(y, v) + L}.$
3. $PQ_\Phi\left((x, y), (u, v)\right) = \log\left(M \cdot p(x, u) + N \cdot q(y, v) + L\right)$ *provided that* $L \geq 1$.

Notice that the preceding example shows that the partial quasi-metric induced by a *pqmg*-function is not, in general, neither a partial metric nor a quasi-metric. Indeed, consider the partial metric space $(\mathbb{R}_+, p_m)$ and the quasi-metric space $(\mathbb{R}, q_u)$. Then we have that $PQ_\Phi((x, y), (u, v)) = \max\{x, u\} + \max\{v - y, 0\}$. Now take the *pqmg*-function $\Phi(a, \alpha) = a + \alpha$. Clearly $PQ_\Phi((1, 1), (1, 1)) = 1$ and, thus, PQ_Φ is not a quasi-metric. Moreover, we have that $PQ_\Phi((1, 0), (0, 1)) = 2$ and $PQ_\Phi((0, 1), (1, 0)) = 1$. So PQ_Φ is not a partial metric.

Observe that this is the reason for which the addition of the Baire partial metric and the complexity quasi-metric, $p_B + q_{\mathcal{C}}$, is neither a partial metric nor a quasi-metric. This fact will be crucial later on in Section 5.2.

In [55], the functions that merge a collection of partial metrics into a new one were characterized. Such functions were called n-partial metric preserving functions ($npmp$-functions for short) and their characterization is given by the next theorem.

Theorem 5.1.6. *A function $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a 2pmp-function if and only if it satisfies the following two properties for all $a, b, c, \alpha, \beta, \gamma, \delta \in \mathbb{R}_+$:*

1) $\Phi(a, \alpha) + \Phi(b, \beta) \leq \Phi(c, \gamma) + \Phi(d, \delta)$ whenever $(a + b, \alpha + \beta) \preceq (c + d, \gamma + \delta)$, $(b, \beta) \preceq (c, \gamma)$ and $(b, \beta) \preceq (d, \delta)$.

2) If $(b, \beta) \preceq (a, \alpha)$, $(c, \gamma) \preceq (a, \alpha)$ and $\Phi(a, \alpha) = \Phi(b, \beta) = \Phi(c, \gamma)$, then $(a, \alpha) = (b, \beta) = (c, \gamma)$.

Note that, the preceding characterization was presented for a collection of partial metrics in [55] (see Subsection 1.1.1). However, we have stated such a result only for two partial metrics with the aim of studying the relationship between 2pmp-functions and $pqmg$-functions. In the next proposition we show that each 2pmp-function is a $pqmg$-function. To this end, let us recall that, according to Proposition 9 in [55], every 2pmp-function is monotone with respect to $\preceq$.

Proposition 5.1.7. *Let $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a 2pmp-function. Then Φ is a pqmg-function.*

Proof. We will show that Φ satisfies conditions (i), (ii) and (iii) in Theorem 5.1.2.

(i) Suppose that $b \geq a$. Since $(a, 0) \preceq (b, \alpha)$ and Φ is monotone we obtain that $\Phi(a, 0) \leq \Phi(b, \alpha)$.

(ii) Now, suppose that $c \geq \max\{a, b\}$ and, $\Phi(a, 0) = \Phi(c, \alpha)$ and $\Phi(b, 0) = \Phi(c, \beta)$. The monotony of Φ gives that $\Phi(c, 0) \leq \Phi(c, \alpha) = \Phi(a, 0) \leq \Phi(c, 0)$. It follows that $\Phi(a, 0) = \Phi(c, \alpha) = \Phi(c, 0)$ with $(a, 0) \preceq (c, \alpha)$ and $(c, 0) \preceq (c, \alpha)$. By assertion 2) in Theorem 5.1.6 we deduce that $(a, 0) = (c, \alpha) = (c, 0)$. Thus $a = c$ and $\alpha = 0$. Analogously, we can prove that $b = c$ and $\beta = 0$.

Thus, $a = b = c$ and $\alpha = \beta = 0$.

(iii) Suppose that $b \leq \min\{c, d\}$, $a + b \leq c + d$ and $\alpha \leq \beta + \gamma$. Then $(a, \alpha) + (b, 0) \preceq (c, \beta) + (d, \gamma)$, $(b, 0) \preceq (c, \beta)$ and $(b, 0) \preceq (d, \gamma)$. Since Φ satisfies assertion 1) in Theorem 5.1.6 we deduce that $\Phi(a, \alpha) + \Phi(b, 0) \leq \Phi(c, \beta) + \Phi(d, \gamma)$.

■

The following example shows that there are *pqmg*-functions which are not 2*pmp*-function.

Example 5.1.8. *Consider the function $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ given by*

$$\Phi(a, \alpha) = \begin{cases} a, & \text{if } \alpha = 0 \\ a + 1, & \text{if } \alpha \neq 0 \end{cases}.$$

It is not hard to check that Φ is a pqmg-function. Nevertheless, Φ is not a 2pmp-function. Indeed, $(1, 1) \preceq (1, 3)$, $(1, 2) \preceq (1, 3)$ and $\Phi(1, 3) = \Phi(1, 1) = \Phi(1, 2) = 2$ but $(1, 3) \neq (1, 2)$. Therefore, Φ does not satisfy assertion 2) in Theorem 5.1.6 and, hence, it is not a 2pmp-function.

In [58], the functions that merge a collection of quasi-metrics into a new one were characterized. Such functions were called n-quasi-metric preserving functions (*nqmp*-functions for short) and their characterization is given by the next theorem.

Theorem 5.1.9. *Let $a, b, c, \alpha, \beta, \gamma \in \mathbb{R}_+$. A function $\Phi : \mathbb{R}^2_+ \to \mathbb{R}_+$ is a 2qmp-function if and only it has the following properties:*

1) $\Phi^{-1}(0) = \{(0,0)\}$.

2) If $(a, \alpha) \preceq (b, \beta) + (c, \gamma)$, *then* $\Phi(a, \alpha) \leq \Phi(b, \beta) + \Phi(c, \gamma)$.

As in the partial metric case, the preceding characterization was presented for a collection of quasi-metrics in [58] (see Subsection 1.1.1). However, we have stated such a result only for two partial quasi-metrics with the aim of studying the relationship between 2*qmp*-functions and *pqmg*-functions.

Concerning the relationship between 2*qmp*-functions and *pqmg*-functions, the situation is different. Indeed, we will show that both classes of functions are not comparable. The next examples show that there are 2*qmp*-functions that are not *pqmg*-functions and vice-versa.

Example 5.1.10. *Consider the function $\Phi : \mathbb{R}^2_+ \to \mathbb{R}_+$ given by $\Phi(a, \alpha) = \max\{a, \alpha\}$. By Corollary 13 in [58], Φ is a 2qmp-function. However, it is not a pqmg-function. Indeed, take $a = b = c = 2$, $\alpha = 1$ and $\beta = 2$. Clearly $\Phi(a, 0) = \Phi(c, \alpha) = 2$ and $\Phi(b, 0) = \Phi(c, \beta) = 2$ but $\alpha = 1 \neq 2 = \beta$. Thus, Φ does not fulfill assertion (ii) in Theorem 5.1.2. Hence, it is not a pqmg-function.*

Example 5.1.11. *According to Proposition 5.1.4, the function $\Phi : \mathbb{R}^2_+ \to \mathbb{R}_+$ given by $\Phi(a, \alpha) = a + \alpha + 1$ is a pqmg-function. Nevertheless it is not a 2qmp-function, since $\Phi(0, 0) = 1$ and so Φ does not satisfy assertion 1) in Theorem 5.1.9.*

The next result provides a sufficient condition that ensures that a *pqmg*-function is a 2*qmp*-function.

Proposition 5.1.12. *Let* $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ *be a pqmg-function such that* $\Phi(0,0) = 0$*. Then,* Φ *is a 2qmp-function.*

Proof. We will see that Φ fulfills assertions 1) and 2) in Theorem 5.1.9. Let $a, \alpha \in \mathbb{R}_+$ such that $\Phi(a, \alpha) = 0$. By Lemma 5.1.1 we deduce that $\Phi(a, \frac{\alpha}{2}) = 0$, since $0 = \Phi(0,0) \leq \Phi(a, \frac{\alpha}{2}) \leq \Phi(a, \alpha) = 0$. Whence we have that $\Phi(a, \alpha) = \Phi(0,0)$ and $\Phi(a, \frac{\alpha}{2}) = \Phi(0,0)$. Then, by assertion (ii) in Theorem 5.1.2, we obtain that $a = 0$ and that $\alpha = 0$. Thus $\Phi^{-1}(0) = \{(0,0)\}$.

Let $a, b, c, \alpha, \beta, \gamma \in \mathbb{R}_+$ such that $(a, \alpha) \preceq (b, \beta) + (c, \gamma)$. It follows that $\alpha \leq \beta + \gamma$, $a + 0 \leq b + c$ and that $0 \leq \min\{b, c\}$. By assertion (iii) in Theorem 5.1.2 and the fact that $\Phi(0,0) = 0$, we have that $\Phi(a, \alpha) + \Phi(0,0) \leq \Phi(c, \beta) + \Phi(d, \gamma)$.

Hence, by Theorem 5.1.9 we have that Φ is a $2qmp$-function. ∎

Example 5.1.10 shows that the converse of Proposition 5.1.12 is not true, in general.

Notice that, by Proposition 5.1.12, every *pqmg*-function such that $\Phi(0,0) = 0$ is a 2-metric preserving function (see [58]).

5.2 An application to analysis of algorithms

As exposed in Subsection 1.1.4, partial metrics have been applied successfully to denotational semantics and quasi-metrics have been used in asymptotic complexity analysis of algorithms via fixed point methods, which have been developed independently without establishing any relationship between them. At first glance it seems difficult to combine two different approaches so that we can build a unique framework which allows us to carry out formally, via fixed point methods, the two aforementioned tasks at the same time. However, taking into account the theory exposed in the aforementioned subsection, in the following we show that the aggregation is an appropriate framework for such an objective.

It seems natural to consider $(\Sigma_\infty \times \mathcal{C}, \Phi(p_B(v,w), q_{\mathcal{C}}|_{\mathcal{C}_c}(f,g)))$, where Φ is a *pqmg*-function, as a first attempt to develop a framework to analyze simultaneously, by means of fixed point methods, the running time of computing of an algorithm that performs a computation using a recursive denotational specification and the meaning of such a specification.

Next, with the aim of achieving our objective, we introduce a fixed point theorem in such a way that the contraction is defined using partial quasi-metrics obtained through the aggregation of a partial metric and a quasi-metric. To this end, let us recall, on account of [51], that a partial quasi-metric space (X, pq) is said to be complete provided the metric space (X, d_{pq}), where d_{pq} is the metric on X defined by $d_{pq}(x,y) = \max\{q_{pq}(x,y), q_{pq}(y,x)\}$ for all $x, y \in X$ with $q_{pq}(x,y) = pq(x,y) - pq(x,x)$ for all $x, y \in X$. Notice that a partial quasi-metric space (X, pq) is complete if and only if the associated quasi-metric space (X, q_{pq}) is bicomplete. Observe that this notion retrieves the notion of completeness, introduced in Subsection 1.1.4, for partial metrics and quasi-metrics as a particular case. Besides, following again [51], a mapping from a partial metric space (X, pq) into itself is said to be a contraction if there exists $c \in [0, 1[$ such that $pq(f(x), f(y)) \leq c pq(x,y)$ for all $x, y \in X$. The preceding constant c is said to be the contractive constant of the contraction f. Again this notion of contraction recovers those given in the quasi-metric and partial metric context and which have been exposed in Subsection 1.1.4.

The announced result is based on the following fixed point theorem in partial quasi-metric spaces which was proved in [51].

Theorem 5.2.1. *Let (X, pq) be a complete partial quasi-metric space and let $f : X \to X$. If f is a contraction from (X, pq) into itself, then f has a unique fixed point x_0. Moreover, $pq(x_0, x_0) = 0$.*

Since every quantitative fixed point theorem needs an appropriate notion of completeness, for the dissimilarity under consideration, in order to guarantee the existence and uniqueness of fixed point we will need the next result, whose easy proof we omit, for the partial quasi-metric obtained via aggregation. In order to state it, let us recall that, according to [42], a function $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is called 2-homogeneous provided that $\Phi(ta, t\alpha) = t\Phi(a, \alpha)$ for all $(a, \alpha) \in \mathbb{R}_+^2$ and $t \in \mathbb{R}_+$.

Lemma 5.2.2. *Let $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a 2-homogeneous pqmg-function. Let (X, p) be a complete partial metric space and let (Y, q) be a bicomplete quasi-metric space. Then the partial quasi-metric space $(X \times Y, PQ_\Phi)$ is complete.*

With the help of the previous result we can prove the next theorem which provides the existence and uniqueness of fixed point for contractions from the partial quasi-metric space obtained through aggregation into itself.

Theorem 5.2.3. *Let $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ be a pqmg-function. Let (X, p) be a complete partial metric space and let (Y, q) be a bicomplete quasi-metric space. If F is a contraction from $(X \times Y, PQ_\Phi)$ into itself, then F has a unique fixed point x_0. Moreover, $PQ_\Phi(x_0, x_0) = 0$.*

Proof. By Lemma 5.2.2 we have that the partial quasi-metric space $(X \times Y, PQ_\Phi)$ is complete. If F is a contraction from the partial quasi-metric space $(X \times Y, PQ_\Phi)$ into itself, then Theorem 5.2.1 guarantees the existence and uniqueness of a fixed point $x_0 \in X \times Y$ of F and, in addition, that $PQ_\Phi(x_0, x_0) = 0$. ■

According to what is stated in Subsection 1.1.4, we consider a recursive algorithm which computes the factorial of a positive integer number and it uses the following recursive denotational specification:

$$fact(n) = \begin{cases} 1 & \text{if } n = 1 \\ nfact(n-1) & \text{if } n > 1 \end{cases}. \tag{5.1}$$

The running time of computing of such an algorithm is the solution to the following recurrence equation

$$T_{fact}(n) = \begin{cases} c & \text{if } n = 1 \\ T_{fact}(n-1) + c & \text{if } n > 1 \end{cases}, \tag{5.2}$$

where $c \in \mathbb{R}_+$ $(c > 0)$ is the time taken by the algorithm to obtain the solution to the problem on the base case.

Consider the non-recursive mapping ϕ_{fact} given as follows:

$$\phi_{fact}(f)(n) = \begin{cases} 1 & \text{if } n = 1 \\ nf(n-1) & \text{if } n > 1 \text{ and } n-1 \in \text{dom} f \end{cases}, \tag{5.3}$$

where ϕ_{fact} is acting over the set of partial functions. Of course, the entire factorial function is the unique fixed point of ϕ_{fact}.

As stated in Subsection 1.1.4, every partial function f can be identified with a word $w^f \in \mathbb{N}_\infty$ such that $w^f = w_1^f w_2^f \dots w_k^f$ with $\text{dom} f = \{1, \dots, k\}$ and $w_i^f = f(i)$ for all $i \in \text{dom} f$. So, hereinafter, we will work with $\mathbb{N}_\infty$.

Next consider the mapping $G_{fact} : \mathcal{C}_c \to \mathcal{C}_c$ by

$$G_{fact}(f)(n) = \begin{cases} c & \text{if } n = 1 \\ f(n-1) + c & \text{if } n \geq 2 \end{cases} \tag{5.4}$$

for all $f \in \mathcal{C}_c$. Clearly $f \in \mathcal{C}_c$ is the solution to the recurrence equation (5.2) if and only if f is the unique fixed point of the mapping G_{fact}.

At first glance it seems difficult to combine the two preceding approaches, $(\mathbb{N}_\infty, p_B)$ and $(\mathcal{C}_c, q_{\mathcal{C}_c})$ so that we can build a unique framework which allows us to carry out formally, via fixed point methods, the meaning of the recursive denotational specification and the running time of computing of the algorithm which performs the computation of such a meaning. However, in the remainder of this section, we show that the aggregation approach is an appropriate framework for the aforementioned target.

To this end, we consider the pair $(\mathbb{N}_\infty \times \mathcal{C}_c, PQ_\Phi)$, where Φ is the *pqmg*-function given by $\Phi(a, \alpha) = a + \alpha$ for all $(a, \alpha) \in \mathbb{R}^2_+$. Clearly Φ is 2-homogeneous and we have that

$$PQ_\Phi((v, f), (w, g)) = \Phi(p_B(v, w), q_\mathcal{C}|_{\mathcal{C}_c}(f, g)) = p_B(v, w) + q_\mathcal{C}|_{\mathcal{C}_c}(f, g)$$

for all $v, w \in \mathbb{N}_\infty$ and $f, g \in \mathcal{C}_c$.

In order to discuss the meaning of the recursive denotational specification (5.1) and the running time of computing of the algorithm performing it via (5.2), we need to apply Theorem 5.2.3.

Next we check if the conditions of Theorem 5.2.3 are hold. According to Subsection 1.1.4, the partial metric space (Σ_∞, p_B) is complete and the quasi-metric space $(\mathcal{C}_c, q_\mathcal{C}|_{\mathcal{C}_c})$ is bicomplete. It follows, by Lemma 5.2.2, that $(\mathbb{N}_\infty \times \mathcal{C}_c, PQ_\Phi)$ is a complete partial quasi-metric space.

Next consider the mapping $F : \mathbb{N}_\infty \times \mathcal{C}_c \to \mathbb{N}_\infty \times \mathcal{C}_c$ defined by

$$F(w, f) = (\phi_{fact}(w), G_{fact}(f))$$

for all $(w, f) \in \mathbb{N}_\infty \times \mathcal{C}_c$.

It is clear that $p_B(\phi_{fact}(v), \phi_{fact}(w)) = \frac{1}{2}p_B(v, w)$ for all $v, w \in \mathbb{N}_\infty$. Then

$$p_B(F_1(v, f), F_1(w, g)) = p_B(\phi_{fact}(v), \phi_{fact}(w)) = 2^{-l(v,w)+1} \leq$$

$$\tfrac{1}{2}\Phi(2^{-l(v,w)}, q_{\mathcal{C}}|_{\mathcal{C}_c}(f, g))$$

for all $(v, f), (w, g) \in \mathbb{N}_\infty \times \mathcal{C}_c$. Moreover, we have that

$$q_{\mathcal{C}}|_{\mathcal{C}_c}(F_2(v, f), F_2(w, g)) = q_{\mathcal{C}}|_{\mathcal{C}_c}(G_{fact}(f), G_{fact}(g)) =$$

$$\textstyle\sum_{n=1}^{\infty} 2^{-(n+1)} \left(\frac{g(n)-f(n)}{(g(n)+c)(f(n)+c)}\right) \leq \frac{1}{2}q_{\mathcal{C}}|_{\mathcal{C}_c}(f, g) \leq \frac{1}{2}\Phi(p_B(v, w), q_{\mathcal{C}}|_{\mathcal{C}_c}(f, g))$$

for all $(v, f), (w, g) \in \mathbb{N}_\infty \times \mathcal{C}_c$. Therefore

$$PQ_\Phi(F(w, f), F(v, g)) \leq \frac{1}{2}PQ_\Phi((w, f), (v, g))$$

and, thus, F is a contraction from $(\mathbb{N}_\infty \times \mathcal{C}_c, PQ_\Phi)$ into itself with contraction constant $\frac{1}{2}$.

Applying Theorem 5.2.3 we deduce the existence of a unique fixed point (w^{fact}, f_{fact}) of F such that $(w^{fact}, f_{fact}) \in \mathbb{N}_\infty \times \mathcal{C}_c$.

By construction of F the fixed point (w^{fact}, f_{fact}) satisfies that w^{fact} is the unique fixed point of ϕ_{fact} and that f_{fact} is the unique fixed point of G_{fact}. Whence $w^{fact} \in \mathbb{N}_\infty$ satisfies that $w_1^{fact} = 1$ and $w_n^{fact} = n!$ for all $n \geq 2$ and f_{fact} is the solution to the recurrence equation (5.2). Consequently w^{fact} represents the meaning of the denotational specification (5.1), that is the meaning of the factorial function $fact$, in such a way that $f_{fact}(n)$ is the time taken by the algorithm to compute the factorial of the non-negative integer number n which is provided by the numerical value w_n^{fact}.

To finish our analysis we need to give an asymptotic upper bound of f_{fact}. To this end, we assume that there exists $(w, g) \in \mathbb{N}_\infty \times \mathcal{C}_c$ such that $PQ_\Phi(F(w,g),(w,g)) = 0$. Then, on the one hand, we have that $q_{\mathcal{C}}|_{\mathcal{C}_c}(G_{fact}(g),g) = 0$ and, hence, that $q_{\mathcal{C}}|_{\mathcal{C}_c}(f_{fact},g) = 0$. So $f_{fact} \in \mathcal{O}(g)$, where $f \in \mathcal{O}(g)$ means that there exist $n_0 \in \mathbb{N}$ and $c \in \mathbb{R}_+$ such that $f(n) \leq cg(n)$ for all $n \in \mathbb{N}$ with $n \geq n_0$.

On the other hand, we have that $PQ_\Phi(F(w,g),(w,g)) = 0$ implies that $p_B(\phi_{fact}(w),w) = 0$. Then it can be easily seen that w must be an infinite word and that $w = w^{fact}$.

Finally, a straightforward computation shows that

$$PQ_\Phi(F(w^{fact},g_c),(w^{fact},g_c)) = 0$$

taking $g_c \in \mathcal{C}_c$ with

$$g_c(n) = \begin{cases} c & \text{if } n = 1 \\ cn & \text{if } n > 1 \end{cases}. \tag{5.5}$$

Whence we conclude that $f_{fact} \in \mathcal{O}(g_c)$.

It seems natural to wonder whether there are another function $h \in \mathcal{C}_c$ such that $PQ_\Phi(F(w^{fact},h),(w^{fact},h)) = 0$ and, thus, $f_{fact} \in \mathcal{O}(h)$ but $h \leq g_c$. Nonetheless, an easy computation shows that $PQ_\Phi(F(w^{fact},h),(w^{fact},h)) = 0$, implies that $g_c \leq h$. Therefore, g_c is the least asymptotic upper bound of f_{fact}. Whence we conclude that the asymptotic upper bound of the running time of computing of the algorithm computing the factorial is $\mathcal{O}(g_c)$. This fact is in line with what is stated in the literature (see, for instance, [6]).

Chapter 6

A technique for fuzzifying metric spaces via metric preserving functions

In this chapter we develop a new technique for constructing fuzzy metric spaces, in the sense of George and Veeramani, from metric spaces and by means of the Lukasievicz t-norm. In particular such a technique is based on the use of metric preserving functions in the sense of J. Doboš. Besides, the new generated fuzzy metric spaces are strong and completable, and if we add an extra condition, they are principal. Appropriate examples of such fuzzy metric spaces are given in order to illustrate the exposed technique. Throughout this chapter the fuzzy metrics are understood as GV-fuzzy metrics. Moreover, usual t-norms are denoted following the tradition in the research field of fuzzy-pseudo metric spaces.

The structure of the chapter is organized as follows. In Section 6.1, we introduce the notion of uniformly continuous mapping between stationary fuzzy metric spaces and metric spaces, and *vice-versa*. Thus we define when they are equivalent. Based on such a notion, we present a technique that allows to construct stationary fuzzy metric spaces from a metric space by means of metric preserving functions with values in $[0, 1[$. Moreover, it is showed that the new constructed stationary fuzzy metric spaces are completable provided that the used metric preserving function is a strongly metric preserving function (in the sense of Doboš). In addition, it is proved that the new stationary fuzzy metric spaces are complete if and only if the metric spaces from which are generated are also complete. Section 6.2 is devoted to generalize the construction presented in Section 6.1 to the non-stationary case. Thus fuzzy metric spaces are generated from metric spaces by means of a family of metric preserving functions that satisfy a distinguished condition which will be specified later on. These fuzzy metric spaces are always strong and, in addition, they are complete if and only if the metric spaces from which are generated are also complete. Furthermore, they are principal and completable whenever all metric preserving functions belonging to the family under consideration are strongly metric preserving functions.

6.1 A technique for inducing stationary fuzzy metric spaces from metric spaces via metric preserving functions

First of all, we introduce two continuity notions that we will need in our subsequent discussion.

Definition 6.1.1. Let $(X, M, *)$ be a stationary fuzzy metric space and let

(Y, ρ) be a metric space. A mapping $f : X \to Y$ is said to be M-ρ uniformly continuous if given $\epsilon > 0$ we can find $\delta \in]0, 1[$ such that $M(x, y) > 1 - \delta$ implies $\rho(f(x), f(y)) < \epsilon$.

Definition 6.1.2. Let (X, d) be a metric space and let $(Y, N, \diamond)$ be a stationary fuzzy metric space. We will say that the mapping $f : X \to Y$ is d-N uniformly continuous if given $\epsilon \in]0, 1[$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $N(f(x), f(y)) > 1 - \epsilon$.

The following examples illustrate the preceding definitions.

Example 6.1.3. *1. Let $(X, M, \mathfrak{L})$ be the stationary fuzzy metric space, where $\mathfrak{L}$ stands for the Luckasievicz t-norm $*_L$, $X = [0, 1[$ and*

$$M(x, y, t) = 1 - |x - y|$$

for each $x, y \in X$ and $t > 0$. Also, let (Y, ρ) be the metric space where $Y = \mathbb{R}^2$ and

$$\rho((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Define the mapping $f : X \to Y$ by

$$f(x) = (4x, 3x + 1)$$

for all $x \in X$. Next we will see that f is M-ρ uniformly continuous.

To this end, fix $\epsilon > 0$ and consider $\delta < \min\{\epsilon/5, 1\}$. Then, for each $x, y \in X$ satisfying $M(x, y) > 1 - \delta$, we have that $\rho(f(x), f(y)) < \epsilon$. Indeed, if $1 - \delta < M(x, y) = 1 - |x - y|$, then $|x - y| < \delta < \epsilon/5$ and so

$$\rho(f(x), f(y)) = \rho((4x, 3x + 1), (4y, 3y + 1)) =$$

$$\sqrt{(4x - 4y)^2 + (3x + 1 - 3y - 1)^2} =$$

$$\sqrt{16(x - y)^2 + 9(x - y)^2} = 5 \cdot |x - y| < 5 \cdot \tfrac{\epsilon}{5} = \epsilon.$$

Therefore, f is M-ρ uniformly continuous.

2. *Let (X, d_e) be the metric space such that $X = [0,1]$ and $d_e(x,y) = |x-y|$ for all $x, y \in X$. Also, let $(Y, N, \cdot)$ be the stationary fuzzy metric space with $Y = [1,2]$ and*

$$N(x,y) = \frac{\min\{x,y\}}{\max\{x,y\}}$$

for all $x, y \in Y$. Notice that the t-norm $*_P$ has been denoted by $\cdot$. Define the mapping $f : X \to Y$ by $f(x) = x+1$ for all $x \in X$. Next we will see that f is d_e-N uniformly continuous.

With this aim, fix $\epsilon > 0$ and consider $\delta < \epsilon$. Then, for each $x, y \in X$ satisfying $d_e(x,y) < \delta$, we have that $N(f(x), f(y)) > 1 - \epsilon$. Indeed, let $x, y \in [0,1]$ such that $d_e(x,y) < \delta$. Since $\max\{x+1, y+1\} \geq 1$ we have that

$$\delta > |x-y| > \frac{|x-y|}{\max\{x+1, y+1\}}.$$

Furthermore,

$$N(f(x), f(y)) = \tfrac{\min\{x+1,y+1\}}{\max\{x+1,y+1\}} =$$

$$1 - \tfrac{|x-y|}{\max\{x+1,y+1\}} > 1 - \delta > 1 - \epsilon.$$

Obviously, note that if f is M-ρ (d-N) uniformly continuous then it is continuous. With the above terminology we can prove the next proposition which will be crucial in the development of our new technique.

Proposition 6.1.4. *Let $(X, M, *)$ be a stationary fuzzy metric space and let (Y, ρ) be a metric space. Let $f : X \to Y$ be an M-ρ uniformly continuous mapping. If $(x_n)_{n\in\mathbb{N}}$ is an M-Cauchy sequence then $(f(x_n))_{n\in\mathbb{N}}$ is a ρ-Cauchy sequence.*

Proof. Let $\epsilon > 0$, and consider a sequence $(x_n)_{n\in\mathbb{N}}$ in X which is M-Cauchy. Since f is M-ρ uniformly continuous we can find $\delta \in]0,1[$ such that $M(x,y) > 1-\delta$ implies $\rho(f(x),f(y)) < \epsilon$. Now, since $\{x_n\}$ is M-Cauchy we can find $n_0 \in \mathbb{N}$ such that $M(x_n,x_m) > 1-\delta$ for all $n,m \geq n_0$, and so $\rho(f(x_n),f(x_m)) < \epsilon$ for all $n,m \geq n_0$. Hence $(f(x_n))_{n\in\mathbb{N}}$ is ρ-Cauchy. ■

Applying a similar reasoning to that given in the proof of Proposition 6.1.4 we can prove the next one.

Proposition 6.1.5. *Let (X,d) be a metric space and let $(Y,N,\diamond)$ be a stationary fuzzy metric space. Let $f : X \to Y$ a d-N uniformly continuous mapping. If $(x_n)_{n\in\mathbb{N}}$ is a d-Cauchy sequence then $(f(x_n))_{n\in\mathbb{N}}$ is an N-Cauchy sequence.*

From now on, if no confusion arises, we will omitt the metric and the fuzzy metric when we refer to a mapping f as uniformly continuous (in the sense of Definitions 6.1.1 and 6.1.2).

In the light of the introduced notions, the next result shows that the composition of uniformly continuous mappings among metric spaces and stationary fuzzy metric spaces is uniformly continuous.

Proposition 6.1.6. *Let $(X,M,*)$ be a stationary fuzzy metric space, and let (Y,ρ) and (Z,d) be two metric spaces. Suppose that $f : X \to Y$ and $g : Y \to Z$ are two uniformly continuous mappings. Then $g \circ f$ is a uniformly continuous mapping.*

Proof. Let $\epsilon > 0$. Since g is uniformly continuous we can find $\delta_1 > 0$ such that $\rho(a,b) < \delta_1$ implies $d(g(a),g(b)) < \epsilon$. Since f is M-ρ uniformly continuous then, given $\delta_1 > 0$, we can find $\delta \in]0,1[$ such that $M(x,y) > 1-\delta$

implies $\rho(f(x), f(y)) < \delta_1$. Therefore, for each $x, y \in X$ such that $M(x, y) > 1 - \delta$ it is satisfied that $\rho(f(x), f(y)) < \delta_1$ and so $d(g(f(x), f(y)) < \epsilon$, hence $g \circ f$ is M-d uniformly continuous. ■

In an analogous way we can prove the next proposition.

Proposition 6.1.7. *Let $(X, M, *)$ be a stationary fuzzy metric space, and let (Y, ρ) and (Z, d) be two metric spaces. Suppose that $f : Z \to X$ and $g : Y \to Z$ are two uniformly continuous mappings. Then $f \circ g$ is a uniformly continuous mapping.*

Taking into account Propositions 6.1.4 and 6.1.5 we extend the classical concept of uniformly equivalent metric spaces to our framework as follows. Let us recall that two metrics d and ρ on X are called uniformly equivalent if both identity mappings $i : (X, d) \to (X, \rho)$ and $i : (X, \rho) \to (X, d)$ are uniformly continuous.

Definition 6.1.8. Let d and $(M, *)$ be a metric and a stationary fuzzy metric on X, respectively. Then d and $(M, *)$ are called uniformly equivalent if both identity mappings are M-d and d-M uniformly continuous, respectively.

Observe that if a metric d is uniformly equivalent to a fuzzy metric $(M, *)$ on X, then $\tau_M = \tau_d$, where τ_d denotes the topology induced by the metric d. Moreover, every d-Cauchy sequence is an M-Cauchy sequence and vice-versa.

The next examples provide instances of metric and fuzzy metric spaces that are uniformly equivalent.

Example 6.1.9. *Assume that (X, d) is a bounded metric space, i.e., there exists $K > 0$ such that $d(x, y) \le K$ for all $x, y \in X$. On account of [33], we* *have that* $(X, M, \mathfrak{L})$ *is a stationary fuzzy metric space, where*

$$M(x,y) = 1 - \frac{d(x,y)}{1+K}$$

for all $x, y \in X$. *Next we show that* (X, d) *and* $(X, M, \mathfrak{L})$ *are uniformly equivalent. Indeed, it is not hard to check that, given* $\varepsilon > 0$, *then* $d(x,y) < \varepsilon$ *provided that* $M(x,y) > 1 - \delta$ *whenever* δ *is taken as follows:*

$$\delta = \begin{cases} \frac{\varepsilon}{1+K} & \text{if } \varepsilon < K \\ \\ \frac{K}{1+K} & \text{if } \varepsilon \geq K \end{cases}$$

Hence the identity mapping is M-d *uniformly continuous. Moreover, given* $\varepsilon \in]0,1[$, *then* $M(x,y) > 1 - \varepsilon$ *provided that* $d(x,y) < \delta$ *whenever* δ *is taken as* $\delta = (1+K)\varepsilon$. *Thus the identity mapping is* d-M *uniformly continuous. So* (X, d) *and* $(X, M, \mathfrak{L})$ *are uniformly equivalent. Of course, it follows that* $\tau_M = \tau_d$.

Example 6.1.10. *Assume that* $(X, M, \mathfrak{L})$ *is a stationary fuzzy metric space. According to [33], the mapping* d_M *defined on* $X \times X$ *by*

$$d_M(x,y) = 1 - M(x,y)$$

for all $x, y \in X$ *is a metric on* X. *A straightforward computation shows that the identity mapping is* M-d_M *and* d_M-M *uniformly continuous. So* $(X, M, \mathfrak{L})$ *and* (X, d_M) *are uniformly equivalent. Of course, it follows that* $\tau_M = \tau_{d_M}$.

Inspired by the preceding examples we will introduce the promised technique for generating stationary fuzzy metric spaces from metric spaces by means of metric preserving functions which, besides, preserves the spirit of the aforementioned examples. To this end, we will denote by $\mathcal{M}^1$ and by

$\mathcal{M}_S^1$ the class of metric preserving functions and strongly metric preserving functions (see Subsection 1.1.1), respectively, satisfying in both cases that $f(x) < 1$ for each $x \in [0, \infty[$.

First, recall the following result proved in [15].

Proposition 6.1.11. *Let f be a metric preserving function and let (X, d) be a metric space. Then,*

(i) If (X, d) is not uniformly discrete, then d_f and d are uniformly equivalent if and only if f is a strongly metric preserving function.

(ii) If (X, d) is uniformly discrete, then d_f and d are uniformly equivalent.

The next result gives a method of obtaining stationary fuzzy metrics deduced from classical metrics and a sufficient condition in order to the topology induced by the stationary fuzzy metric coincides with the topology induced by the metric.

Proposition 6.1.12. *Let (X, d) be a metric space and let $f \in \mathcal{M}^1$. Then:*

(i) $(X, M_f, \mathfrak{L})$ is a stationary fuzzy metric space, where $M_f(x, y) = 1 - d_f(x, y)$ for all $x, y \in X$. Moreover, $\tau_{M_f} = \tau_{d_f}$.

(ii) If, in addition $f \in \mathcal{M}_S^1$, then $(M_f, \mathfrak{L})$ and d are uniformly equivalent and, thus, $\tau_{M_f} = \tau_d$.

Proof. Let (X, d) be a metric space and consider $f \in \mathcal{M}^1$.

(i) It is straightforward.

(ii) Suppose that $f \in \mathcal{M}^1_S$. Then, from the previous theorem we obtain that d and d_f are uniformly equivalent, and so the identity mapping $i_1 : (X, d_f) \to (X, d)$ is uniformly continuous. Next we show that the identity mapping $i_2 : (X, M_f) \to (X, d_f)$ is uniformly continuous. To this end, we can consider $\epsilon \in]0, 1[$, since the metric d_f is bounded with $d_f(x, y) < 1$ for all $x, y \in X$. Then, $M_f(x, y) > 1 - \epsilon$ if and only if $1 - d_f(x, y) > 1 - \epsilon$, or equivalently, if and only if $d_f(x, y) < \epsilon$. Therefore $i_2 : (X, M) \to (X, d_f)$ is uniformly continuous. Thus, by Proposition 6.1.6, the identity mapping $i : (X, M) \to (X, d)$ is uniformly continuous, since $i = i_1 \circ i_2$. Following similar arguments, but now with the help of Proposition 6.1.7, we can show that $i : (X, d) \to (X, M)$ is uniformly continuous. Therefore M_f and d are uniformly equivalent and, thus, $\tau_{M_f} = \tau_d$.

■

The next example shows that the condition "$f \in \mathcal{M}^1_S$" cannot be relaxed in the assertion (ii) in the statement of Proposition 6.1.12.

Example 6.1.13. *Consider $f : [0, \infty[\to [0, \infty[$ given by*

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ \frac{x+1}{x+2}, & \text{if } x > 0. \end{cases}$$

Obviously, f is monotone. We will see that it is also subadditive. Let $a, b \in [0, \infty[$. If one of them is 0 the subadditive condition is obvious. Suppose that $a, b \in]0, \infty[$. Then,

$$f(a + b) = \frac{a + b + 1}{a + b + 2} < 1 \leq \frac{a + 1}{a + 2} + \frac{b + 1}{b + 2} = f(a) + f(b).$$

Therefore, f is a monotone and subadditive function such that $f \in \mathcal{O}$, so f is metric preserving. Moreover, $f(x) < 1$ for each $x \in [0, \infty[$ and it *is clearly not continuous at* 0. *Then, by Theorem 1.1.4, f is not strongly metric preserving, i.e., there exists (X, d) which is not topologically equivalent to (X, d_f). Then, by assertion (i) in the statement of Proposition 6.1.12, $\tau_M = \tau_{d_f}$ but τ_{d_f} is not equivalent to τ_d.*

Since every stationary fuzzy metric space is always principal we have the following result as a consequence of Proposition 6.1.12.

Corollary 6.1.14. *Let (X, d) be a metric space and let $f \in \mathcal{M}^1$. Then the fuzzy metric space $(X, M_f, \mathfrak{L})$ is principal, where $M_f(x, y) = 1 - d_f(x, y)$ for all $x, y \in X$.*

Attending to Proposition 6.1.12, we introduce the next definition.

Definition 6.1.15. Let $f \in \mathcal{M}^1$ and (X, d). The stationary fuzzy metric $(M_f, \mathfrak{L})$ defined on X by $M_f(x, y) = 1 - d_f(x, y)$ for all $x, y \in X$, will be called the stationary fuzzy metric induced by f and (X, d) or, simply, induced by f if no confusion arises.

In the following example we show that some well-known instances of stationary fuzzy metric spaces can be obtained applying the technique introduced in Proposition 6.1.12. Observe that such an example illustrates Definition 6.1.15 and, in addition, complementes the examples furnished by Examples 6.1.9 and 6.1.10 about fuzzy metric spaces uniformly equivalent to metric spaces.

Example 6.1.16. *Let (X, d) be a metric space and let $K > 0$. Consider the functions f, g and h defined for all $x \in [0, \infty[$ by*

1. $f(x) = \min\left\{\frac{x}{1+K}, \frac{K}{1+K}\right\}$,

2. $g(x) = \frac{x}{K+x}$,
3. $h(x) = 1 - \exp^{-\frac{x}{K}}$.

Note that the preceding functions are monotone, subadditive and belong to $\mathcal{O}$, so they are metric preserving. Moreover, it is not hard to see that they are continuous and, thus, by Theorem 1.1.4, we have that they are strongly metric preserving. Since, in addition, they take values into $[0,1[$ we have that, in fact, they belong to $\mathcal{M}_S^1$.

The corresponding stationary fuzzy metrics $(M_f, \mathfrak{L})$, $(M_g, \mathfrak{L})$ and $(M_h, \mathfrak{L})$ induced by f, g and h and (X,d) are given, respectively, by

1. $M_f(x,y) = 1 - d_f(x,y) = \max\left\{1 - \frac{d(x,y)}{1+K}, \frac{1}{1+K}\right\}$,
2. $M_g(x,y) = 1 - d_g(x,y) = \frac{K}{K+d(x,y)}$,
3. $M_h(x,y) = 1 - d_h(x,y) = \exp^{-\frac{d(x,y)}{K}}$,

for each $x, y \in X$.

Once the technique for generating stationary fuzzy metric spaces have been introduced we are able to discuss their completion. To this end, we recall the pertinent notions and results concerning the fuzzy metric completion.

In the study of completion of fuzzy metric space the notion of isometry introduced by V. Gregori and S. Romaguera in [36] plays a central role. It is recalled below.

Definition 6.1.17. Let $(X, M, *_1)$ and $(Y, N, *_2)$ be two fuzzy metric spaces. A mapping f from X to Y is called an *isometry* if for each $x, y \in X$ and each $t > 0$, $M(x, y, t) = N(f(x), f(y), t)$. Moreover if f is a bijective isometry, then $(X, M, *_1)$ and $(Y, N, *_2)$ (or simply X and Y) are called *isometric*. Thus a *fuzzy metric completion* of $(X, M, *)$ is a complete fuzzy metric space $(X^*, M^*, \diamond)$ such that $(X, M, *)$ is isometric to a dense subspace of X^*. Furthermore, $(X, M, *)$ (or simply X) is called *completable* if it admits a fuzzy metric completion.

In [37], the following characterization about completion of a fuzzy metric space is given, such a characterization will be useful when we discuss the completion of those fuzzy metric spaces induced by our new technique.

Theorem 6.1.18. *Let $(X, M, *)$ be a fuzzy metric space, and let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two Cauchy sequences in $(X, M, *)$. Then $(X, M, *)$ is completable if and only if it satisfies the following conditions:*

(C1) The assignment $t \to \lim_n M(a_n, b_n, t)$ is a continuous function from $]0, \infty[$ into $]0, 1]$.

(C2) If $\lim_n M(a_n, b_n, s) = 1$ for some $s > 0$ then $\lim_n M(a_n, b_n, t) = 1$ for all $t > 0$.

Remark 6.1.19. Obviously, a stationary fuzzy metric space $(X, M, *)$ is completable if and only if $\lim_n M(a_n, b_n) > 0$ for every two Cauchy sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$.

Taking into account the preceding notions Gregori and Romaguera proved in [36] the following useful result.

Proposition 6.1.20. *If a fuzzy metric space has a fuzzy metric completion then it is unique up to isometry.*

Remark 6.1.21. Attending to the last proposition, let us recall the construction of the completion of a fuzzy metric space given in [37]. Suppose $(X^*, M^*, \diamond)$ is a fuzzy metric completion of $(X, M, *)$. Then we have that:

1. $X \subseteq X^*$, where X^* is the quotient set on the set of M-Cauchy sequences induced by the equivalence relation $\sim$ defined by

$$(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}} \Leftrightarrow \lim_n M(x_n, y_n, t) = 1 \text{ for all } t > 0.$$

2. $\diamond$ can be assumed to be $*$.

3. M^* is defined on X^* by

$$M^*(x^*, y^*, t) = \lim_n M(x_n, y_n, t)$$

   for all $x^*, y^* \in X^*$ and for all $t > 0$, where $\{x_n\} \in x^*$ and $\{y_n\} \in y^*$.

Now, we are able to approach the study of the the completion of the generated stationary fuzzy metric spaces constructed above.

Theorem 6.1.22. *Let (X, d) be a metric space and let $f \in \mathcal{M}_S^1$. The following assertions hold:*

(i) *$(X, M_f, \mathfrak{L})$ is complete if and only if (X, d) is complete.*

(ii) *$(X, M_f, \mathfrak{L})$ is completable and the completion of $(X, M_f, \mathfrak{L})$ is $(X^*, M_f^*, \mathfrak{L})$, where M_f^* is the stationary fuzzy metric given by $M_f^*(a^*, b^*) = 1 - d_f^*(a^*, b^*)$ for each $a^*, b^* \in X^*$ and, in addition, (X^*, d^*) is the completion of (X, d).*

Proof. Let (X, d) be a metric space and let $f \in \mathcal{M}_S^1$. Consider the stationary fuzzy metric M_f induced by f and (X, d).

(i) By Proposition 6.1.12 d and M_f are uniformly equivalent. Hence $\tau_{M_f} = \tau_d$ and, by Propositions 6.1.4 and 6.1.5, a sequence in X is M_f-Cauchy if and only if it is d-Cauchy.

(ii) First, we will show that (X, M_f) is completable. With this aim, let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two M_f-Cauchy sequences. By assertion (ii) in the statement of Proposition 6.1.12 and by Proposition 6.1.4, they are d-Cauchy. Consider $a^*, b^* \in X^*$ such that $\{a_n\} \in a^*$ and $\{b_n\} \in b^*$. By Theorem 1.1.4 f is continuous and so we have that

$$\lim_n M_f(a_n, b_n) = \lim_n (1 - d_f(a_n, b_n)) = 1 - \lim_n f(d(a_n, b_n))$$
$$= 1 - f(\lim_n d(a_n, b_n)) = 1 - f(d^*(a^*, b^*)).$$

Since $f \in \mathcal{M}^1$ we have that

$$\lim_n M_f(a_n, b_n) = 1 - f(d^*(a^*, b^*)) > 0.$$

Therefore, by Remark 6.1.19 we have that (X, M_f) is completable.

Next suppose that $(\tilde{X}, M_f^*, \mathfrak{L})$ is the completion of $(X, M_f, \mathfrak{L})$. We will see that $\tilde{X} = X^*$.

By Proposition 6.1.12, d and M_f are uniformly equivalent and, thus, $(x_n)_{n\in\mathbb{N}}$ is an M_f-Cauchy sequence in X if and only if $(x_n)_{n\in\mathbb{N}}$ is a d-Cauchy sequence in X. On the other hand, given two d-Cauchy sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in X we have that

$$\lim_n d_f(x_n, y_n) = \lim_n f(d(x_n, y_n) = f(\lim_n d(x_n, y_n)),$$

because, by Theorem 1.1.4, f is continuous. Hence

$$\lim_n M_f(x_n, y_n) = 1 \Leftrightarrow \lim_n d_f(x_n, y_n) = 0 \Leftrightarrow \lim_n d(x_n, y_n) = 0,$$

since $f \in \mathcal{O}$. Thus, $\tilde{X} = X^*$.

Finally, consider $a^*, b^* \in X^*$ such that $\{a_n\} \in a^*$ and $\{b_n\} \in b^*$. Attending to Remark 6.1.21, we have that

$$M_f^*(a^*, b^*, t) = \lim_n M_f(a_n, b_n, t),$$

for each $t > 0$. Since $M_f(a_n, b_n, t) = M_f(a_n, b_n)$ for each $t > 0$ and

$$\lim_n M_f(a_n, b_n) = 1 - \lim_n d_f(a_n, b_n) =$$

$$1 - f(\lim_n d(a_n, b_n)) = 1 - f(d^*(a^*, b^*)) = 1 - d_f^*(a^*, b^*),$$

we conclude that

$$M_f^*(a^*, b^*, t) = 1 - d_f^*(a^*, b^*)$$

for all $\{a_n\} \in a^*$ and $\{b_n\} \in b^*$ and $t > 0$.

■

Remark 6.1.23. Notice that the notion of uniformly discrete metric space can be adapted to the stationary fuzzy metric context in the following easy way. Indeed, a stationary fuzzy metric space $(X, M, *)$ will be said to be uniformly discrete provided that there exists $\epsilon \in]0, 1[$ such that $M(x, y) < \epsilon$ for all $x, y \in X$ such that $x \neq y$. Of course we have omitted the t in the expression of M because of its stationary nature.

Taking into account the preceding notion we have the following reasoning regarding the completion of the stationary fuzzy metric $(X, M_f, \mathfrak{L})$.

If the metric preserving function f in the statement of Theorem 6.1.22 is assumed to be discontinuous, then, by (ii) in Proposition 6.1.11, the metric d_f is uniformly discrete and, consequently, the stationary fuzzy metric M_f induced by f and (X, d) is also uniformly discrete. Therefore, the unique

M_f-Cauchy sequences on X are the eventually constant sequences and so $(X, M_f, \mathfrak{L})$ is complete.

In the light of Theorem 6.1.22 and Remark 6.1.23 we can assert that for each $f \in \mathcal{M}^1$ and each metric space (X, d) the stationary fuzzy metric space $(X, M_f, \mathfrak{L})$ induced by f and (X, d) is always completable.

6.2 A technique for inducing non-stationary fuzzy metric spaces from metric spaces via metric preserving functions

In Section 6.1, we have provided a technique which is able to induce stationary fuzzy metric spaces from classical metric spaces by means of metric preserving functions and, in addition, we have studied the completion and completeness of such fuzzy metric spaces. In this section we will extend the aforementioned technique in order to construct non-stationary fuzzy metric spaces induced by classical metric spaces and, now, by a family of metric preserving functions. Moreover we will study the completion and completeness of such fuzzy metric spaces. We begin such a study introducing the next concept.

Definition 6.2.1. Consider a family $F = \{f_t : t > 0\}$ of real functions defined on $[0, \infty[$. We will say that F is decreasing if $t < s$ implies $f_t(x) \geq f_s(x)$ for each $x \in [0, \infty[$.

Example 6.2.7 gives a few instances of decreasing families of functions in the sense of Definition 6.2.1.

In order to introduce the announced technique let us recall a few known facts. According to [33] we have the following:

Proposition 6.2.2. *Let* $\{(X, M_t, \mathfrak{L}) : t > 0\}$ *be a family of stationary fuzzy metric spaces associated to a strong fuzzy metric space* $(X, M, \mathfrak{L})$*. Then the following assertions hold:*

(i) The real function d*, defined by* $d(x, y) = 1 - \bigwedge_t M_t(x, y)$ *for all* $x, y \in X$*, is a metric on* X *such that* $\tau_d \supseteq \bigvee_t \tau_{M_t} = \tau_M$*, where* $\bigwedge_t M_t$ *is the real function defined on* $X \times X$ *by* $\bigwedge_t M_t(x, y) = \inf\{M_t(x, y) : t > 0\}$ *for all* $x, y \in X$*.*

(ii) The real function d_t*, defined by* $d_t(x, y) = 1 - M_t(x, y)$ *for all* $x, y \in X$*, is a metric on* X *for all* $t > 0$*. Moreover,* $d(x, y) = \bigvee_t d_t(x, y)$ *for all* $x, y \in X$*, where* $\bigvee_t d_t(x, y) = \sup\{d_t(x, y) : t > 0\}$ *for all* $x, y \in X$*, is a metric on* X *and* $\tau_d \supseteq \tau_{d_t}$ *for all* $t > 0$*.*

Taking into account the preceding proposition we introduce the new technique in the following result.

Theorem 6.2.3. *Let* (X, d) *be a metric space and let* $F = \{f_t : t > 0\}$ *be a decreasing family of functions included in* $\mathcal{M}^1$ *such that the function* f^x *is continuous on* $]0, \infty[$ *for each* $x \in [0, \infty[$*, where* $f^x(t) = f_t(x)$ *for all* $t > 0$*. Then the following assertions hold:*

(i) $(X, M_F, \mathfrak{L})$ *is a fuzzy metric space, where* $M_F(x, y, t) = 1 - d_{f_t}(x, y)$ *for each* $x, y \in X$ *and each* $t > 0$*.*

(ii) $(X, M_F, \mathfrak{L})$ *is strong.*

(iii) $\tau_{M_F} = \bigvee\{\tau_{M_{F_t}} : t > 0\} = \bigvee\{\tau_{d_{f_t}} : t > 0\}$*, where* $M_{F_t}(x, y) = M_F(x, y, t)$ *for each* $x, y \in X$ *and* $t > 0$*.*

(iv) *The function* d_F *is a metric on* X*, where* d_F *is defined by* $d_F(x,y) = 1 - \bigwedge_t M_{F_t}(x,y)$ *for all* $x, y \in X$*. Besides,* $d_F(x,y) = \bigvee_t d_{f_t}(x,y)$ *for all* $x, y \in X$ *and* $\tau_{d_F} \supseteq \tau_{M_F}$*.*

(v) *If* $F \subseteq \mathcal{M}^1_S$*, then* $(X, M_F, \mathfrak{L})$ *is principal and* $\tau_{d_F} \supseteq \tau_{M_F} = \tau_d$*.*

Proof. Consider a decreasing family $F = \{f_t : t > 0\}$ of functions in $\mathcal{M}^1$ such that for each $x \in [0, \infty[$ we have that f^x is continuous on $]0, \infty[$. Define $M_F(x,y,t) = 1 - d_{f_t}(x,y)$ for each $x, y \in X$ and each $t > 0$.

(i) Next we will see that $(X, M_F, \mathfrak{L})$ is a fuzzy metric space.

It is obvious that M satisfies axioms $(GV1), (GV2)$ and $(GV3)$. Furthermore, the assumption that the function f^x is continuous on $]0, \infty[$ for all $x \in [0, \infty[$ ensures that $(GV5)$ is fulfilled. We will show that $(GV4)$ is satisfied too.

First, note that for each $x, y \in X$ we have that $M_{F_{x,y}}$ is a monotone function on $]0, \infty[$, where $M_{F_{x,y}}(t) = M_F(x,y,t)$ for each $t > 0$. Indeed, since the family $\{f_t : t > 0\}$ is decreasing, given $0 < t < s$ then

$$M_{F_{x,y}}(s) = M_F(x,y,s) = 1 - d_{f_s}(x,y) \geq$$

$$1 - d_{f_t}(x,y) = M_F(x,y,t) = M_{F_{x,y}}(t).$$

Moreover, on the one hand, $M_F(x,z,t) = 1 - d_{f_t}(x,z) > 0$, since $f_t \in \mathcal{M}_1$. On the other hand, since for each $t > 0$ we have that d_{f_t} is a metric on X, then for each $x, y, z \in X$ and each $t > 0$ we have that $M_F(x,z,t) = 1 - d_{f_t}(x,z) \geq 1 - d_{f_t}(x,y) - d_{f_t}(y,z) = 1 - d_{f_t}(x,y) + 1 - d_{f_t}(y,z) - 1$. Therefore,

$$M_F(x,z,t) \geq M_F(x,y,t)\mathfrak{L}M_F(y,z,t). \tag{6.1}$$

Finally, given $x, y, z \in X$ and $t, s > 0$, by these two last observations we have

$$M_F(x,z,t+s) \geq M_F(x,z,\max\{t,s\}) \geq$$

$$M_F(x,y,\max\{t,s\})\mathfrak{L}M_F(y,z,\max\{t,s\}) \geq M(x,y,t)\mathfrak{L}M(y,z,s)$$

and so $(GV4)$ is fulfilled. Therefore, $(X, M_F, \mathfrak{L})$ is a fuzzy metric space.

(ii) The inequality (6.1) shows that the fuzzy metric space $(X, M_F, \mathfrak{L})$ is strong, i.e., that it holds the condition $(GV4')$.

(iii) Since $(X, M_F, \mathfrak{L})$ is strong we deduce, by Remark 6.2.9, that $\tau_{M_F} = \bigvee\{\tau_{M_{F_t}} : t > 0\}$, where $M_{F_t}(x,y) = M_F(x,y,t)$ for each $x, y \in X$ and $t > 0$. Proposition 6.1.12 guarantees that $\tau_{M_{F_t}} = \tau_{d_t}$ for each $t > 0$. It follows that $\tau_{M_F} = \bigvee\{\tau_{M_{F_t}} : t > 0\} = \bigvee\{\tau_{d_t} : t > 0\}$.

(iv) By assertion (i) in the statement of Proposition 6.2.2 we have that the function d_F is a metric on X. By assertion (ii) in the statement of the aforesaid proposition we obtain that $d_F(x,y) = \bigvee_t d_{f_t}(x,y)$ for all $x, y \in X$ is also a metric on X and that $\tau_{d_F} \supseteq \tau_{M_F}$.

(v) Next we see that $(X, M_F, \mathfrak{L})$ is principal provided $f_t \in \mathcal{M}_S^1$ for all $t > 0$. By assertion (ii) in the statement of Proposition 6.1.12 we have that $\tau_{M_{F_t}} = \tau(d_t) = \tau(d)$ for each $t > 0$. Whence we have that $\tau_{M_F} = \bigvee\{\tau_{d_t} : t > 0\} = \tau(d)$. Thus, $\tau_{M_F} = \tau_{M_{F_t}}$ for each $t > 0$. By Remark 6.2.9 we conclude that $(X, M_F, \mathfrak{L})$ is principal. Hence $\tau_{d_F} \supseteq \tau_{M_F} = \tau_d$.

■

The next example shows that the condition "f^x is continuous on $]0,\infty[$ for each $x \in [0,\infty[$" cannot be deleted in the statement of Theorem 6.2.3 in order to guarantee the introduced technique induces a fuzzy metric.

Example 6.2.4. *Consider the family $F = \{f_t : t > 0\}$, where*

$$f_t(x) = \begin{cases} \frac{x}{t+x}, & \text{if } 0 < t \leq 1 \text{ and } x \in [0,\infty[, \\ \frac{x}{2t+x}, & \text{if } t > 1 \text{ and } x \in [0,\infty[. \end{cases}$$

It is easy to verify that F is a decreasing family of functions included in $\mathcal{M}^1$. Besides,

$$f^x(t) = \begin{cases} \frac{x}{t+x}, & \text{if } 0 < t \leq 1 \\ \frac{x}{2t+x}, & \text{if } t > 1, \end{cases}$$

for each $x \in X$, which, obviously, is not continuous at $t = 1$.

Let (X,d) be a metric space, if we define the fuzzy set M_F on $X \times X \times]0,\infty[$ as in assertion (i) in the statement of Theorem 6.2.3, i.e.,

$$M_F(x,y,t) = \begin{cases} \frac{t}{t+d(x,y)}, & \text{if } 0 < t \leq 1 \text{ and } x,y \in X, \\ \\ \frac{2t}{2t+d(x,y)}, & \text{if } t > 1 \text{ and } x,y \in X, \end{cases}$$

it is easy to verify that M_F does not satisfy axiom $(GV5)$ in definition of fuzzy metric space.

The next example shows that the assumption "$F \subseteq \mathcal{M}_S^1$" cannot be deleted in the statement of Theorem 6.2.3 in order to guarantee that the induced fuzzy metric is principal.

Example 6.2.5. *Let (X,d) be a metric space. Consider the family of functions $F = \{f_t : t > 0\}$ defined on $[0,\infty[$ by*

$$f_t(x) = \begin{cases} 0, & \text{if } x = 0; \\ 1 - \frac{t^2}{t+x}, & \text{if } x \in]0,\infty[, t \in]0,1[; \\ \frac{x}{t+x}, & \text{elsewhere.} \end{cases}$$

Note that $f_t \in \mathcal{M}^1$ *for each* $t \in [1, \infty[$*. Now, we will see that* $f_t \in \mathcal{M}^1$ *for each* $t \in]0, 1[$*. To this end, note that* $f_t \in \mathcal{O}$ *and* $f_t(x) < 1$ *for each* $x \in [0, \infty[$ *and each* $t \in]0, 1[$*. Besides, it is easy to see that* f_t *is monotone for* $t \in]0, 1[$*.*

Next, we will see that f_t *is also subadditive for each* $t \in]0, 1[$*.*

With this aim, we fix $t \in]0, 1[$*.*

If $a = 0$ *or* $b = 0$*, then it is obvious that* $f(a + b) \leq f(a) + f(b)$*. Now, suppose that* $a, b \in]0, \infty[$*. Then it is easy to verify that*

$$\frac{t}{t+a+b} \geq \frac{t}{t+a} \cdot \frac{t}{t+b}.$$

So

$$\frac{t^2}{t+a+b} \geq \frac{t^2}{t+a} \cdot \frac{t^2}{t+b},$$

since $t \in]0, 1[$*. Moreover,* $\frac{t^2}{t+a}, \frac{t^2}{t+b} \in [0, 1]$ *and taking into account that* $x \cdot y \geq x \mathfrak{L} y$ *for each* $x, y \in [0, 1]$*, we have that*

$$\frac{t^2}{t+a+b} \geq \frac{t^2}{t+a} \mathfrak{L} \frac{t^2}{t+b} \geq \frac{t^2}{t+a} + \frac{t^2}{t+b} - 1.$$

Whence we deduce that

$$f(a+b) = 1 - \frac{t^2}{t+a+b} \leq 1 - \frac{t^2}{t+a} + 1 - \frac{t^2}{t+b} = f(a) + f(b).$$

Therefore, f_t *is subadditve and so* $f_t \in \mathcal{M}^1$*.*

It is not hard to check that f^x *is continuous on* $]0, \infty[$ *for each* $x \in [0, \infty[$*, since* $f^0(t) = f_t(0) = 0$ *for each* $t \in]0, \infty[$ *and, for each* $x \in]0, \infty[$*, we have that*

$$f^x(t) = \begin{cases} 1 - \frac{t^2}{t+x}, & \text{if } t \in]0,1[; \\ \\ \frac{x}{t+x}, & \text{if } t \in [1,\infty[. \end{cases}$$

Clearly the family F satisfies all hypothesis in the statement of Theorem 6.2.3. Thus $(X, M_F, \mathfrak{L})$ is a strong fuzzy metric on X, where M_F is given by

$$M_F(x,y,t) = 1 - f_t(d(x,y)) = \begin{cases} 1, & \text{if } x = y; \\ \\ \frac{t^2}{t+d(x,y)}, & \text{if } x,y \in X \text{ with } x \neq y \text{ and } t \in]0,1[; \\ \\ \frac{t}{t+d(x,y)}, & \text{if } x,y \in X \text{ with } x \neq y \text{ and } t \in [1,\infty[. \end{cases}$$

According to [30], $(X, M_F, \mathfrak{L})$ is not a principal fuzzy metric space. Besides, notice that f_t is not continuous at 0 for any $t \in]0,1[$ and, thus, by Theorem 1.1.4 we have that $f_t \notin \mathcal{M}_S^1$ for any $t \in]0,1[$.

The following notion has been inspired by Theorem 6.2.3.

Definition 6.2.6. Let (X,d) be a metric space and let $F = \{f_t : t > 0\}$ be a decreasing family of functions included in $\mathcal{M}^1$ such that for each $x \in [0,\infty[$ we have that f^x is continuous on $]0,\infty[$, where $f^x(t) = f_t(x)$ for all $t > 0$. Then the fuzzy metric space $(X, M_F, \mathfrak{L})$, where $M_F(x,y,t) = 1 - d_{f_t}(x,y)$ for each $x,y \in X$ and each $t > 0$, will be called the fuzzy metric space induced by the family F and the metric space (X,d). We will also say that $(M_F, \mathfrak{L})$ is the fuzzy metric induced by F and (X,d).

In the next example we show that some well-known instances of strong and principal fuzzy metric spaces can be obtained applying the technique introduced in Theorem 6.2.3. Such examples illustrate Definition 6.2.6.

Example 6.2.7. *Let (X, d) be a metric space. Consider the three families of functions $F = \{f_t : t > 0\}$, $G = \{g_t : t > 0\}$ and $H = \{h_t : t > 0\}$ defined on $[0, \infty[$ by:*

1. $f_t(x) = \min\{\frac{x}{1+t}, \frac{t}{1+t}\}$,
2. $g_t(x) = \frac{x}{t+x}$,
3. $h_t(x) = 1 - \exp^{\frac{-x}{t}}$.

It is not hard to check that these families of functions fulfill all hypothesis, even that they are included in $\mathcal{M}_S^1$, in the statement of Theorem 6.2.3. The corresponding strong and principal fuzzy metric spaces induced by F, G and H and the metric space (X, d) are given, respectively, by

1. $M_F(x, y, t) = \max\{1 - \frac{d(x,y)}{1+t}, \frac{1}{t+1}\}$,
2. $M_G(x, y, t) = 1 - \frac{d(x,y)}{t+d(x,y)} = \frac{t}{t+d(x,y)}$,
3. $M_H(x, y, t) = 1 - (1 - \exp^{\frac{-d(x,y)}{t}}) = \exp^{\frac{-d(x,y)}{t}}$,

for each $x, y \in X$ and each $t > 0$. Observe that $(M_G, \mathfrak{L})$ is the standard fuzzy metric induced by the metric d.

After introducing the technique for generating non-stationary fuzzy metric spaces we end the paper focusing our discussion on their completeness and their completion. Before, we give two observations on the class of strong fuzzy metrics that will be useful in our work.

Remark 6.2.8. In [32], it was shown that the assignment in condition $(C1)$, in the statement of Theorem 6.1.18, is always a continuous function whenever M is strong. So, as it was pointed out in Theorem 4.7 of the cited paper, a strong fuzzy metric space $(X, M, *)$ is completable if and only if for each pair of Cauchy sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in X the following conditions are fulfilled:

(c1) $\lim_n M(a_n, b_n, s) = 1$ for some $s > 0$ implies $\lim_n M(a_n, b_n, t) = 1$ for all $t > 0$.

(c2) $\lim_n M(a_n, b_n, t) > 0$ for all $t > 0$.

Remark 6.2.9. Observe that if $(X, M, *)$ is a non-stationary fuzzy metric space, then we can define the family of fuzzy sets $\{M_t : t > 0\}$ where, for each $t > 0$, $M_t : X \times X \times]0, \infty[\to]0, 1]$ is given by $M_t(x, y, s) = M(x, y, t)$ for all $x, y \in X$ and for all $s > 0$. According to [32], $(X, M, *)$ is strong if and only if $(X, M_t, *)$ is a stationary fuzzy metric space for each $t > 0$. In this case the family $\{M_t : t > 0\}$ is called the *family of stationary fuzzy metrics associated* to M. Note that if $(X, M_t, *)$ is a stationary fuzzy metric space, then we can identify the value $M_t(x, y, s)$ with the value $M_t(x, y)$. Moreover, $\tau_M = \bigvee\{\tau_{M_t} : t > 0\}$ provided that $(X, M, *)$ is strong (see [33]). Furthermore, if M is strong and $\tau_M = \tau_{M_t}$ for all $t > 0$, then M is principal.

Now, we are able to tackle the completeness of the fuzzy metrics constructed by the introduced technique.

Proposition 6.2.10. *Let (X, d) be a metric space and let $F = \{f_t : t > 0\}$ be a decreasing family of functions included in $\mathcal{M}^1_S$ such that f^x is continuous on $]0, \infty[$ for each $x \in [0, \infty[$, where $f^x(t) = f_t(x)$ for all $t > 0$. Then the fuzzy metric space $(X, M_F, \mathfrak{L})$ induced by F and (X, d) is complete if and only if (X, d) is complete.*

Proof. Let $(M, \mathfrak{L})$ be the fuzzy metric induced by F and (X, d). We first note that a sequence $(x_n)_{n\in\mathbb{N}}$ in X is M_F-Cauchy if and only if it is d-Cauchy. Indeed, since M_F is strong, then a sequence $(x_n)_{n\in\mathbb{N}}$ is M_F-Cauchy if and only if it is M_{F_t}-Cauchy for all $t > 0$. Moreover, d-Cauchy sequences coincide with M_{F_t}-Cauchy sequences for each $t > 0$, since d and M_{F_t} are uniformly equivalent by assertion (ii) in the statement of Proposition 6.1.12. Thus, M_F-Cauchy sequences coincide with d-Cauchy sequences. Furthermore, assertion (v) in the statement of Theorem 6.2.3 gives that $\tau_M = \tau(d)$.

Therefore, every M_F-Cauchy sequence converges in τ_{M_F} if and only if every d-Cauchy sequence converges in τ_d. Thus $(X, M_F, \mathfrak{L})$ is complete if and only if (X, d) is complete. ■

Theorem 6.2.11. *Let (X, d) be a metric space and let $F = \{f_t : t > 0\}$ be a decreasing family of functions included in $\mathcal{M}_S^1$ such that f^x is continuous on $]0, \infty[$ for each $x \in [0, \infty[$, where $f^x(t) = f_t(x)$ for all $t > 0$. Then the fuzzy metric space $(X, M_F, \mathfrak{L})$ induced by F and (X, d) is completable and $(X^*, M_F^*, \mathfrak{L})$ is its completion, where $M_F^*(x^*, y^*, t) = 1 - d_{f_t}^*(x^*, y^*)$ for each $x^*, y^* \in X^*$ and each $t > 0$ and, in addition, (X^*, d^*) is the completion of (X, d).*

Proof. Consider the fuzzy metric space $(X, M_F, \mathfrak{L})$ induced by F and (X, d). Let (X^*, d^*) be the completion of (X, d). We begin showing that $(X, M_F, \mathfrak{L})$ is completable. To this end, let $t > 0$ and, in addition, let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two Cauchy sequences in X, where $(a_n)_{n\in\mathbb{N}} \in a^*$ and $(b_n)_{n\in\mathbb{N}} \in b^*$. Then taking into account that $F \subseteq \mathcal{M}_S^1$, we have that

$$\lim_n M_F(a_n, b_n, t) = \lim_n (1 - d_{f_t}(a_n, b_n)) =$$

$$1 - f_t(\lim_n d(a_n, b_n)) = 1 - d_{f_t}^*(a^*, b^*) > 0.$$

By Remark 6.2.8 we deduce that assertion $(C1)$ in the statement of Theorem 6.1.18 is fulfilled.

Next, suppose that $\lim_n M_F(a_n, b_n, s) = 1$ for two M_F-Cauchy sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in X and for some $s > 0$. Then, by continuity of f_s, we have that

$$1 = \lim_n M_F(a_n, b_n, s) = \lim_n (1 - d_{f_s}(a_n, b_n)) = 1 - f_s(\lim_n d(a_n, b_n)).$$

So $\lim_n M_F(a_n, b_n, s) = 1$ if and only if $f_s(\lim_n d(a_n, b_n)) = 0$. Since $f_s \in \mathcal{M}_S^1$ we have that $f_s(\lim_n d(a_n, b_n)) = 0$ if and only if $\lim_n d(a_n, b_n) = 0$.

Therefore if $\lim_n M_F(a_n, b_n, s) = 1$ for some $s > 0$, then $f_t(\lim_n d(a_n, b_n) = 0$ for each $t > 0$. Hence $\lim_n M_F(a_n, b_n, t) = 1$ for each $t > 0$. Therefore, assertion $(C2)$ in the statement of Theorem 6.1.18 is fulfilled. Consequently, Theorem 6.1.18 yields that $(X, M_F, \mathfrak{L})$ is completable.

Next, we will construct the completion of $(X, M_F, \mathfrak{L})$. Suppose that $(\tilde{X}, M_F^*, \mathfrak{L})$ is the completion of $(X, M_F, \mathfrak{L})$.

First we will see that $\tilde{X} = X^*$. The set of M_F-Cauchy sequences in X coincides with the set of d-Cauchy sequences in X, as we have seen in the proof of Proposition 6.2.10. Then, given two Cauchy sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in X, we have that $\lim_n M_F(x_n, y_n, t) = 1$, for each $t > 0$, if and only if $\lim_n d_{f_t}(x_n, y_n) = 0$, for each $t > 0$. So, $\lim_n d(x_n, y_n) = 0$ if and only if $\lim_n d_{f_t}(x_n, y_n) = 0$, for each $t > 0$, since $f_t \in \mathcal{M}_S^1$ for each $t > 0$. Thus, $\tilde{X} = X^*$.

Finally, given $x^*, y^* \in X^*$ and $t > 0$, on account of Remark 6.1.21, we have that

$$M_F^*(x^*, y^*, t) = \lim_n M_F(x_n, y_n, t) = 1 - \lim_n d_{f_t}(x_n, y_n)$$

$$= 1 - f_t(\lim_n d(x_n, y_n)) = 1 - f_t(d^*(x^*, y^*)) = 1 - d^*_{f_t}(x^*, y^*),$$

where $(x_n)_{n\in\mathbb{N}} \in x^*$ and $(y_n)_{n\in\mathbb{N}} \in y^*$.

■

Chapter 7

A duality relationship between fuzzy metrics and metrics

Based on the duality relationship between indistinguishability operators and (pseudo-)metrics exposed in Subsection 1.1.3, we address the problem of establishing whether there is a relationship between the last ones and fuzzy (pseudo-)metrics in this chapter. We give a positive answer to the posed question. Concretely, we yield methods for generating fuzzy (pseudo-)metrics from (pseudo)-metrics and vice-versa. The aforementioned methods involve the use of the pseudo-inverse of the additive generator of a continuous Archimedean t-norm. As a consequence we get a technique to generate non-strong fuzzy (pseudo-)metrics from (pseudo-)metrics. Examples that illustrate the exposed methods are also given. Finally, we show that the classical duality relationship between indistinguishability operators and (pseudo)-metrics can be retrieved as a particular case of our results when continuous Archimedean t-norms are under consideration. Throughout this chapter the fuzzy pseudo-metrics are understood as KM-fuzzy pseudo-metrics.

The structure of the chapter is as follows: in Section 7.1, we introduce the method for inducing fuzzy (pseudo-)metrics from (pseudo)-metrics using the pseudo-inverse of the additive generator of a continuous t-norm. Moreover, in the same section, we discuss conditions under which our method gives as a result non-strong fuzzy pseudo-metrics. Furthermore, we give examples in order to illustrate the introduced method. Finally, in Section 7.2, we study the converse of the aforesaid method. Thus we generate (pseudo-)metrics from fuzzy (pseudo-)metrics by means of an additive generator of the continuous t-norm under consideration.

7.1 A method for generating fuzzy pseudo-metrics from pseudo-metrics

In this section we will construct a fuzzy metric space from a given metric space (X, d). Our method will be based on the pseudo-inverse of a continuous Archimedean t-norm preserving the spirit of the construction given in Theorem 1.1.28.

First, we will try to motivate the way of constructing our fuzzy metric from the classical one.

Given a fuzzy (pseudo-)metric space $(X, M, *)$, the value $M(x, y, t)$ can be interpreted as the degree of nearness or similarity between x and y, with respect to a positive real parameter t. Under this interpretation if we consider a (pseudo-)metric space (X, d) we can consider the parameter t as a threshold from which x and y would be indistinguishable, and so the degree of nearness between them will be 1. Before that threshold, the degree of nearness is fuzzified, taking values in $[0,1)$ smaller as t decreases, since $M_{x,y}$ is a decreasing function on $]0,\infty[$ (the fact that $M_{x,y}$ is a monotone function was proved in [23]).

The following fuzzy metric illustrates the aforementioned idea. Notice that it is a generalization of a well-known example of probabilistic metric space introduced in [89].

It is not hard to see that, given a (pseudo-)metric space (X,d), $(X,M^d,\wedge)$ is a fuzzy (pseudo-)metric space, where M^d is given by

$$M^d(x,y,t) = \begin{cases} 0, & \text{if } 0 < t \leq d(x,y) \\ 1, & \text{if } t > d(x,y) \end{cases} .$$

Observe that the preceding example constitutes a drastic fuzzification of the classical (pseudo-)metric from which it is defined. Indeed, the degree of nearness between two points $M(x,y,t)$ is 0, x and y are totally distinguishable before a threshold parameter value t ($t = d(x,y)$) and, in addition, x and y are indistinguishable from the threshold, i.e., $M(x,y,t) = 1$ when $t > d(x,y)$.

Taking into account the method of construction of indistinguishability operators given in Theorem 1.1.28 and the fact that every indistinguishability operator, for a continuous and Archimedean t-norm, can be considered as a stationary fuzzy pseudo-metric, we introduce a method to fuzzify a classical pseudo-metric in such a way that the spirit of the construction of the fuzzy metric M^d is preserved.

Theorem 7.1.1. *Let (X,d) be a pseudo-metric space and let $*$ be a continuous t-norm with additive generator f_*. Then, $(M_{d,f_*},*)$ is a fuzzy pseudo-metric on X, where M_{d,f_*} is the fuzzy set defined on $X\times X\times]0,\infty[$ as follows:*

$$M_{d,f_*}(x,y,t) = f^{(-1)}\left(\max\{d(x,y)-t,0\}\right),$$

for all $x,y \in X$ and for all $t \in]0,\infty[$. Furthermore, $(M_{d,f_*},*)$ is a fuzzy metric on X if and only if d is a metric on X.

Proof. Let (X,d) be a pseudo-metric space and let $*$ be a continuous Archimedean t-norm with additive generator f_*. We define, for each $x,y\in X$ and each $t\in]0,\infty[$, the mapping

$$M_{d,f_*}(x,y,t) = f^{(-1)}\left(\max\{d(x,y)-t,0\}\right).$$

We will see that $(M_{d,f_*},*)$ is a fuzzy pseudo-metric on X.

First, note that the axiom $(KM3)$ is obviously fulfilled by definition of M_{d,f_*} and by the fact that $d(x,y)=d(y,x)$ for all $x,y\in X$. So, we only need to see that, for each $x,y\in X$ and each $t\in]0,\infty[$, the fuzzy set M_{d,f_*} also satisfies $(KM2),(KM4)$ and $(KM5)$.

Next we show that M_{d,f_*} satisfies $(KM2)$. To this end, let $x\in X$ and $t\in]0,\infty[$. Since (X,d) is a pseudo-metric space we have that $d(x,x)=0$ and so

$$M_{d,f_*}(x,x,t) = f^{(-1)}\left(\max\{0-t,0\}\right) = f^{(-1)}(0) \text{ for each } t\in]0,\infty[.$$

Since $f^{(-1)}(0)=1$ we deduce that $M(x,x,t)=1$ for each $t\in]0,\infty[$ and $(KM2)$ is hold. In order to show that M_{d,f_*} satisfies $(KM4)$, let $x,y,z\in X$

and $t, s \in]0, \infty[$. First, note that $d(x,z) - t - s \leq d(x,y) - t + d(y,z) - s$, since d is a pseudo-metric on X. Then,

$$\max\{d(x,y) - t, 0\} + \max\{d(y,z) - s, 0\} \geq \max\{d(x,z) - t - s, 0\}.$$

Hence, we obtain that

$$M_{d,f_*}(x,z,t+s) = f^{(-1)}(\max\{d(x,z) - t - s, 0\}) \geq$$

$$f^{(-1)}(\max\{d(x,y) - t, 0\} + \max\{d(x,y) - t, 0\}) \qquad ,$$

since $f^{(-1)}$ is decreasing.

Moreover, we have that

$$M_{d,f_*}(x,y,t) * M_{d,f_*}(y,z,s) =$$

$$f^{(-1)}(\max\{d(x,y) - t, 0\}) * f^{(-1)}(\max\{d(x,y) - s, 0\}) =$$

$$f^{(-1)}\left(f\left(f^{(-1)}(\max\{d(x,y) - t, 0\})\right) + f\left(f^{(-1)}(\max\{d(x,y) - s, 0\})\right)\right),$$

since f_* is an additive generator of $*$.

Since

$$f^{(-1)}(\max\{d(x,y) - t, 0\} + \max\{d(x,y) - s, 0\}) \geq$$

$$f^{(-1)}\left(f\left(f^{(-1)}(\max\{d(x,y) - t, 0\})\right) + f\left(f^{(-1)}(\max\{d(x,y) - s, 0\})\right)\right)$$

we deduce that

$$M_{d,f_*}(x,z,t+s) \geq M_{d,f_*}(x,y,t) * M_{d,f_*}(y,z,s).$$

Thus, $(KM4)$ is satisfied.

Next we show that $(KM5)$ is hold. Fix $x, y \in X$ and consider the function $M_{x,y} :]0,\infty[\to [0,1]$ given by $M_{x,y}(t) = M_{d,f_*}(x,y,t)$ for all $t \in]0,\infty[$. Then,

$$M_{x,y}(t) = \begin{cases} f^{(-1)}\left(d(x,y) - t\right), & \text{if } 0 < t \leq d(x,y) \\ 1, & \text{if } t > d(x,y) \end{cases}.$$

An straightforward computation, and taking into account that $f^{(-1)}$ is decreasing and continuous, gives that $M_{x,y}$ is continuous on $]0,\infty[$ and so left-continuous on $]0,\infty[$.

Therefore, $(M_{d,f_*}, *)$ is a fuzzy pseudo-metric on X.

It remains to prove that M_{d,f_*} is a fuzzy metric on X if and only if, d is a metric on X. To this end, note that M_{d,f_*} satisfies $(KM2')$ if and only if $M_{d,f}(x,y,t) = 1$ for all $t \in]0,\infty[$ implies $x = y$. Moreover, $M_{d,f_*}(x,y,t) = 1$ for all $t \in]0,\infty[$ is equivalent to $f^{(-1)}(\max\{d(x,y) - t, 0\}) = 1$ for all $t \in]0,\infty[$. Since $f^{(-1)}$ is the pseudo-inverse of an additive generator of $*$, then $f^{(-1)}(a) = 1$ if and only if $a = 0$. Therefore, $M(x,y,t) = 1$ for each $t \in]0,\infty[$ if and only if $\max\{d(x,y) - t, 0\} = 0$ for each $t \in]0,\infty[$, or equivalently, if and only if $d(x,y) = 0$. Thus, the fuzzy pseudo-metric $(M_{d,f_*}, *)$ is a fuzzy metric on X if and only if d is a metric space on X.

■

In the following two corollaries, we specify the method given in Theorem 7.1.1 for the case of the usual product $*_P$ and the Luckasievicz t-norm $*_L$.

Corollary 7.1.2. *Let (X,d) be a pseudo-metric space. Then, $(M_{d,f_{*_P}},*_P)$ is a fuzzy pseudo-metric on X, where $M_{d,f_{*_P}}$ is the fuzzy set defined on $X\times X\times]0,\infty[$ as follows:*

$$M_{d,f_{*_P}}(x,y,t)=\begin{cases} \mathsf{e}^{t-d(x,y)}, & \text{if } t\leq d(x,y) \\ 1, & \text{if } t>d(x,y) \end{cases},$$

for all $x,y\in X$ and for all $t\in]0,\infty[$. Furthermore, $(M_{d,f_{*_P}},*_P)$ is a fuzzy metric on X if and only if d is a metric on X.

Corollary 7.1.3. *Let (X,d) be a pseudo-metric space. Then, $(M_{d,f_{*_L}},*_L)$ is a fuzzy pseudo-metric on X, where $M_{d,f_{*_L}}$ is the fuzzy set defined on $X\times X\times]0,\infty[$ as follows:*

$$M_{d,f_{*_L}}(x,y,t)=\begin{cases} 0, & \text{if } t\leq d(x,y)-1 \\ 1+t-d(x,y) & \text{if } d(x,y)-1<t\leq d(x,y) \\ 1, & \text{if } t\geq d(x,y) \end{cases},$$

for all $x,y\in X$ and for all $t\in]0,\infty[$. Furthermore, $(M_{d,f_{*_L}},*_L)$ is a fuzzy metric on X if and only if d is a metric on X.

7.1.1 Stationary fuzzy metric spaces and indistinguishability operators

In this subsection, we will show that the method yielded in Theorem 1.1.28 can be retrieved as a particular case of our construction provided in Theorem 7.1.1. The following result, that gives a method to construct an indistinguishability operator from a given fuzzy pseudo-metric, will be crucial to this end.

Lemma 7.1.4. *Let $(M, *)$ be a fuzzy pseudo-metric on X and let E_M be the fuzzy set defined on $X \times X$ as follows:*

$$E_M(x,y) = \bigwedge_{s>0} M(x,y,s) \text{ for each } x,y \in X.$$

Then, E_M is an indistinguishability operator for $*$ on X. Moreover, E_M separates points if and only if $(M, *)$ is a fuzzy metric on X.

Proof. First of all, we note that the numerical value $\bigwedge_{s>0} M(x,y,s)$ exists. Indeed, the set $\{M(x,y,s)\}_{s\in]0,\infty[}$ is bounded below by 0.

Next we show that the fuzzy set E_M satisfies axioms $(E1)$, $(E2)$ and $(E3)$. Clearly, $(E2)$ is fulfilled by definition of E_M and the fact that $M(x,y,s) = M(y,x,s)$ for all $x,y \in X$ and for all $s \in]0,\infty[$.

In order to show that E_M satisfies $(E1)$, let $x \in X$. Since $(M,*)$ is a fuzzy pseudo-metric on X we have that $M(x,x,s) = 1$ for each $s \in]0,\infty[$ and so $E_M(x,x) = \bigwedge_{s>0} M(x,x,s) = 1$. Thus, $(E1)$ is hold.

With the aim of showing that E_M satisfies $(E3)$, let $x,y,z \in X$. Since $(M,*)$ is a fuzzy pseudo-metric on X it is hold that

$$M(x,z,s) \geq M(x,y,s/2) * M(y,z,s/2), \text{for each } s \in]0,\infty[.$$

Then,

$$E_M(x,z) = \bigwedge_{s>0} M(x,z,s) \geq \bigwedge_{s>0} \left(M(x,y,s/2) * M(y,z,s/2)\right).$$

Since $\bigwedge_{s>0} M(u,v,s/2) \leq M(u,v,t/2)$ for all $u,v \in X$ and $t \in]0,\infty[$ we have that

$$\left(\bigwedge_{s>0} M(x,y,s/2)\right) * \left(\bigwedge_{s>0} M(y,z,s/2)\right) \leq M(x,y,s/2) * M(y,z,s/2)$$

for all $s \in]0,\infty[$ and, hence, that

$$\left(\bigwedge_{s>0} M(x,y,s/2)\right) * \left(\bigwedge_{s>0} M(y,z,s/2)\right) \leq \bigwedge_{s>0} \left(M(x,y,s/2) * M(y,z,s/2)\right).$$

Whence we deduce that

$$E_M(x,z) \geq \left(\bigwedge_{s>0} M(x,y,s/2)\right) * \left(\bigwedge_{s>0} M(y,z,s/2)\right) = E_M(x,y) * E_M(y,z).$$

Therefore, E_M satisfies axiom $(E3)$ too and so E_M is an indistinguishability operator for $*$ on X.

Finally, it remains to prove that E_M separates points if and only $(M,*)$ is a fuzzy metric space on X. It is easy to check that, given $x,y \in X$, $E_M(x,y) = 1 \Leftrightarrow M(x,y,s) = 1$ for each $s \in]0,\infty[$. Whence we immediately obtain that $E_M(x,y) = 1 \Leftrightarrow x = y$ if and only if $M(x,y,s) = 1$ for each $s \in]0,\infty[\Leftrightarrow x = y$.

■

In the light of Lemma 7.1.4 and Theorem 7.1.1 we are bale to achieve our promised target, i.e., that the method given in in Theorem 1.1.28 can be retrieved from the method provided in Theorem 7.1.1.

Corollary 7.1.5. *Let (X,d) be a pseudo-metric space and let $*$ be a continuous Archimedean t-norm with additive generator $f_* : [0,1] \to [0,\infty]$. Then, the fuzzy set $E_{d,f_*} : X \times X \to [0,1]$ is an indistinguishability operator for $*$ on X, where $E_{d,f_*}(x,y) = f^{(-1)}(d(x,y))$ for each $x,y \in X$. Furthermore, E_{d,f_*} separates points if and only if d is a metric on X.*

Proof. Let (X, d) be a pseudo-metric space and let $*$ be a continuous Archimedean t-norm with additive generator f_*.

On the one hand, Theorem 7.1.1 ensures that $(M_{d,f_*}, *)$ is a fuzzy pseudo-metric, where M_{d,f_*} is given by

$$M_{d,f_*}(x, y, t) = f^{(-1)}\left(\max\{d(x, y) - t, 0\}\right), \text{ for each } x, y \in X, t \in]0, \infty[.$$

On the other hand, we define the fuzzy set $E_{M_{d,f_*}}$ on $X \times X\times$ given by

$$E_{M_{d,f_*}}(x, y) = \bigwedge_{s>0} M_{d,f_*}(x, y, s), \text{ for each } x, y \in X.$$

Then, by Lemma 7.1.4, we have that $E_{M_{d,f_*}}$ is an indistinguishability operator for $*$ on X. In addition, $E_{M_{d,f_*}}$ separates points if and only if $(M_{d,f_*}, *)$ is a fuzzy metric on X.

Now, observe that for each $x, y \in X$ we have that

$$E_{M_{d,f_*}}(x, y) = \bigwedge_{s>0} M_{d,f_*}(x, y, s) = \bigwedge_{s>0} \left(f^{(-1)}(d(x, y) - s)\right) = f^{(-1)}(d(x, y)),$$

since $f^{(-1)}$ is a decreasing function. Thus, the fuzzy set E_{d,f_*}, given by $E_d(x, y) = f^{(-1)}(d(x, y))$ for each $x, y \in X$, matches up with $E_{M_{d,f_*}}$ on $X \times X$ and, therefore, it is an indistinguishability operator for $*$ on X. Furthermore, $E_{M_{d,f_*}}$ separates points if and only if $(M_{d,f_*}, *)$ is a fuzzy metric on X. By Theorem 7.1.1 we have that $(M_{d,f_*}, *)$ is a fuzzy metric on X if and only if d is a metric on X. Therefore we conclude that E_{d,f_*} separates points if and only if d is a metric on X.

■

7.1.2 Non-strong fuzzy (pseudo-)metric spaces

Many examples of fuzzy pseudo-metrics that can be found in the literature are strong (see Subsection 1.1.3). So, taking into account that each stationary fuzzy pseudo-metric on a non-empty set X is an indistinguishability operator, each strong fuzzy pseudo-metrics can be seen as a parametric family of indistinguishability operators.

Fuzzy metric spaces satisfying the property of being strong constitute a large class. In fact, in the literature it is very difficult to find examples of non-strong fuzzy (pseudo-)metrics. Inspired by this handicap a few authors have focused their efforts on finding examples of non-strong fuzzy (pseudo-)metrics (see, for instance, [31, 41]). As an instance of this kind of fuzzy pseudo-metrics we have the fuzzy pseudo-metric $(M^d, \wedge)$ introduced in Section 7.1. Indeed, it is easily seen that $(M^d, \wedge)$ is a non-strong fuzzy metric on $\mathbb{R}$ when d is taken as the Euclidean metric.

Based on the preceding observation and motivated by the fact lack of examples of non-strong fuzzy pseudo-metrics, our purpose in this subsection is, on the one hand, to show that the method given in Theorem 7.1.1 does not yield in general strong fuzzy pseudo-metrics and, on the other hand, to provide conditions that guarantee when our construction gives a strong fuzzy pseudo-metric.

The next example gives an instance of non-strong fuzzy pseudo-metric which is obtained by means of Theorem 7.1.1.

Example 7.1.6. *Consider the Euclidean metric d_e on $\mathbb{R}$. Attending to Corollary 7.1.2 we have that $M_{d_e,f_{*P}}$ is a fuzzy metric on $\mathbb{R}$, where recall that*

$M_{d_e,f_{*P}}$ *is given by*

$$M_{d_e,f_{*P}}(x,y,t)=\begin{cases} \mathsf{e}^{t-|y-x|}, & \text{if } 0<t\leq |y-x| \\ 1, & \text{if } t>|y-x| \end{cases}.$$

Next we show that $M_{d_e,f_{*P}}(0,2,1) < M_{d_e,f_{*P}}(0,1,1) *_P M_{d_e,f_{*P}}(1,2,1)$. *Indeed, it is clear that* $|1-0|=1$, $|2-1|=1$ *and* $|2-0|=2$. *Thus we have that*

$$M_{d_e,f_{*P}}(0,2,1)=\mathsf{e}^{1-2}=\mathsf{e}^{-1}<1=\mathsf{e}^{1-1} *_P \mathsf{e}^{1-1}=$$

$$M_{d_e,f_{*P}}(0,1,1) *_P M_{d_e,f_{*P}}(1,2,1).$$

Thus, the fuzzy metric $(M_{d_e,f_{*P}}, *_P)$ *is not strong.*

The next result ensures that our method, given by Theorem 7.1.1, allows always to construct non-strong fuzzy pseudo-metrics when we consider that the metric fulfills an extra condition.

Theorem 7.1.7. *Let* $*$ *be a continuous Archimedean* t*-norm with additive generator* f_* *and let* (X,d) *be a pseudo-metric space such that there exist* $a,b,c\in X$ *and* $t_0\in]0,\infty[$ *satisfying* $d(a,c)\in]t_0,f_*(0)[$ *and* $d(a,b),d(b,c)\in[0,t_0]$. *Then* $(M_{d,f_*},*)$ *is a non-strong fuzzy pseudo-metric on* X, *where* M_{d,f_*} *is the fuzzy set defined on* $X\times X\times]0,\infty[$ *as follows:*

$$M_{d,f_*}(x,y,t)=f^{(-1)}\left(\max\{d(x,y)-t,0\}\right),$$

for all $x,y\in X$ *and for all* $t\in]0,\infty[$. *Furthermore,* $(M_{d,f_*},*)$ *is a non-strong fuzzy metric on* X *if and only if* d *is a metric on* X.

Proof. By Theorem 7.1.1 we have that $(M_{d,f_*}, *)$ is a fuzzy pseudo-metric space on X. Furthermore, $(M_{d,f_*}, *)$ is a fuzzy metric on X if and only if d is a metric on X.

Now, we will see that in both cases the fuzzy (pseudo-)metric is not strong. With this aim, we will show that $(KM4')$ is not fulfilled. Indeed, let $a, b, c \in X$ and $t_0 \in]0, \infty[$ such that $d(a, c) \in]t_0, f_*(0)[$ and $d(a, b), d(b, c) \in [0, t_0]$. Since $f^{(-1)}$ is strictly monotone on $[0, f_*(0)]$ then

$$M_{d,f_*}(a, c, t_0) = f^{(-1)}(\max\{d(a, c) - t_0, 0\}) =$$

$$f^{(-1)}(d(a, c) - t_0) < f^{(-1)}(0) = 1$$

and

$$M_{d,f_*}(a, b, t_0) * M_{d,f_*}(b, c, t_0) =$$

$$f^{(-1)}(\max\{d(a, b) - t_0, 0\}) * f^{(-1)}(\max\{d(b, c) - t_0, 0\}) =$$

$$= f^{(-1)}(0) * f^{(-1)}(0) = 1.$$

Whence we conclude that

$$M_{d,f_*}(a, c, t_0) < M_{d,f_*}(a, b, t_0) * M_{d,f_*}(b, c, t_0)$$

and, thus, that M_{d,f_*} is non-strong, as we claimed. ∎

In [41], two examples of non-strong fuzzy metrics were provided, one for the product t-norm and another one for the Lukasievicz t-norm. Besides they posed the question of finding examples of non-strong fuzzy metrics when the continuous t-norms that are under consideration are greater than the product but different from minimum. It must be stressed that they posed the aforesaid question in the framework of GV-fuzzy metrics (fuzzy metrics in the sense of George and Veeramani, see [23]). Motivated by the fact that GV-fuzzy metrics are closely related to the fuzzy metrics studied in the present paper, we introduce in the following example an instance of non-strong fuzzy pseudo-metric when the considered continuous t-norm is greater than the product t-norm and different form the minimum.

Example 7.1.8. *Consider the Hamacher t-norm* $*_H$ *defined by* $a *_H b = \frac{ab}{a+b-ab}$ *for each* $a, b \in (0,1]$ *and* $0 *_H 0 = 0$ *(see [49] for more information about the family of Hamacher t-norms). It is clear that* $*_H$ *is a continuous Archimedean t-norm.*

Moreover, it is easy to check that for each $a, b \in [0,1]$ *we have that* $a *_H b \geq a *_P b$, *so* $*_H$ *is greater than* $*_P$.

An additive generator of $*_H$ *is given by the function* $f_{*_H} : [0,1] \to [0,\infty]$ *given by* ${}_{*_H}(x) = \frac{1-x}{x}$ *for all* $x \in [0,1]$. *Hence an easy computation shows that the pseudo-inverse* $f_{*_H}^{(-1)}$ *of* f_{*_H} *is given as follows:*

$$f_{*_H}^{(-1)}(y) = \frac{1}{1+y}, \text{ for each } y \in [0,\infty].$$

Furthermore, note that $f_{*_H}(0) = \infty$.

Now, consider the Euclidean metric d_e *on* $\mathbb{R}$. *Taking* $x = 0$, $y = 1$, $z = 2$ *and* $t_0 = 1$ *we have that the condition in the statement of Theorem 7.1.7 is hold, since* $|2-0| \in]1,\infty[$, $|1-0| \in [0,1]$ *and* $|2-1| \in [0,1]$.

By Theorem 7.1.1, $(M_{d_e, f_{*_H}}, *_H)$ *is a fuzzy metric on* $\mathbb{R}$, *where the fuzzy set* $M_{d_e, f_{*_H}}$ *on* $\mathbb{R} \times \mathbb{R} \times]0,\infty[$ *is given by*

$$M_{d_e, f_{*_H}}(x,y,t) = \begin{cases} \frac{1}{1+d_e(x,y)-t}, & \text{if } t \leq d_e(x,y); \\ 1, & \text{if } t > d_e(x,y). \end{cases} .$$

Similar arguments to those given in Example 7.1.6 remain valid to show that $(M_{d_e, f_{*_H}}, *_H)$ *is non-strong.*

7.2 A method for generating pseudo-metrics from fuzzy pseudo-metrics

The problem of obtaining metrics from fuzzy metrics has been treated in the literature by several authors. Some approaches to the aforementioned problem have been obtained in [43, 76, 77]. Recently, the aforementioned results given in the cited references have been generalized in [10]. In all results given in the aforesaid references, a metric is constructed from a fuzzy metric using an additional function which is not related to the continuous t-norm under consideration and must satisfy many constraints.

Taking into account the mentioned results, we continue the study on the duality relationship between pseudo-metrics and fuzzy pseudo-metrics in this section. Thus our goal is twofold. On the one hand, we have interested in generating pseudo-metrics from fuzzy pseudo-metrics by means of the pseudo-inverse of an additive generator of the continuous t-norm under consideration. On the other hand, we aspire to retrieve the method provided in Theorem 1.1.27 as a particular case when stationary fuzzy pseudo-metrics are under consideration.

Our method, in contrast to those mentioned before, presents the advantage of needing only to consider the pseudo-inverse of an additive generator of the t-norm under consideration. Despite the aforesaid benefit, it must be pointed out that our method should be restricted to take under consideration only Archimedean continuous t-norms which excludes, for instance, the minimum t-norm.

The next result provides a method to obtain a pseudo-metric from a fuzzy pseudo-metric. The theorem is proved following similar arguments to those used in the proof of Theorem 1.1.30 which was proved in [10]. First we point out a comment on the notation that we use, which is inspired by [49].

Remark 7.2.1. In a lattice $(L, \preceq)$, the meet (the greatest lower bound) and the join (the least upper bound) of a subset A of L are denoted by $\inf A$ and $\sup A$, respectively. Observe that each element of L is both an upper and a lower bound of the empty set $\emptyset$, so $\inf \emptyset$ and $\sup \emptyset$ depend on the underlying set L. In particular, in the lattice $(]a, b[, \leq)$ (with $[a, b] \subseteq [-\infty, \infty]$ and where $\leq$ is the usual order on the (extended) real line) we obtain $\inf \emptyset = b$ and $\sup \emptyset = a$.

Now, we are able to show the announced theorem.

Theorem 7.2.2. *Let $(M, *)$ be a fuzzy pseudo-metric on X, where $*$ is a continuous Archimedean t-norm. Then the function $d_{M,f_*} : X \times X \to [0, \infty]$ defined as*

$$d_{M,f_*}(x, y) = \sup\{t \in]0, f_*(0)[: M(x, y, t) \leq f^{(-1)}(t)\},$$

is a pseudo-metric on X, where f_* is an additive generator of $*$. Moreover, d is a metric on X if and only if $(M, *)$ is a fuzzy metric on X.

Proof. First we show that $d_{M,f_*}(x, x) = 0$. To this end, let $x \in X$. Since $(M, *)$ is a fuzzy pseudo-metric on X we have that $M(x, x, t) = 1$ for all $t \in]0, f_*(0)[$. It follows that $M(x, x, t) > f^{(-1)}(t)$ for all $t \in]0, f_*(0)[$, since $f^{(-1)}(t) \in [0, 1[$ for each $t \in]0, f_*(0)[$. Thus, $d_{M,f_*}(x, x) = 0$.

The symmetry of d_{M,f_*} is obvious attending to its definition and the fact that $M(x,y,t) = M(y,x,t)$ for all $x, y \in X$ and $t \in]0, \infty[$.

Next we show that $d_{M,f_*}(x,z) \leq d_{M,f_*}(x,y) + d_{M,f_*}(y,z)$ for all $x, y, z \in X$. To this end, we will assume that $d_{M,f_*}(x,y) < f_*(0)$ and $d_{M,f_*}(y,z) < f_*(0)$ because otherwise the triangle inequality is hold trivially.

By definition of d_{M,f_*} we have

$$M(x,y,d_{M,f_*}(x,y)+\epsilon) > f^{(-1)}(d_{M,f_*}(x,y)+\epsilon)$$

and

$$M(y,z,d_{M,f_*}(y,z)+\epsilon) > f^{(-1)}(d_{M,f_*}(y,z)+\epsilon)$$

for each $x, y, z \in X$ and for each $\epsilon \in]0, K[$, where

$$K = \min\{f_*(0) - d_{M,f_*}(x,y), f_*(0) - d_{M,f_*}(y,z)\}.$$

Then, for each $\epsilon \in]0, \infty[$, we have that

$$f_*(M(x,y,d_{M,f_*}(x,y)+\epsilon)) < d_{M,f_*}(x,y)+\epsilon \tag{7.1}$$

and

$$f_*(M(y,z,d_{M,f_*}(y,z)+\epsilon)) < d_{M,f_*}(y,z)+\epsilon, \tag{7.2}$$

since f_* is a strictly decreasing function and $(f_* \circ f^{(-1)})(a) \leq a$ for each $a \in [0, \infty]$.

Attending to axiom $(KM4)$ and taking into account that f_* is an additive generator of $*$, we have

$$M(x, z, d_{M,f_*}(x, y) + d_{M,f_*}(y, z) + 2\epsilon) \geq$$

$$M(x, y, d_{M,f_*}(x, y) + \epsilon) * M(y, z, d_{M,f_*}(y, z) + \epsilon) =$$

$$f^{(-1)}\left(f_*(M(x, y, d_{M,f_*}(x, y) + \epsilon)) + f_*(M(y, z, d_{M,f_*}(y, z) + \epsilon))\right) >$$

$$f^{(-1)}(d_{M,f_*}(x, y) + \epsilon + d_{M,f_*}(y, z) + \epsilon)).$$

Thus, by definition of $d_{M,f_*}(x, z)$, it is hold that

$$d_{M,f_*}(x, z) \leq d_{M,f_*}(x, y) + d_{M,f_*}(y, z) + 2\epsilon.$$

Taking into account that $\epsilon \in]0, \infty[$ is arbitrary, we obtain

$$d_{M,f_*}(x, z) \leq d_{M,f_*}(x, y) + d_{M,f_*}(y, z).$$

■

In the following two corollaries, we specify the method given in Theorem 7.2.2 for the case of the usual product $*_P$ and the Luckasievicz t-norm $*_L$.

Corollary 7.2.3. *Let $(M, *_P)$ be a fuzzy pseudo-metric on X. Then the function $d_{M,f_{*_P}} : X \times X \to [0, \infty]$ defined as*

$$d_{M,f_{*_P}}(x, y) = \sup\{t \in]0, \infty[: M(x, y, t) \leq \mathsf{e}^{-t}\},$$

is a pseudo-metric on X*. Moreover,* $d_{M,f_{*P}}$ *is a metric on* X *if and only if* $(M, *_P)$ *is a fuzzy metric on* X*.*

Corollary 7.2.4. *Let* $(M, *_L)$ *be a fuzzy pseudo-metric on* X*. Then the function* $d_{M,f_{*L}} : X \times X \to [0, \infty]$ *defined as*

$$d_{M,f_{*L}}(x, y) = \sup\{t \in]0, 1[: M(x, y, t) \leq 1 - t\},$$

is a pseudo-metric on X*. Moreover,* $d_{M,f_{*L}}$ *is a metric on* X *if and only if* $(M, *_L)$ *is a fuzzy metric on* X*.*

Next we are able to show that the method given in Theorem 1.1.27 can be retrieved from the method provided in Theorem 7.2.2.

Corollary 7.2.5. *Let* X *be a non-empty set and* $*$ *a continuous Archimedean* t*-norm with additive generator* $f_* : [0, 1] \to [0, \infty]$*. If* E *is an indistinguishability operator for* $*$ *on* X*, then the function* $d_E : X \times X \to [0, \infty]$ *is a pseudo-metric on* X*, where* $d_E(x, y) = f_*(E(x, y))$ *for each* $x, y \in X$*. In addition,* d_E *is a metric on* X *if and only if* E *separates points.*

Proof. Define the mapping $M_E(x, y, t) = E(x, y)$ for each $x, y \in X$ and each $t \in]0, \infty[$. Then $(M_E, *)$ is a fuzzy pseudo-metric on X, where $M_E(x, y, t) = E(x, y)$ for each $x, y \in X$ and $t \in]0, \infty[$.

We will show that $\sup\{t \in]0, f_*(0)[: M_E(x, y, t) \leq f^{(-1)}(t)\} = f_*(E(x, y))$ for each $x, y \in X$.

Fix $x, y \in X$. Since $(f^{(-1)} \circ f)(a) = a$ for all $a \in [0, 1]$ then

$$f^{(-1)}(f_*(E(x, y)) = E(x, y) = M_E(x, y, t), \text{ for all } t \in]0, \infty[.$$

Thus we deduce that $f^{(-1)}(f_*(E(x, y)) = E(x, y) = M_E(x, y, f_*(E(x, y))$ and $f_*(E(x, y)) \in \{t \in]0, f_*(0)[: M_E(x, y, t) \leq f^{(-1)}(t)\}$. The fact that $f^{(-1)}$ is an strictly decreasing function on $]0, f_*(0)[$ guarantees $M_E(x, y, t) > f^{(-1)}(t)$ for all $t > f_*(E(x, y))$. Therefore

$$\sup\{t \in]0, f_*(0)[: M_E(x, y, t) \leq f^{(-1)}(t)\} = f_*(E(x, y)),$$

as we claimed.

Since $d_{M_E, f_*}(x, y) = d_E(x, y)$ for all $x, y \in X$, by Theorem 7.2.2, we have that the function $d_E : X \times X \to [0, \infty]$ defined by

$$d_E(x, y) = \sup\{t \in]0, f_*(0)[: M_E(x, y, t) \leq f^{(-1)}(t)\} = f_*(E(x, y)),$$

is a pseudo-metric on X. In addition, applying the aforementioned theorem, we have that d_E is a metric on X if and only if $(M_E, *)$ is a fuzzy metric on X. ■

According to Theorem 7.2.2 we infer that the pseudo-metric d_{M,f_*} will not take the value ∞ whenever the t-norm is nilpotent (notice that the t-norm is continuous and Archimedean). Contrarily, d_{M,f_*} can take the value ∞, when a continuous Archimeden t-norm $*$ is strict because in that case each additive generator f_* satisfies $f_*(0) = \infty$. In order to guarantee the finiteness of the pseudo-metric we provide a necessary condition through the next result.

Corollary 7.2.6. *Let $(M, *)$ be a fuzzy pseudo-metric on X such that for each $x, y \in X$ there exists $t_0 \in]0, \infty[$ satisfying $M(x, y, t_0) > 0$, where $*$ is a strict continuous Archimedean t-norm. Then the function $d_{M,f_*} : X \times X \to [0, \infty]$ defined as*

$$d_{M,f_*}(x, y) = \sup\{t \in]0, \infty[: M(x, y, t) \leq f^{(-1)}(t)\},$$

is a pseudo-metric on X such that $d_{M,f_*}(x, y) < \infty$ for each $x, y \in X$, where f_* is an additive generator of $*$. Moreover, d_{M,f_*} is a metric on X if and only if $(M, *)$ is a fuzzy metric on X.

Proof. By Theorem 7.2.2 de deduce that d_{M,f_*} is a pseudo-metric on X. Then we only need to show that $d_{M,f_*}(x,y) < \infty$ for each $x, y \in X$. To this end, consider $x, y \in X$ and we will see that $\sup\{t \in]0,\infty[: M(x,y,t) \leq f^{(-1)}(t)\} < \infty$.

On the one hand, by our hypothesis, given $x, y \in X$ there exists $t_0 \in]0,\infty[$ such that $M(x,y,t_0) > 0$. Furthermore, since $M_{x,y}$ is monotone then $M(x,y,t) > 0$ for all $t \in [t_0,\infty[$.

On the other hand, $f^{(-1)}$ is decreasing and continuous. Then, for each $\epsilon \in]0,\infty[$ there exists $t_\epsilon \in]0,\infty[$ such that $f^{(-1)}(t) < \epsilon$ for all $t \in [t_\epsilon,\infty[$. In particular, if we take $\epsilon = M(x,y,t_0) \in]0,\infty[$ there exists $t_\epsilon \in]0,\infty[$ such that $f^{(-1)}(t) < M(x,y,t_0)$ for all $t \in [t_\epsilon,\infty[$.

Therefore, $M(x,y,t) > f^{(-1)}(t)$ for all $t \in [t_1,\infty[$, where $t_1 = \max\{t_0, t_\epsilon\}$. So

$$\sup\{t \in]0,\infty[: M(x,y,t) \leq f^{(-1)}(t)\} \leq t_1.$$

Hence, $d_{M,f_*}(x,y) < \infty$ as we claimed. ■

The following example shows that we cannot delete the additional condition on the fuzzy pseudo-metric imposed in the preceding corollary to construct a pseudo-metric which does not take the value ∞.

Example 7.2.7. *Define the fuzzy set M_0 on $\mathbb{R} \times \mathbb{R} \times]0,\infty[$ as follows*

$$M_0(x,y,t) = \begin{cases} 0, & \text{if } x \neq y \\ 1, & \text{if } x = y \end{cases},$$

for all $t \in]0,\infty[$. It is not hard to check that $(M_0, *_P)$ is a fuzzy metric on $\mathbb{R}$ and, obviously, the additional condition imposed in Corollary 7.2.6 is not fulfilled. Notice that $*_P$ is a continuous Archimedean t-norm which is strict.

Next we show that $d_{M,f_{*P}}$ *can take the value* ∞. *Indeed, let* $x, y \in \mathbb{R}$ *with* $x \neq y$. *Then* $M(x,y,t) = 0$ *for all* $t \in]0,\infty[$ *and so*

$$0 = M(x,y,t) \leq f_{*P}^{(-1)}(t) = \mathsf{e}^{-t}$$

for all $t \in]0,\infty[$. *Therefore,*

$$d_{M,f_{*P}}(x,y) = \sup\{t \in]0,\infty[: M(x,y,t) \leq \mathsf{e}^{-t}\} = \infty.$$

Chapter 8

On indistinguishability operators, fuzzy metrics and modular metrics

The notion of indistinguishability operator was introduced by E. Trillas, in 1982, with the aim of fuzzifying the crisp notion of equivalence relation ([94]). Such operators allow us to measure the similarity between objects when there is a limitation on the accuracy of the performed measurement or a certain degree of similarity can be only determined between the objects being compared. Since Trillas introduced such kind of operators, many authors have studied their properties and applications. In particular, an intensive research line is focused on the metric behavior of indistinguishability operators ([12, 25, 46, 49, 67, 78, 95]). Specifically, it has been explored the existence of a duality between metrics and indistinguishability operators. In this direction a technique to generate metrics from indistinguishability operators, and vice-versa, has been developed by several authors in the literature. Nowadays, such a measurement of similarity is provided by the so-called fuzzy metrics when the degree of similarity between objects is measured relative to a parameter. The main purpose of this chapter is to extend the notion of indistinguishability operator in such a way that the measurements of similarity are relative to a parameter and, thus, classical indistinguishability operators and fuzzy metrics can be retrieved as a particular case. Moreover, we discuss the relationship between the new operators and metrics. Concretely, we prove the existence of a duality between them and the so-called modular metrics which provide a dissimilarity measurement between objects relative to a parameter. The new duality relationship allows us, on the one hand, to introduce a technique for generating the new indistinguishability operators from modular metrics and vice-versa and, on the other hand, to derive, as a consequence, a technique for generating fuzzy metrics from modular metrics and vice-versa. Furthermore, we yield examples which illustrate the new results. Throughout this chapter the fuzzy pseudo-metrics are understood as KM-fuzzy pseudo-metrics.

8.1 Modular indistinguishability operators

As mentioned above, we are interested in proposing a new type of operator that unifies the notion of fuzzy (pseudo-)metric and indistinguishability operator in such a way that a unique theoretical basis can be supplied to develop a wide range of applications. To this end, we introduce the notion of modular indistinguishability operator as follows:

Definition 8.1.1. Let X be a non-empty set and let $*$ be a t-norm, we will say that a fuzzy set $F : X \times X \times]0, \infty[\to [0, 1]$ is a *modular indistinguishability operator* for $*$ if for each $x, y, z \in X$ and $t, s > 0$ the following axioms are satisfied:

(ME1) $F(x, x, t) = 1$;

(ME2) $F(x, y, t) = F(y, x, t)$;

(ME3) $F(x, z, t + s) \geq F(x, y, t) * F(y, z, s)$.

If in addition, F satisfies for all $x, y \in X$, the following condition:

(ME1') $F(x, y, t) = 1$ for all $t > 0$ implies $x = y$,

we will say that F separates points.

Moreover, we will say that F is *stationary* provided that the function $F_{x,y} :]0, \infty[\to [0, 1]$ defined by $F_{x,y}(t) = F(x, y, t)$ is constant for each $x, y \in X$.

Notice that the numerical value $F(x, y, t)$ can understood as the degree up to which x is indistinguishable from y or equivalent to y relative to the value t of the parameter. Moreover, the greater $F(x, y, t)$ the more similar are x and y relative to the value t of the parameter. Clearly, $F(x, y, t) = 1$ for all $t > 0$ when $x = y$.

It is worth mentioning that the classical notion of indistinguishability operator is recovered when the modular indistinguishability operator F is stationary. Besides, it is clear that a modular indistinguishability operator can be considered as a generalization of the concept of fuzzy (pseudo-)metric. However, there are examples of modular indistinguishability operators that are not fuzzy (pseudo-)metrics such as the next example shows.

Example 8.1.2. *Consider a metric d on a non-empty set X. Define the fuzzy set F_d on $X \times X \times]0, \infty[$ as follows*

$$F_d(x, y, t) = \begin{cases} 0, & \text{if} \quad 0 < t < d(x, y); \\ 1, & \text{if} \quad t \geq d(x, y). \end{cases}$$

It is easy to check that F_d is a modular indistinguishability operator for the product t-norm $*_P$. Nevertheless, $(F_d, *_P)$ is not a fuzzy (pseudo-)metric because the function $F_{d_{x,y}} :]0, \infty[\to [0, 1]$, defined by $F_{d_{x,y}}(t) = F_d(x, y, t)$ is not left-continuous.

The concept of modular indistinguishability operator also generalizes the notion of fuzzy (pseudo-)metric in another outstanding aspect. Observe that in Definition 8.1.1 it is not required the continuity on the t-norm. Naturally the assumption of continuity of the t-norm is useful from a topological viewpoint, since the continuity is necessary in order to define a topology by means of a family of balls in a similar way like in the pseudo-metric case. However, such an assumption could be limiting the range of applications of such fuzzy measurements in those case where (classical) indistinguishability operators works well. In this direction, modular indistinguishability operators present an advantage with respect to fuzzy (pseudo-)metrics because the involved t-norms are not assumed to be continuous.

The following example illustrates the preceding remark providing an instance of modular indistinguishability operator for the Drastic t-norm $*_D$ which is not a modular indistinguishability operator for any continuous t-norm.

Example 8.1.3. *Let φ be the function defined on $]0, \infty[$ by $\varphi(t) = \frac{t}{1+t}$. We* *define the fuzzy set* F_D *on* $[0,1[\times[0,1[\times]0,\infty[$ *as follows*

$$F_D(x,y,t) = \begin{cases} 1, & \text{for each } t>0, \quad \text{if} \quad x=y \\ \max\{x,y,\varphi(t)\}, & \text{for each } t>0, \quad \text{if} \quad x\neq y \end{cases}.$$

First of all, note that for each $x,y \in [0,1[$ *and* $t>0$ *we have that* $F_D(x,y,t) \in [0,1[$*, since* $x,y,\varphi(t) \in [0,1[$*. Hence,* F_D *is a fuzzy set on* $[0,1[\times[0,1[\times]0,\infty[$.

Now, we will see that F_D *is a modular indistinguishability operator on* $[0,1[$ *for* $*_D$*. To this end, let us recall that* $*_D$ *is defined a by*

$$a *_D b = \begin{cases} 0, & \text{if } a,b \in [0,1[; \\ \min\{a,b\}, & \text{elsewhere.} \end{cases}$$

It is clear that F_D *satisfies axioms* $(ME1)$ *and* $(ME2)$*. Next we show that* F_D *satisfies* $(ME3)$*, i.e.,*

$$F_D(x,z,t+s) \geq F_D(y,z,s) = F_D(x,y,t) *_D F_D(y,z,s)$$

for all $x,y,z \in [0,1[$ *and* $t,s>0$.

Notice that we can assume that $x \neq z$*. Otherwise the preceding inequality is hold trivially. Next we distinguish two cases:*

1. *Case 1.* $x \neq y$ *and* $y \neq z$*. Then* $F_D(x,y,t) = \max\{x,y,\varphi(t)\} < 1$ *and* $F_D(y,z,s) = \max\{y,z,\varphi(s)\} < 1$*, since* $x,y,z \in [0,1[$ *and* $\varphi(t) < 1$ *for each* $t>0$*. Thus,* $F_D(x,y,t) *_D F_D(y,z,s) = 0$ *attending to the definition of* $*_D$*. It follows that* $F_D(x,z,t+s) \geq F_D(x,y,t) *_D F_D(y,z,s)$.

2. *Case 2. $x = y$ or $y = z$ (suppose, without loss of generality, that $x = y$). Then $F_D(x, y, t) = 1$ and so*

$$F_D(x, z, t + s) = F_D(y, z, t + s) = \max\{y, z, \varphi(t + s)\} \geq$$

$$\geq \max\{y, z, \varphi(s)\} = F_D(y, z, s),$$

since φ is a monotone function. Thus

$$F_D(x, z, t + s) \geq F_D(y, z, s) = F_D(x, y, t) *_D F_D(y, z, s).$$

Furthermore, the modular indistinguishability operator F_D separates points. Indeed, let $x, y \in [0, 1[$ and $t > 0$. Since $x, y, \varphi(t) \in [0, 1[$ for each $t > 0$ we have that if $x \neq y$ then $F_D(x, y, t) = \max\{x, y, \varphi(t)\} < 1$. Thus, $F_D(x, y, t) = 1$ implies $x = y$.

Finally, we will prove that F_D is not a modular indistinguishability operator for any continuous t-norm. To this end, we will show that axiom $(ME3)$ is not fulfilled for any t-norm continuous at $(1, 1)$.

Let $*$ be a continuous t-norm at $(1, 1)$. Then, for each $\epsilon \in]0, 1[$ we can find $\delta \in]0, 1[$ such that $\delta * \delta > 1 - \epsilon$.

Now, consider $x = 0$, $z = \frac{1}{2}$ and $t = s = 1$. Then,

$$F_D(x, z, t + s) = \max\left\{0, \frac{1}{2}, \frac{2}{3}\right\} = \frac{2}{3}.$$

Taking $\epsilon = \frac{1}{3}$ we can find $\delta \in]0, 1[$ such that $\delta * \delta > \frac{2}{3}$. Note that, in this case, $\delta > \frac{2}{3}$. Therefore, if we take $y = \delta$ we have that

$$F_D(x, y, t) * F_D(y, z, s) = \max\left\{0, y, \frac{1}{2}\right\} * \max\left\{y, \frac{1}{2}, \frac{1}{2}\right\} =$$

$$= y * y > \frac{2}{3} = F_D(x, z, t + s).$$

Thus, $(ME3)$ *is not satisfied.*

We end the section with a reflection on axiom (KM1). When such an axiom is considered in the definition of fuzzy (pseudo-)metric (i.e., the fuzzy (pseudo-)metric is considered as a fuzzy set on $X \times X \times [0, \infty[$ instead on $X \times X \times]0, \infty[$), one could wonder whether modular indistinguishability operators would be able to extend the notion of fuzzy (pseudo-)metric in that case. The answer to the posed question is affirmative. In fact, in order to define a new indistinguishability operator for that purpose we only need to include in the axiomatic in Definition 8.1.1 the following axiom:

(ME0) $F(x, y, 0) = 0$ for all $x, y \in X$.

Notice that even in such a case there exist modular indistinguishability operators which are not fuzzy (pseudo-)metrics. An example of such a kind of operators is given by an easy adaptation of the fuzzy set F_d introduced in Example 8.1.2. Indeed, we only need consider such a fuzzy set defined as in the aforesaid example and, in addition, satisfying $F_d(x, y, 0) = 0$ for all $x, y \in X$. Of course, it is easy to check that F_d is a modular indistinguishability operator for the product t-norm $*_P$ which satisfies (ME0) but $(F_d, *_P)$ is not a fuzzy (pseudo-)metric.

8.2 The duality relationship

This section is devoted to explore the metric behavior of the new indistinguishability operators. Concretely, we extend, on the one hand, the technique through which a metric can be generated from an indistinguishability operator by means of an additive generator of a t-norm (in Subsection 8.2.1) and, on the other hand, the technique that allows to induce an indistinguishability operator from a metric by means of the pseudo-inverse of the additive generator of a t-norm (in Subsection 8.2.2). The same results are also explored when fuzzy (pseudo-)metrics are considered instead of modular indistinguishability operators.

8.2.1 From modular indistinguishability operators to (pseudo-)metrics

In order to extend Theorem 1.1.27 to the modular framework we need to propose a metric class as candidate to be induced by a modular indistinguishability operator. We have found that such a candidate is known in the literature as *modular metric*. Let us recall a few basics about this type of metrics.

According to V.V. Chytiakov (see [11]), a function $w :]0, \infty[\times X \times X \to [0, \infty]$ is a *modular metric* on a non-empty set X if for each $x, y, z \in X$ and each $\lambda, \mu > 0$ the following axioms are fulfilled:

(MM1) $w(\lambda, x, y) = 0$ for all $\lambda > 0$ if and only if $x = y$;

(MM2) $w(\lambda, x, y) = w(\lambda, y, x)$;

(MM3) $w(\lambda + \mu, x, z) \leq w(\lambda, x, y) + w(\mu, y, z)$.

If the axiom $(MM1)$ is replaced by the following one

(MM1') $w(\lambda, x, x) = 0$ for all $\lambda > 0$,

then w is a called *modular pseudo-metric* on X.

Of course, the value $w(\lambda, x, y)$ can be understood as a dissimilarity measurement between objects relative to the value λ of a parameter.

Following [11], given $x, y \in X$ and $\lambda > 0$, we will denote, from now on, the value $w(\lambda, x, y)$ by $w_\lambda(x, y)$.

Notice that, as was pointed out in [11], a (pseudo-)metric is a modular (pseudo-)metric which is "stationary", i.e., it does not depends on the value λ of the parameter. Thus (pseudo-)metrics on X are modular (pseudo-)metrics $w :]0, \infty[\times X \times X \to [0, \infty]$ such that the assignment $w_{x,y} :]0, \infty[\to [0, \infty]$, given by $w_{x,y}(\lambda) = w_\lambda(x, y)$ is a constant function for each $x, y \in X$.

The following are well-known examples of modular (pseudo-)metrics.

Example 8.2.1. *Let d be a (pseudo-)metric on X and let $\varphi :]0, , \infty[\to]0, \infty[$ be a non-decreasing function. The functions defined on $]0, \infty[\times X \times X$ as follows*

(i) $$w^1_\lambda(x, y) = \begin{cases} \infty, & \text{if } \quad x \neq y \\ 0, & \text{if } \quad x = y \end{cases};$$

(ii) $$w^2_\lambda(x, y) = \begin{cases} \infty, & \text{if } \quad 0 < \lambda < d(x, y) \text{ and } d(x, y) > 0 \\ 0, & \text{if } \quad \lambda \geq d(x, y) \text{ and } d(x, y) > 0 \\ 0, & \text{if } \quad d(x, y) = 0 \end{cases};$$

(iii) $w^3_\lambda(x, y) = \frac{d(x,y)}{\varphi(\lambda)}$,

are modular (pseudo-)metrics on X.

Next we provide an example of modular metric that will be crucial in Subsection 8.2.2.

Proposition 8.2.2. *Let (X, d) be a metric space. Then the function $w :]0, \infty[\times X \times X \to [0, \infty]$ is a modular metric on X, where*

$$w_\lambda(x, y) = \frac{d^2(x, y)}{\lambda}$$

for each $x, y \in X$ and $\lambda \in]0, \infty[$ (in the last expression, $d^2(x, y)$ denotes $(d(x, y))^2$, as usual).

Proof. It is clear that axioms $(MM1)$ and $(MM2)$ are satisfied. It remains to show that axiom $(MM3)$ is hold. Let $x, y, z \in X$ and $\lambda, \mu \in]0, \infty[$. Note that

$$d^2(x, z) \leq \Big(d(x, y) + d(y, z)\Big)^2 = d^2(x, y) + 2d(x, y)d(y, z) + d^2(y, z),$$

since d is a metric and satisfies the triangle inequality.

From the preceding inequality we deduce the following one:

$$\frac{d^2(x,y)}{\lambda} + \frac{d^2(y,z)}{\mu} - \frac{d^2(x,z)}{\lambda+\mu} = \frac{\mu(\lambda+\mu)d^2(x,y)+\lambda(\lambda+\mu)d^2(y,z)-\lambda\mu d^2(x,z)}{\lambda\mu(\lambda+\mu)} =$$

$$\frac{\mu\lambda d^2(x,y)+\mu^2 d^2(x,y)+\lambda^2 d^2(y,z)+\lambda\mu d^2(y,z)-\lambda\mu d^2(x,z)}{\lambda\mu(\lambda+\mu)} \geq$$

$$\frac{\mu\lambda d^2(x,y)+\mu^2 d^2(x,y)+\lambda^2 d^2(y,z)+\lambda\mu d^2(y,z)-\lambda\mu(d^2(x,y)+2d(x,y)d(y,z)+d^2(y,z))}{\lambda\mu(\lambda+\mu)} =$$

$$\frac{\mu^2 d^2(x,y)+\lambda^2 d^2(y,z)-2\lambda\mu d(x,y)d(y,z)}{\lambda\mu(\lambda+\mu)} = \frac{(\mu d(x,y)-\lambda d(y,z))^2}{\lambda\mu(\lambda+\mu)} \geq 0.$$

Therefore,

$$w_{\lambda+\mu}(x,z) = \frac{d^2(x,z)}{\lambda+\mu} \leq \frac{d^2(x,y)}{\lambda} + \frac{d^2(y,z)}{\mu} = w_\lambda(x,y) + w_\mu(y,z).$$

Hence w satisfies $(MM3)$. ■

After a brief introduction to modular metric spaces we are able to yield a modular version of Theorem 1.1.27.

Theorem 8.2.3. *Let X be a non-empty set and let $*$ be a continuous t-norm with additive generator $f_* : [0,1] \to [0,\infty]$. If $\diamond$ is a t-norm, then the following assertions are equivalent:*

1) $* \leq \diamond$ (i.e., $x * y \leq x \diamond y$ for all $x, y \in [0,1]$).

2) For any modular indistinguishability operator F on X for $\diamond$, the function $(w^{F,f_*}) :]0,\infty[\times X \times X \to [0,\infty]$ defined by

$$(w^{F,f_*})_\lambda(x,y) = f_*(F(x,y,\lambda)),$$

for each $x, y \in X$ and $\lambda > 0$, is a modular pseudo-metric on X.

3) For any modular indistinguishability operator F on X for $\diamond$ that separates points, the function $(w^{F,f_*}) :]0,\infty[\times X \times X \to [0,\infty]$ defined by

$$(w^{F,f_*})_\lambda(x,y) = f_*(F(x,y,\lambda)),$$

for each $x, y \in X$ and $\lambda > 0$, is a modular metric on X.

Proof.

1) $\Rightarrow$ 2) Suppose that $* \leq \diamond$ and let F be a modular indistinguishability operator on X for $\diamond$. We will see that (w^{F,f_*}) is a modular pseudo-metric on X.

(MM1') Let $x \in X$. Since $F(x,x,\lambda) = 1$ for each $\lambda > 0$, then $(w^{F,f_*})_\lambda(x,x) = f_*(F(x,x,\lambda)) = f_*(1) = 0$ for each $\lambda > 0$.

(MM2) It is obvious because $F(x,y,\lambda) = F(y,x,\lambda)$ for all $x, y \in X$ and $\lambda > 0$.

(MM3) Let $x, y, z \in X$ and $\lambda, \mu > 0$. We will show that the following inequality

$$(w^{F,f_*})_{\lambda+\mu}(x,z) \leq (w^{F,f_*})_\lambda(x,y) + (w^{F,f_*})_\mu(y,z)$$

is hold. Firs of all, note that F is also a modular indsitinguishability operator for $*$ on X due to $\diamond \geq *$. Then, it is satisfied the following inequality

$$F(x,z,\lambda+\mu) \geq F(x,y,\lambda) * F(y,z,\mu) =$$

$$f_*^{(-1)}\left(f_*(F(x,y,\lambda)) + f_*(F(y,z,\mu))\right).$$

Taking into account that f_* is an additive generator, and thus a decreasing function, we have that

$$f_*(F(x,z,\lambda+\mu)) \leq f_*\left(f_*^{(-1)}\left(f_*(F(x,y,\lambda)) + f_*(F(y,z,\mu))\right)\right).$$

Now, we will distinguish two different cases:

(a) Suppose that $f_*(F(x,y,\lambda)) + f_*(F(y,z,\mu)) \in Ran(f_*)$. Since f_* is an additive generator of the t-norm $*$ we have that $f_* \circ f_*^{(-1)}|_{Ran(f_*)} = id|_{Ran(f_*)}$. Then

$$f_*\left(f_*^{(-1)}\left(f_*(F(x,y,\lambda)) + f_*(F(y,z,\mu))\right)\right) =$$

$$f_*(F(x,y,\lambda)) + f_*(F(y,z,\mu)).$$

It follows that

$$(w^{F,f_*})_{\lambda+\mu}(x,z) = f_*(F(x,z,\lambda+\mu)) \leq$$

$$f_*(F(x,y,\lambda)) + f_*(F(y,z,\mu)) == (w^{F,f_*})_\lambda(x,y) + (w^{F,f_*})_\mu(y,z).$$

(b) Suppose that $f_*(F(x,y,\lambda)) + f_*(F(y,z,\mu)) \notin Ran(f_*)$. Since f_* is an additive generator of the t-norm $*$ we have that $f_*(a)+f_*(b) \in Ran(f_*) \cup [f_*(0), \infty]$ for each $a, b \in [0,1]$. Then

$$f_*(F(x,y,\lambda)) + f_*(F(y,z,\mu)) > f_*(0).$$

So we obtain

$$f_*(F(x,z,\lambda+\mu)) \leq f_*(0) < f_*(F(x,y,\lambda)) + f_*(F(y,z,\mu)).$$

Whence we have that

$$(w^{F,f_*})_{\lambda+\mu}(x,z) \leq (w^{F,f_*})_\lambda(x,y) + (w^{F,f_*})_\mu(y,z),$$

as we claimed.

Therefore, (w^{F,f_*}) is a modular pseudo-metric on X.

2) $\Rightarrow$ 3) Let F be a modular indistinguishability operator on X for $\diamond$ that separates points. By our assumption, (w^{F,f_*}) is a pseudo-modular metric on X. We will see that (w^{F,f_*}) is a modular metric on X.

Let $x, y \in X$ such that $(w^{F,f_*})_\lambda(x,y) = 0$ for all $\lambda > 0$. By definition, we have that $f_*(F(x,y,\lambda)) = 0$ for all $\lambda > 0$. Then, $F(x,y,\lambda) = 1$ for all $\lambda > 0$, since f_* is an additive generator of $*$. Therefore $x = y$, since F is a modular indistinguishability operator on X for $\diamond$ that separates points.

3) $\Rightarrow$ 1) Suppose that for any modular indistinguishability operator F on X for $\diamond$ that separates points the function (w^{F,f_*}) is a modular metric on X. We will show that $\diamond \geq *$. To this end, we will prove that $a \diamond b \geq a * b$ provided $a, b \in [0,1[$. Note that the preceding inequality is obvious whenever either $a = 1$ or $b = 1$.

Let $a, b \in [0,1[$. Consider a set constituted by three distinct points $X = \{x, y, z\}$. We define a fuzzy set F on $X \times X \times]0, \infty[$ as follows:

$$F(u,v,t) = F(v,u,t) = \begin{cases} 1, & \text{if } u = v \\ a \diamond b, & \text{if } u = x \text{ and } v = z \\ a, & \text{if } u = x \text{ and } v = y \\ b, & \text{if } u = y \text{ and } v = z \end{cases},$$

for all $t > 0$.

It is easy to verify, attending to its definition, that F is a modular indistinguishability operator on X for $\diamond$ that separates points. So (w^{F,f_*}) is a modular metric on X. Therefore, given $\lambda > 0$ we have that

$$f_*(a \diamond b) = (w^{F,f_*})_{2\lambda}(x,z) \leq (w^{F,f_*})_\lambda(x,y) + (w^{F,f_*})_\lambda(y,z) =$$

$$f_*(a) + f_*(b).$$

Notice that for each $c \in [0,1]$ we have that $(f_*^{(-1)} \circ f_*)(c) = c$, $a * b = f_*^{(-1)}(f_*(a) + f_*(b))$ and that $f_*^{(-1)}$ is decreasing, since f_* is an additive generator of the t-norm $*$. Taking into account the preceding facts and from the above inequality we deduce that

$$a \diamond b = f_*^{(-1)}(f_*(a \diamond b)) \geq f_*^{(-1)}(f_*(a) + f_*(b)) = a * b,$$

as we claimed.

This last implication concludes the proof. ■

In order to illustrate the technique introduced in the above theorem, we provide two corollaries which establish the particular cases for the Luckasievicz t-norm and the usual product. With this aim we recall that an additive generator f_{*_L} of $*_L$ and f_{*_P} of $*_P$ is given by

$$\begin{aligned} f_{*_L}(a) &= 1 - a \\ f_{*_P}(a) &= -\log(a) \end{aligned}$$

for each $a \in [0,1]$, respectively. Of course, we have adopted the convention that $\log(0) = -\infty$.

Corollary 8.2.4. *Let X be a non-empty set. If $\diamond$ is a t-norm, then the following assertions are equivalent:*

1) $*_L \leq \diamond$.

2) *For any modular indistinguishability operator F on X for $\diamond$, the function $(w^{F,f_{*L}}) :]0,\infty[\times X \times X \to [0,\infty]$ defined by*

$$(w^{F,f_{*L}})_\lambda(x,y) = 1 - F(x,y,\lambda),$$

for each $x,y \in X$ and $\lambda > 0$, is a modular pseudo-metric on X.

3) *For any modular indistinguishability operator F on X for $\diamond$ that separates points, the function $(w^{F,f_{*L}}) :]0,\infty[\times X \times X \to [0,\infty]$ defined by*

$$(w^{F,f_{*L}})_\lambda(x,y) = 1 - F(x,y,\lambda),$$

for each $x,y \in X$ and $\lambda > 0$, is a modular metric on X.

Corollary 8.2.5. *Let X be a non-empty set. If $\diamond$ is a t-norm, then the following assertions are equivalent:*

1) $*_P \leq \diamond$.

2) *For any modular indistinguishability operator F on X for $\diamond$, the function $(w^{F,f_{*P}}) :]0,\infty[\times X \times X \to [0,\infty]$ defined by*

$$(w^{F,f_{*P}})_\lambda(x,y) = -log(F(x,y,\lambda)),$$

for each $x,y \in X$ and $\lambda > 0$, is a modular pseudo-metric on X.

3) For any modular indistinguishability operator F on X for $\diamond$ that separates points, the function $(w^{F,f_{*P}}) :]0,\infty[\times X \times X \to [0,\infty]$ *defined by*

$$(w^{F,f_{*P}})_\lambda(x,y) = -log(F(x,y,\lambda)),$$

for each $x, y \in X$ and $\lambda > 0$, is a modular metric on X.

Theorem 8.2.3 also gives a specific method to generate modular metrics when we focus our attention on fuzzy (pseudo-)metrics instead of modular indistinguishability operators in general.

Corollary 8.2.6. *Let X be a non-empty set and let $*$ be a t-norm with additive generator $f_* : [0,1] \to [0,\infty]$. If $\diamond$ is a continuous t-norm, then the following assertions are equivalent:*

1) $* \leq \diamond$.

2) For any fuzzy pseudo-metric $(M, \diamond)$ on X, the function $(w^{M,f_*}) :]0,\infty[\times X \times X \to [0,\infty]$ *defined by*

$$(w^{M,f_*})_\lambda(x,y) = f_*(M(x,y,\lambda)),$$

for each $x, y \in X$ and $\lambda > 0$, is a modular pseudo-metric on X.

3) For any fuzzy metric $(M, \diamond)$ on X, the function $(w^{M,f_*}) :]0,\infty[\times X \times X \to [0,\infty]$ *defined by*

$$(w^{M,f_*})_\lambda(x,y) = f_*(M(x,y,\lambda)),$$

for each $x, y \in X$ and $\lambda > 0$, is a modular metric on X.

As a consequence of the preceding result we obtain immediately the following one.

Corollary 8.2.7. *Let X be a non-empty set and let $*$ be a continuous t-norm with additive generator $f_* : [0,1] \to [0,\infty]$. Then the following assertions are equivalent:*

1) *For any fuzzy pseudo-metric $(M,*)$ on X, the function $(w^{M,f_*}) :]0,\infty[\times X \times X \to [0,\infty]$ defined by*

$$(w^{M,f_*})_\lambda(x,y) = f_*(M(x,y,\lambda)),$$

for each $x, y \in X$ and $\lambda > 0$, is a modular pseudo-metric on X.

2) *For any fuzzy metric $(M,*)$ on X, the function $(w^{M,f_*}) :]0,\infty[\times X \times X \to [0,\infty]$ defined by*

$$(w^{M,f_*})_\lambda(x,y) = f_*(M(x,y,\lambda)),$$

for each $x, y \in X$ and $\lambda > 0$, is a modular metric on X.

It is clear that when we consider stationary modular indistinguishability operators in statement of Theorem 8.2.3 we obtain as a particular case Theorem 1.1.27 and, thus, the classical technique to induce a metric from an indistinguishability operator by means of an additive generator. Clearly, if we replace modular indistinguishability operators by stationary fuzzy metrics we obtain a more restrictive version of the classical technique, provided by Theorem 8.2.3, because it only remains valid for continuous t-norms.

8.2.2 From modular (pseudo-)metrics to modular indistinguishability operators

The main goal of this subsection is to provide a version of Theorem 1.1.28 when we consider a modular (pseudo-)metric instead of a (pseudo-)metric.

Thus we give a technique to induce a modular indistinguishability operator from a modular (pseudo-)metric by means of the pseudo-inverse of the additive generator of a t-norm. To this end, let us recall the following representation result for continuous t-norms ([49]), which will be crucial in our subsequent discussion:

Theorem 8.2.8. *A binary operator* $*$ *in* $[0,1]$ *is a continuous Archimedean t-norm if and only if there exists a continuous additive generator* f_* *such that*

$$x * y = f_*^{(-1)}(f_*(x) + f_*(y)), \tag{8.1}$$

where the pseudo-inverse $f_*^{(-1)}$ *is given by*

$$f_*^{(-1)}(y) = f^{-1}(\min\{f_*(0), y\}) \tag{8.2}$$

for all $y \in [0,\infty]$.

In the next result we introduce the promised technique.

Theorem 8.2.9. *Let* $*$ *be a continuous t-norm with additive generator* $f_* : [0,1] \to [0,\infty]$. *If* w *is a modular pseudo-metric on* X, *then the function* $F^{w,f_*} : X \times X \times]0,\infty[\to [0,1]$ *defined, for all* $x, y \in X$ *and* $t > 0$, *by*

$$F^{w,f_*}(x,y,t) = f_*^{(-1)}(w_t(x,y))$$

is a modular indistinguishability operator for $*$. *Moreover, the modular indistinguishability operator* F^{w,f_*} *separates points if and only if* w *is a modular metric on* X.

Proof. Let $*$ be a continuous Archimedean t-norm with additive generator $f_* : [0,1] \to [0,\infty]$ and consider w a modular pseudo-metric on X.

We define the function $F^{w,f_*} : X \times X \times]0,\infty[\to [0,1]$ as follows

$$F^{w,f_*}(x,y,t) = f_*^{(-1)}(w_t(x,y)),$$

for all $x, y \in X$ and $t > 0$. We will see that F^{w,f_*} is a modular inidistinguishability operator for $*$.

(ME1) Let $x \in X$. Since w is a modular pseudo-metric on X we have that $w_t(x,x) = 0$ for all $t > 0$. Therefore, $F^{w,f_*}(x,x,t) = f_*^{(-1)}(w_t(x,x)) = f_*^{(-1)}(0) = 1$ for all $t > 0$.

(ME2) Is a consequence of the definition of F^{w,f_*}, since w is a modular pseudo-metric and so it satisfies that $w_t(x,y) = w_t(y,x)$ for each $x, y \in X$ and $t > 0$.

(ME3) Let $x, y, z \in X$ and $t, s > 0$. On the one hand, by (8.2), we deduce that

$$F^{w,f_*}(x,z,t+s) = f_*^{(-1)}(w_{t+s}(x,z)) = f_*^{-1}(\min\{f_*(0), w_{t+s}(x,z)\}).$$

Now, since w is a modular pseudo-metric on X, then

$$w_{t+s}(x,z) \leq w_t(x,y) + w_s(y,z)$$

and, hence,

$$F^{w,f_*}(x,z,t+s) \geq f_*^{-1}(\min\{f_*(0), w_t(x,y) + w_s(y,z)\}).$$

On the other hand, we have that

$$F^{w,f_*}(x,y,t) * F^{w,f_*}(y,z,s) =$$

$$f_*^{(-1)}\left(f_*\left(F^{w,f_*}(x,y,t)\right) + f_*\left(F^{w,f_*}(y,z,s)\right)\right) =$$

$$f_*^{-1}\left(\min\left\{f_*(0), f_*\left(F^{w,f_*}(x,y,t)\right) + f_*\left(F^{w,f_*}(y,z,s)\right)\right\}\right)$$

Moreover, by (8.2), we obtain that

$$f_*\left(F^{w,f_*}(x,y,t)\right) = f_*\left(f_*^{(-1)}(w_t(x,y))\right) = \min\{f_*(0), w_t(x,y)\}$$

and

$$f_* \left(F^{w,f_*}(y,z,s) \right) = f_* \left(f_*^{(-1)}\left(w_s(y,z) \right) \right) = \min\{f_*(0), w_s(y,z)\}.$$

To finish the proof, we will see that

$$\min\{f_*(0), w_t(x,y) + w_s(y,z)\} =$$

$$= \min\{f_*(0), \min\{f_*(0), w_t(x,y)\} + \min\{f_*(0), w_s(y,z)\}\}.$$

To this end, we will distinguish three cases:

Case 1. $f_*(0) \leq w_t(x,y)$ and $f_*(0) \leq w_s(y,z)$. Then we have that

$$\min\{f_*(0), w_t(x,y) + w_s(y,z)\} = f_*(0)$$

and

$$\min\{f_*(0), \min\{f_*(0), w_t(x,y)\} + \min\{f_*(0), w_s(y,z)\}\} =$$

$$= \min\{f_*(0), f_*(0) + f_*(0)\} = f_*(0).$$

Case 2. $f_*(0) > w_t(x,y)$ and $f_*(0) \leq w_s(y,z)$ (the case $f_*(0) \leq w_t(x,y)$ and $f_*(0) > w_s(y,z)$ runs following the same arguments). It follows that

$$\min\{f_*(0), w_t(x,y) + w_s(y,z)\} = f_*(0)$$

and

$$\min\{f_*(0), \min\{f_*(0), w_t(x,y)\} + \min\{f_*(0), w_s(y,z)\}\} =$$

$$= \min\{f_*(0), w_t(x,y) + f_*(0)\} = f_*(0).$$

Case 3. $f_*(0) > w_t(x,y)$ and $f_*(0) > w_s(y,z)$. Then we have that

$$\min\{f_*(0), \min\{f_*(0), w_t(x,y)\} + \min\{f_*(0), w_s(y,z)\}\} =$$

$$= \min\{f_*(0), w_t(x,y) + w_S(y,z)\}.$$

Therefore,

$$F^{w,f_*}(x,z,t+s) \geq f_*^{-1}\left(\min\left\{f_*(0), f_*\left(F^{w,f_*}(x,y,t)\right) + f_*\left(F^{w,f_*}(y,z,s)\right)\right\}\right)$$
$$= F^{w,f_*}(x,y,t) * F^{w,f_*}(y,z,s).$$

Whence we deduce that F^{w,f_*} is a modular indistinguishability operator for $*$ on X.

Finally, ir is clear that $F^{w,f_*}(x,y,t) = 1$ for all $x,y \in X$ and $t > 0$ if, and only if, $f_*^{(-1)}(w_t(x,y)) = 1$ for all $x,y \in X$ and $t > 0$. Since $f_*^{(-1)}(w_t(x,y)) = 1$ for all $x,y \in X$ and $t > 0$ if, and only if, $w_t(x,y) = 0$ for all $x,y \in X$ and $t > 0$ we immediately obtain that F^{w,f_*} is a modular indistinguishability operator that separates points if, and only if, w is a modular metric on X.

■

Next we specify the method given in Theorem 8.2.9 for the t-norms $*_L$ and $*_P$. Note that the pseudo-inverse of the additive generator f_{*_L} and f_{*_P} is given by

$$f_{*_L}^{(-1)}(b) = \begin{cases} 1-b & \text{if } b \in [0,1[\\ 0, & \text{if } b \in [1,\infty] \end{cases}$$

and

$$f_{*_P}^{(-1)}(b) = \mathsf{e}^{-b}$$

for each $b \in [0,\infty]$, respectively, where we have adopted the convention that $\mathsf{e}^{-\infty} = 0$.

Corollary 8.2.10. *If w is a modular pseudo-metric on X, then the function $F^{w,f_{*_L}} : X \times X \times]0,\infty[\to [0,1]$ defined, for all $x,y \in X$ and $t > 0$, by*

$$F^{w,f_{*_L}}(x,y,t) = \begin{cases} 1 - w_t(x,y) & \textit{if } w_t(x,y) \in [0,1[\\ 0, & \textit{if } w_t(x,y) \in [1,\infty] \end{cases},$$

is a modular indistinguishability operator for $*_L$*. Moreover, the modular indistinguishability operator* $F^{w,f_{*L}}$ *separates points if and only if* w *is a modular metric on* X*.*

Corollary 8.2.11. *If* w *is a modular pseudo-metric on* X*, then the function* $F^{w,f_{*P}} : X \times X \times]0,\infty[\to [0,1]$ *defined, for all* $x, y \in X$ *and* $t > 0$*, by*

$$F^{w,f_{*P}}(x,y,t) = \mathsf{e}^{-w_t(x,y)},$$

is a modular indistinguishability operator for $*_P$*. Moreover, the modular indistinguishability operator* $F^{w,f_{*L}}$ *separates points if and only if* w *is a modular metric on* X*.*

In the light Theorem 8.2.9, it seems natural to ask if the continuity of the t-norm can be eliminated from the assumptions of such a result. The next example gives a negative answer to that question. In particular it proves that there are fuzzy sets F^{w,f_*}, given by Theorem 8.2.9, that are not modular indistinguishability operators when the t-norm $*$ under consideration is not continuous.

Example 8.2.12. *Consider the Euclidean metric* d_e *on* $\mathbb{R}$*. By Proposition 8.2.2, the function* w^e *is a modular metric on* $\mathbb{R}$*, where*

$$w_\lambda^e(x,y) = \frac{(d_e(x,y))^2}{\lambda}$$

for all $x, y \in \mathbb{R}$ *and* $\lambda > 0$*. Consider the additive generator* f_{*D} *of the non-continuous* t*-norm* $*_D$*. Recall that* f_{*D} *is given by*

$$f_{*D}(x) = \begin{cases} 0, & \text{if } x = 1; \\ 2 - x, & \text{if } x \in [0,1[\end{cases}$$

An easy computation shows that its pseudo-inverse is given by

$$f_{*_D}^{(-1)}(x) = \begin{cases} 1, & \textit{if } x \in [0,1]; \\ 2-x, & \textit{if } x \in]1,2]; \\ 0, & \textit{if } x \in]2,\infty[. \end{cases}$$

Next we show that we always can find $x, y, z \in \mathbb{R}$ and $\lambda, \mu \in]0, \infty[$ such that

$$F^{w^e, f_{*_D}}(x, z, \lambda + \mu) < F^{w^e, f_{*_D}}(x, y, \lambda) *_D F^{w^e, f_{*_D}}(y, z, \mu).$$

Let $x = 0$, $y = 1$ and $z = 2$, and consider $\lambda = \mu = 1$. Then,

$$w^e_{\lambda+\mu}(x, z, \lambda) = \frac{(d_e(x,z))^2}{\lambda + \mu} = \frac{2^2}{2} = 2,$$

$$w^e_{\lambda}(x, y) = \frac{(d_e(x,y))^2}{\lambda} = \frac{1^2}{1} = 1$$

and

$$w^e_{\lambda}(y, z) = \frac{(d_e(y,z))^2}{\mu} = \frac{1^2}{1} = 1.$$

Therefore,

$$0 = f_{*_D}^{(-1)}(2) = F^{w^e, f_{*_D}}(x, z, \lambda + \mu) <$$

$$< F^{w^e, f_{*_D}}(x, y, \lambda) *_D F^{w^e, f_{*_D}}(y, z, \mu) = f_{*_D}^{(-1)}(1) *_D f_{*_D}^{(-1)}(1) = 1.$$

Since the continuity is a necessary hypothesis in the statement of Theorem 8.2.9 one could expect that the following result would be true.

"Let $*$ be a continuous Archimedean t-norm with additive generator $f_* : [0,1] \to [0, \infty]$. If w is a modular pseudo-metric on X, then the pair $(M^{w,f_*}, *)$ is a fuzzy (pseudo-)metric, where the fuzzy set M^{w,f_*} : $X \times X \times]0, \infty[$ is given, for all $x, y \in X$ and $t > 0$, by

$$M^{w,f_*}(x,y,t) = f_*^{(-1)}(w_t(x,y)).$$

Moreover, $(M^{w,f_*}, *)$ is a fuzzy metric if and only if w is a modular metric on X."

Nevertheless the following example proves that such a result does not hold. In fact the technique provided by Theorem 8.2.9 does not give in general a fuzzy (pseudo-)metric.

Example 8.2.13. *Let d be a metric on a non-empty set X. Consider the modular metric w^2 on X introduced in Example 8.2.1, that is,*

$$w_t^2(x,y) = \begin{cases} \infty, & \text{if} \quad 0 < t < d(x,y) \text{ and } d(x,y) > 0 \\ 0, & \text{if} \quad t \geq d(x,y) \text{ and } d(x,y) > 0 \\ 0, & \text{if} \quad d(x,y) = 0 \end{cases}$$

for all $x, y \in X$ and $t > 0$. Then it is not hard to check that the pair $(M^{w^2,f_{*P}}, *_P)$ is not a fuzzy (pseudo-)metric, where the fuzzy set $M^{w^2,f_{*P}}$ is given by $M^{w^2,f_{*P}}(x,y,t) = f_{*P}^{(-1)}(w_t^2(x,y))$ for all $x, y \in X$ and $t > 0$, where

$$f_{*P}^{(-1)}(w_t^2(x,y)) = \begin{cases} 0, & \text{if} \quad 0 < t < d(x,y) \text{ and } d(x,y) > 0 \\ 1, & \text{if} \quad t \geq d(x,y) \text{ and } d(x,y) > 0 \\ 1, & \text{if} \quad d(x,y) = 0 \end{cases}$$

for all $x, y \in X$ and $t > 0$. Notice that $(M^{w^2,f_{*P}}, *_P)$ fails to fulfill axiom (KM5), i.e., the function $M_{x,y}^{w^2,f_{*P}} :]0, \infty[\to [0,1]$ is not left-continuous.

The preceding example suggest the study of those conditions that a modular (pseudo-)metric must satisfy in order to induce a fuzzy (pseudo-) metric by means of the technique exposed in Theorem 8.2.9. The following lemma, whose proof was given in [11], will help us to find it.

Lemma 8.2.14. *Let w be a modular (pseudo-)metric on X. Then, for each $x, y \in X$ we have that $w_s(x,y) \geq w_t(x,y)$ whenever $s,t \in]0,\infty[$ with $s < t$.*

Taking into account the preceding lemma, the next result provides a condition which is useful for our target.

Proposition 8.2.15. *Let w be a modular pseudo-metric on X. The function $\tilde{w} :]0,\infty[\times X \times X \to [0,\infty]$ given, for each $x, y \in X$ and $t > 0$, by*

$$\tilde{w}_\lambda(x,y) = \inf_{0<t<\lambda} w_t(x,y)$$

is a modular pseudo-metric on X such that for each $x, y \in X$ the function $\tilde{w}_{x,y} :]0,\infty[\to]0,\infty[$ is left continuous, where $\tilde{w}_{x,y}(\lambda) = \tilde{w}_\lambda(x,y)$ for each $\lambda \in]0,\infty[$. Furthermore, $\tilde{w}$ is a modular metric on X if and only if w it is so.

Proof. It is obvious that $\tilde{w}$ satisfies axiom $(MM2)$. Next we show that $\tilde{w}$ satisfies axioms $(MM1')$ and $(MM3)$.

(MM1') Fix $x \in X$ and let $\lambda \in]0,\infty[$. Since w is a modular pseudo-metric on X then $w_t(x,x) = 0$ for each $t > 0$. Therefore,

$$\tilde{w}_\lambda(x,x) = \inf_{0<t<\lambda} w_t(x,x) = 0.$$

(MM3) Let $x, y, z \in X$ and $\lambda, \mu \in]0,\infty[$. Next we prove that

$$\tilde{w}_{\lambda+\mu}(x,z) \leq \tilde{w}_\lambda(x,y) + \tilde{w}_\mu(y,z).$$

With this aim note that, given $u, v \in X$ and $\alpha \in]0,\infty[$, we have that for each $\epsilon \in]0,\infty[$ we can find $t \in]0,\alpha[$ satisfying $w_t(u,v) < \tilde{w}_\alpha(u,v) + \epsilon$.

Fix an arbitrary $\epsilon \in]0,\infty[$, then we can find $t \in]0,\lambda[$ and $s \in]0,\mu[$ such that $w_t(x,y) < \tilde{w}_\lambda(x,y)+\epsilon/2$ and $w_s(y,z) < \tilde{w}_\mu(y,z)+\epsilon/2$. Therefore,

$$\tilde{w}_{\lambda+\mu}(x,z) \leq w_{t+s}(x,z) \leq w_t(x,y)+w_s(y,z) < \tilde{w}_\lambda(x,y)+\tilde{w}_\mu(y,z)+\epsilon,$$

since w is a pseudo-metric on X. Taking into account that $\epsilon \in]0,\infty[$ is arbitrary we conclude that

$$\tilde{w}_{\lambda+\mu}(x,z) \leq \tilde{w}_\lambda(x,y) + \tilde{w}_\mu(y,z).$$

Thus $\tilde{w}$ is a modular pseudo-metric on X.

We will continue showing that for each $x,y \in X$ the function $\tilde{w}_{x,y} :]0,\infty[\to]0,\infty[$ is left continuous. Fix $x,y \in X$ and consider an arbitrary $\lambda_0 \in]0,\infty[$. Then given $\epsilon \in]0,\infty[$ we can find $\delta \in]0,\infty[$ such that

$$\tilde{w}_\lambda(x,y) - \tilde{w}_{\lambda_0}(x,y) < \epsilon,$$

for each $\lambda \in]\lambda_0-\delta,\lambda_0]$ (note that $\tilde{w}_\lambda(x,y) \geq \tilde{w}_{\lambda_0}(x,y)$ for each $\lambda \in]\lambda_0-\delta,\lambda_0]$ by Lemma 8.2.14). Indeed, let $\epsilon \in]0,\infty[$. As before, we can find $t \in]0,\lambda_0[$ such that

$$w_t(x,y) < \tilde{w}_{\lambda_0}(x,y) + \epsilon$$

and, again by Lemma 8.2.14, we have that $w_s(x,y) < \tilde{w}_{\lambda_0}(x,y) + \epsilon$ for each $s \in]t,\lambda_0]$. Therefore, taking $\delta = \lambda_0 - t$ we have that

$$\tilde{w}_\lambda(x,y) - \tilde{w}_{\lambda_0}(x,y) \leq w_\lambda(x,y) - \tilde{w}_{\lambda_0}(x,y) < \epsilon,$$

for each $\lambda \in]\lambda_0-\delta,\lambda_0]$, as we claimed. Thus, $\tilde{w}_{x,y}$ is left-continuous on $]0,\infty[$ since λ_0 is arbitrary.

Finally, it is easy to verify that $\tilde{w}$ is a modular metric on X if and only if w it is so. Indeed, $\tilde{w}$ is a modular metric on X if and only if $\tilde{w}_\lambda(x,y) = 0$ for each $\lambda \in]0,\infty[$ implies $x = y$, but $\tilde{w}_\lambda(x,y) = \inf_{0<t<\lambda} w_t(x,y) = 0$ for each $\lambda \in]0,\infty[$ if and only if $w_t(x,y) = 0$ for each $t \in]0,\infty[$, which concludes the proof. ■

Observe that in the preceding result $\tilde{w}$ coincides with w, whenever $w_{x,y}$ is a left-continuous function, for each $x, y \in X$.

Proposition 8.2.15 and Theorem 8.2.9 allow us to give the searched method for constructing a fuzzy pseudo-metric from a modular pseudo-metric.

Theorem 8.2.16. *Let $*$ be a continuous t-norm with additive generator $f_* : [0,1] \to [0,\infty]$. If w is a modular pseudo-metric on X, then the pair $(M^{w,f_*}, *)$ is a fuzzy pseudo-metric on X, where the fuzzy set $M^{w,f_*} : X \times X \times [0,\infty[$ is defined, for all $x, y \in X$, by*

$$M^{w,f_*}(x,y,t) = f_*^{(-1)}\left(\tilde{w}_t(x,y)\right),$$

where $\tilde{w}_t(x,y) = \inf_{0<\lambda<t} w_t(x,y)$. Moreover, $(M^{w,f_*}, *)$ is a fuzzy metric on X if and only if w is a modular metric on X.

Proof. By Proposition 8.2.15 we deduce that $\tilde{w}_{x,y}$ is a modular pseudo-metric on X. Theorem 8.2.9 guarantees that M^{w,f_*} is a modular indistinguishability operator for $*$ on X. Moreover, continuity of $f_*^{(-1)}$ and the left-continuity, provided by Proposition 8.2.15, of the function $\tilde{w}_{x,y}$ guarantee that axiom $(KM5)$ is fulfilled. Thus the pair $(M^{w,f_*}, *)$ is a fuzzy pseudo-metric on X. Finally, by Proposition 8.2.15 and Theorem 8.2.9, it is obvious that $(M^{w,f_*}, *)$ is a fuzzy metric on X if and only if w is a modular metric on X. ■

Chapter 9

Indistinguishability Operators Applied to Task Allocation Problem in Multi-Agent Systems

This chapter addresses the multi-robot task allocation problem. In particular, given a collection of tasks and robots (agents), we focus on how to select the best robot to execute each task by means of the so-called response threshold method. In the aforesaid method, each robot decides to leave a task and to perform another one (decides to transit) following a probability (response functions) that depends mainly on a stimulus and the current task. The probabilistic approaches used to model the transitions present several handicaps which will be detailed later on. To solve these problems we introduced the use of indistinguishability operators to model response functions and possibility theory instead of probability. In particular, we propose this kind of operators to represent a response function when the stimulus only depends on the distance between the agent and a determined task, since we prove that two celebrated response functions used in the literature can be reproduced by appropriate indistinguishability operators when the stimulus only depends on the distance to each task that must be carried out. Nowadays there is not a systematic method to generate response functions in the literature, this chapter provides, for the first time, a theoretical foundation to generate them and study their properties. To validate the theoretical results, the aforementioned indistinguishability operators have been used to simulate under MATLAB the allocation of a set of tasks in a multi-robot system with fuzzy Markov chains instead of probabilistic Markov chains. Such simulations show how the possibilistic Markov chains outperform their probabilistic counterpart.

9.1 Swarm task allocation and the Response Threshold method

In this section we will introduce the main concepts of classical (probabilistic) Response Threshold Method (RTM) and we will motivate that the involved response functions can be assimilated to indistinguishability operators. It must be recall that the classical RTM is modelled using probabilistic Markov chains.

As it was mentioned in Subsection 1.1.4, one way to model the probability transition function of the Markov chain is by means of the so-called stimulus and response thresholds. Concretely, the stimulus expresses the need perceived by the agent to develop a task and the threshold determines the tendency of an agent to respond to an stimulus intensity and, therefore, to make the task. In [5], it was proposed a method, based on response functions, to model the aforesaid probability when the response threshold is fixed over time. In the aforesaid reference, the so-called semi-logarithmic probability response function $P(s,\theta)$ can be defined by

$$P(s,\theta) = \frac{s^n}{s^n + \theta^n}, \tag{9.1}$$

where s denotes for each agent the intensity of a stimulus to carry out a particular task and θ denotes the threshold for each agent and task. Notice that, according to [5], $n > 1$ (with $n \in \mathbb{N}$) determines the steepness of the threshold. Of course, the numerical value $P(s,\theta)$ can be interpreted as follows: On the one hand, values of the stimulus intensity much smaller than threshold (denoted by $s << \theta$), implies response values (probabilities of engaging task perfomance) close to 0. On the other hand, stimulus intensity much greater than the threshold (denoted by $s >> \theta$), means probability of engaging task performance close to 1.

Other authors have used response functions of type (9.1) in order to model probability of engaging tasks performance in multi-robot task allocation. We can find an instance in [48], where it was proposed a mathematical model to assign particular events to individual robots in such a way that each robot is limited to one task at time. Concretely, they assume that each robot senses the need to handle the closest task. In this direction, the stimulus produced by a task e for a robot r was taken as the inverse of the distance between the task and the robot, i.e., $s = \sigma(r,e) = \frac{1}{d(r,e)}$. Then, the probability response function is formulated as follows:

$$P(s,\theta) = \frac{\sigma(r,e)^n}{\sigma(r,e)^n + \theta^n}. \tag{9.2}$$

After exploring different thresholds, As was pointed out in [48], the best performance was achieved with the inverse of the expected distance between tasks D, i.e., $\theta = \frac{1}{D}$. In this case, $s << \theta$, or equivalently $d(r,e) >> D$, implies low response to engage the task and $s >> \theta$, or equivalently $d(r,e) << D$, implies high motivation to take on the task.

A straightforward computation yields that (9.2) can be transformed into the response function

$$P(s,D) = \frac{D^n}{D^n + d(r,e)^n}, \tag{9.3}$$

Notice that expression (9.3) maintains the essential properties of the response function (9.2). It must be stressed that this kind of response functions have been recently applied to possibilistic multi-robot task allocation problems (see [38] for more details, although the possibilistic multi-robot task allocation problem will be exposed in Sections 9.3 and 9.4).

Expression (9.3) has motivated this chapter, since as we will show in Section 9.2, $P(s,D)$ is an indistinguishability operator.

9.2 T-indistinguishability operators, distances and response functions

In this section we will apply Theorem 1.1.28 for some particular t-norms in order to construct two indistinguishability operators that allow to reproduce two celebrated response functions that appear in [5], in particular the semi-logarithmic one. In this direction we first provide a few examples of T-indistinguishability operators that separate points with the aim of illustrating such a technique, due to the lack of this sort of examples in the literature.

9.2.1 Examples

We begin applying the aforesaid construction to the Luckasievicz t-norm T_L which is continuous and Archimedean.

A $*_L$-indistinguishability operator

Let d be a distance on a nonempty set X and consider the Luckasiewicz t-norm. It is clear that the function $f_{*_L} : [0,1] \to [0,\infty]$, given by $f_{*_L}(x) = 1 - x$ for all $x \in [0,1]$, is an additive generator of $*_L$. Applying (1.1), as pointed out before, an easy computation shows that the pseudo-inverse $f_{*_L}^{(-1)}$ of the additive generator f_{*_L} is given by

$$f_{*_L}^{(-1)}(y) = \max\{0, 1 - y\}$$

for all $y \in [0,\infty]$. Then, using the construction of Theorem 1.1.28, i.e. $E_{*_L}(x,y) = f_{*_L}^{(-1)}(d(x,y))$ for all $x, y \in X$, we obtain a $*_L$-indistinguishability operator on X that separates points, which has the following expression:

$$E_{*_L}(x,y) = \begin{cases} 1 - d(x,y), & \text{if } 0 \leq d(x,y) < 1; \\ 0, & \text{elsewhere,} \end{cases}$$

for all $x, y \in X$.

A $*_P$-indistinguishability operator

Let d be a distance on a nonempty set X and consider the product t-norm. It is clear, as exposed before, that the function $f_{*_P} : [0,1] \to [0,\infty]$, given by $f_{*_P}(x) = -\log(x)$ for all $x \in [0,1]$, is an additive generator of $*_P$. The pseudo-inverse $f_{*_P}^{(-1)}$ of f_{*_P} is given by

$$f_{*_P}^{(-1)}(y) = \mathsf{e}^{-y}$$

for all $y \in [0,\infty]$. Then, using the construction of Theorem 1.1.28, i.e., $E_{*_P}(x,y) = f_{*_P}^{(-1)}(d(x,y))$ for all $x, y \in [0,1]$, we obtain the following expression:

$$E_{*_P}(x,y) = \mathsf{e}^{-d(x,y)}$$

for all $x, y \in X$. As in the above example, Theorem 1.1.28 ensures that E_{*_P} is a $*_P$-indistinguishability operator on X that separates points.

After presenting these two easy, but illustrative, preceding examples, which are constructed by means of the most commonly continuous Archimedean t-norms used in Fuzzy Logic, we will continue showing that the semi-logarithmic response function P, given by (9.3), is an indistinguishability operator which opens a wide range of potential applications from a mixed framework based on indistinguishability operators and distances to task allocation problems in multi-agent systems.

9.2.2 The semi-logarithmic response function as an indistinguishability operator

Consider the family of t-norms $(*^{\lambda}_{Dom})_{\lambda\in[0,\infty]}$ due to Dombi (see [49]). Recall, according to [49], that such a family of t-norms is given by:

$$*^{\lambda}_{Dom}(x,y)=\begin{cases} *_D(x,y) & \text{if } \lambda=0\\ *_M(x,y) & \text{if } \lambda=\infty\\ \frac{1}{1+\left(\left(\frac{1-x}{x}\right)^{\lambda}+\left(\frac{1-y}{y}\right)^{\lambda}\right)^{\frac{1}{\lambda}}} & \text{elsewhere,}\end{cases}.$$

According to [49], the t-norm $*^{\lambda}_{Dom}$ is continuous and Archimedean for each $\lambda\in]0,\infty[$. Moreover an additive generator of $*^{\lambda}_{Dom}$ is given by

$$f_{*^{\lambda}_{Dom}}(x)=\left(\frac{1-x}{x}\right)^{\lambda}$$

for all $x\in[0,1]$ and for each $\lambda\in]0,\infty[$. It is not hard to verify that the pseudo-inverse of this additive generator $f^{(-1)}_{*^{\lambda}_{Dom}}$ is given by

$$f^{(-1)}_{*^{\lambda}_{Dom}}(y)=\frac{1}{1+y^{\frac{1}{\lambda}}}$$

for all $y\in[0,\infty]$ and for each $\lambda\in]0,\infty[$.

Taking into account the preceding facts we are able to prove that P is in fact an indistinguishability operator constructed from a Dombi t-norm. To this end, assume that d is a distance on X and let $*^{\lambda}_{Dom}$ be a Dombi t-norm for an arbitrary $\lambda\in]0,\infty[$. By Theorem 1.1.28 we obtain a $*^{\lambda}_{Dom}$-

indistinguishability operator $E_{*^{\lambda}_{Dom}}$ that separates points by means of

$$E_{*^{\lambda}_{Dom}}(x,y) = f^{(-1)}_{*^{\lambda}_{Dom}}(d(x,y)),$$

for all $x, y \in X$. It follows that

$$E_{*^{\lambda}_{Dom}}(x,y) = \frac{1}{1 + d(x,y)^{\frac{1}{\lambda}}}$$

for all $x, y \in X$.

Next fix $n \in \mathbb{N}$ and take $\lambda = \frac{1}{n}$. Then we have that $f_{*^{\frac{1}{n}}_{Dom},\theta}$ is also an additive generator of $*^{\frac{1}{n}}_{Dom}$, where

$$f_{*^{\frac{1}{n}}_{Dom},\theta}(x) = \theta \cdot f_{*^{\frac{1}{n}}_{Dom}}(x) = \theta \cdot \left(\frac{1-x}{x}\right)^{\frac{1}{n}}$$

for all $x \in X$ and for each $\theta \in]0, \infty[$.

Now, we will apply Theorem 1.1.28 through $f_{*^{\frac{1}{n}}_{Dom},\theta}$. Since the pseudo-inverse $f^{(-1)}_{*^{\frac{1}{n}}_{Dom},\theta}$ of $f_{*^{\frac{1}{n}}_{Dom},\theta}$ is given by

$$f^{(-1)}_{*^{\frac{1}{n}}_{Dom},\theta}(y) = f^{(-1)}_{*^{\frac{1}{n}}_{Dom}}\left(\frac{y}{\theta}\right) = \frac{1}{1 + \left(\frac{y}{\theta}\right)^n} = \frac{\theta^n}{\theta^n + y^n}.$$

Therefore, $E_{*^{\frac{1}{n}}_{Dom}}$ is a $*^{\frac{1}{n}}_{Dom}$-indsitinguishability operator with

$$E_{*^{\frac{1}{n}}_{Dom}}(x,y) = f^{(-1)}_{*^{\frac{1}{n}}_{Dom},\theta}(d(x,y)) = \frac{\theta^n}{\theta^n + d(x,y)^n} \tag{9.4}$$

for all $x, y \in X$. Since P, given by (9.3), matches up with the preceding expression, given by (9.4), we conclude that the response function P is a $*^{\frac{1}{n}}_{Dom}$-indsitinguishability operator that separates points.

9.2.3 An exponential response function and indistinguishability operators

In [73] (see also [5]) an exponential response function was introduced in order to model honey bee division of labour by means of response thresholds. In particular, the exponential response function of an agent taken under consideration was the following:

$$P_{exp}(s,\theta) = 1 - \mathsf{e}^{-\frac{s}{\theta}}, \tag{9.5}$$

where s denotes the intensity of the stimulus for an agent to carry out a task and θ is the threshold. Note that, as in the case of response function (9.1), the probability of engaging task performance is small for $s << \theta$, and is close to 1 for $s >> \theta$.

Our final goal of this section is twofold. On the one hand, we provide an example of indistinguishability operator which exhibits a behavior similar to response function (9.5) and, thus, it could be used in task allocation problems. On the other hand, we are able to retrieve exactly, from the generated indistinguishability operator and through the technique stated in the statement of Theorem 1.1.27, the response function (9.5) when it depends on the distance between an agent and a task.

Next consider the family of t-norms $(*^{\lambda}_{AA})_{\lambda\in[0,\infty]}$ introduced by Aczél and Alsina. Recall, according to [49], that such a family is given as follows:

$$*_{AA}^{\lambda}(x,y)=\begin{cases} *_D(x,y) & \text{if } \lambda=0 \\ *_M(x,y) & \text{if } \lambda=\infty \\ \mathsf{e}^{-\left((-\log x)^{\lambda}+(-logy)^{\lambda}\right)^{\frac{1}{\lambda}}} & \text{elsewhere.} \end{cases},.$$

Following [49], for each $\lambda\in]0,\infty[$ we have that $*_{AA}^{\lambda}$ is continuous and Archimedean. Moreover, an additive generator of $*_{AA}^{\lambda}$ is given by

$$f_{*_{AA}^{\lambda}}(x)=(-\log(x))^{\lambda}$$

for all $x\in[0,1]$. A straightforward computation shows that the pseudo-inverse $f_{*_{AA}^{\lambda}}^{(-1)}$ of $f_{*_{AA}^{\lambda}}$ is given as follows:

$$f_{*_{AA}^{\lambda}}^{(-1)}(y)=e^{-\left(y^{\frac{1}{\lambda}}\right)},$$

for all $y\in[0,\infty]$ and for each $\lambda\in]0,\infty[$.

In the light of the exposed facts we are able to introduce the announced indistinguishability operator. To this end, assume that d is a distance on a nonempty set X and let $*_{AA}^{\lambda}$ be an Aczél-Alsina t-norm for $\lambda\in]0,\infty[$. Applying Theorem 1.1.28 we obtain the $*_{AA}^{\lambda}$-indistinguishability operator $E_{*_{AA}^{\lambda}}$ given by

$$E_{*_{AA}^{\lambda}}(x,y)=e^{-\left(d(x,y)^{\frac{1}{\lambda}}\right)}$$

for all $x,y\in X$. Notice that $E_{*_{AA}^{\lambda}}$ separates points.

Now, following similar arguments to those given in Subsection 9.2.2 we obtain, for each $n \in \mathbb{N}$ and $\theta \in]0,\infty[$, the following indistinguishability operator from the preceding one:

$$E_{*_{AA}^{\frac{1}{n}}}(x,y) = e^{-\frac{d(x,y)^n}{\theta^n}}. \tag{9.6}$$

Of course, one can observe that the last operator presented involves the same elements of function (9.3). Indeed, this operator depends simultaneously on the distance $d(x,y)$, on a threshold parameter θ and it contains the non-linearity constant n. Besides, the nature of this indistinguishability operator is an exponential function as response function (9.5). Nevertheless, a slightly difference between them must be stressed with the aim of interpreting the operator given in (9.6) as a response function. It is clear that in (9.6), s must be considered as the inverse of the distance win order to interpret the indistinguishability operator in (9.6) as a response function. Indeed, s must be understood as the inverse of the distance in order to preserve the essence of the impact of the stimulus s in the expression of a response function. Thus we have that the operator given by (9.6) acts as response function, since it satisfies the following: $d(x,y) >> \theta$ implies probability response close to 0 and $d(x,y) << \theta$ returns a probability response close to 1. It follows that the idea of "an agent is high motivated for performing closer tasks" is preserved.

The fact that the operator in (9.6) can be interpreted as a response function inspires that several families of indistinguishability operators can be proposed and tested with a large number of experiments in order to be compared with previous response functions used in the literature and, thus, to determine if indistinguishability operators are an appropriate mathematical tool for task allocation problems.

Finally, we show that, in addition, the indistinguishability operator given

(9.6) allows to retrieve exactly the response function (9.5). Hence, indistinguishability operators can still be used for generating response function even in those cases in which the stimulus s cannot be understood as the inverse of the distance. Indeed, note that $*_L \leq *^1_{AA} = *_P$ and, thus, Theorem 1.1.27 guarantees that the function given by $d_{E_{*^1_{AA}}} = f_{*_L}(E_{*^1_{AA}}) = 1 - E_{*^1_{AA}}$ is a distance on X. But such a function matches up with the exponential response function given by (9.5).

9.3 Possibilistic Markov chains: theory

As was proved in [38], possibilistic Markov chains provide a lot of advantages and outperform its probabilistic counterpart when they are applied to task allocation problems. This section summarizes the main theoretical concepts of possibilistic (fuzzy) Markov chains.

Following [3, 101], we can define a possibility Markov (memoryless) process as follows: let $S = \{s_1, \ldots, s_m\}$ $(m \in \mathbb{N})$ denote a finite set of states. If the system is in the state s_i at time τ $(\tau \in \mathbb{N})$, then the system will move to the state s_j with possibility p_{ij} at time $\tau + 1$. Let $x(\tau) = (x_1(\tau), ..., x_m(\tau))$ be a fuzzy state set, where $x_i(\tau)$ is defined as the possibility that the state s_i will occur at time τ for all $i = 1, \ldots, m$. Notice that $\bigvee_{i=1}^{m} x_i(\tau) \leq 1$ where $\vee$ stands for the maximum operator on $[0, 1]$. In the light of the preceding facts, the evolution of the fuzzy Markov chain in time is given by

$$x_i(\tau) = \bigvee_{j=1}^{m} p_{ji} \wedge x_j(\tau - 1),$$

where $\wedge$ stands for the minimum operator on $[0, 1]$. The preceding expression admits a matrix formulated as follows:

$$x(\tau) = x(\tau - 1) \circ M = x(0) \circ M^{\tau}, \tag{9.7}$$

where $M = \{p_{ij}\}_{i,j=1}^{m}$ is the fuzzy transition matrix, $\circ$ is the matrix product in the max-min algebra $([0,1], \vee, \wedge)$ and $x(\tau) = (x_1(\tau), \ldots, x_m(\tau))$ is the possibility distribution at time τ.

Taking into account the preceding matrix notation and following [3], a possibility distribution $x(\tau)$ of the system states at time τ is said to be stationary, or stable, whenever $x(\tau) = x(\tau) \circ M$. During the experiments, explained in Section 9.5, each state will be a task to execute and, therefore, m will stand for the number of tasks.

One of the main advantages of the possibilistic Markov chains with respect to their probabilistic counterpart is given by the fact that under certain conditions, provided in [17] by J. Duan, the system converges to a stationary possibility distribution in at most $m - 1$ steps.

9.4 Possibilsitic multi-robot task allocation

In this section we will see how to use possibilistic Markov chains for developing a RTM in order to allocate a set of robots to tasks using the aforementioned indistinguishability operators (see (9.1) and (9.6)). Although the implementation proposed in this section only considers robots, it can be easily extended to more generic multi-agent scenarios.

Formally, the problem to solve could be defined as follows: Let $l, m \in \mathbb{N}$. Denote by R the set of robots with $R = \{r_1, ..., r_l\}$ and by T the set of tasks to carry out with $T = \{t_1, ..., t_m\}$. Both, tasks and robots are placed in an environment.

According to the classical RTM (see Section 9.1), for each robot r_k and for each task t_j, a stimulus $s_{r_k,t_j} \in \mathbb{R}$ that represents how suitable t_j is for r_k is defined. Besides, a threshold value θ is assigned to each robot r_k. Thus, a robot r_k, allocated at task t_i, will select a task t_j to execute with a possibility $E_{*_{Dom}^{\frac{1}{n}}}(t_i, t_j)$ according to a fuzzy Markov decision chain. In the following, the stimulus of each robot r_k to transit from task t_i to task t_j only depends on the distance between the tasks which will be denoted by $d(t_i, t_j)$. So, the stimulus of each robot r_k to transit from t_i to task t_j is given as follows:

$$s_{t_i,t_j} = \begin{cases} \frac{1}{d(t_i,t_j)} & \text{if } d(t_i,t_j) \neq 0 \\ \infty & \text{if } d(r_k,t_j) = 0 \end{cases}. \tag{9.8}$$

This stimulus s_{r_k,t_j} allows us to obtain, by means of the indistinguishability operator (9.4), the following semi-logarithmic possibilistic transition function (response function),

$$p_{ij} = E_{*_{Dom}^{\frac{1}{n}}}(t_i, t_j) = \frac{\theta^n}{\theta^n + d(t_i,t_j)^n}. \tag{9.9}$$

If the same stimulus (distance) is applied to the indistinguishability operator (9.6), then the following exponential possibilistic transition function (response function), is obtained:

$$p_{ij} = E_{*_{AA}^{\frac{1}{n}}}(t_i, t_j) = e^{-\frac{d(t_i,t_j)^n}{\theta^n}}. \tag{9.10}$$

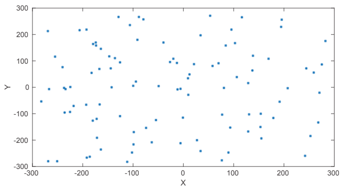
From now on, we will reference the response function given by (9.10) as Exponential Possibility Response Function (EPRF for short) and the response function given by (9.9) as Semi-Logarithmic Possibility Response Function (SLPRF for short). Furthermore, as the transition functions EPRF and SLPRF are indistinguishability operators, we will use both terms, transition possibility and indistinguishability operator equally.

As was proved in [39] (see also [38]), when the response function (or indistinguishability operator), (9.9) or (9.10), is used as a possibility transition, the obtained fuzzy Markov chain holds the Duan's convergence requirements (we refer the reader to [17] for a detailed treatment of the topic), that is column diagonally dominance and power dominance conditions. Therefore, we can ensure that the system converges to a stationary state in at most $m-1$ steps. It must be recall that, in general, the convergence of the probabilistic Markov chains is only guaranteed asymptotically.

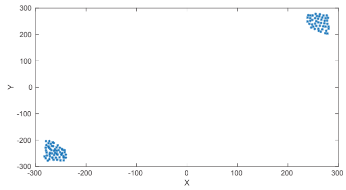
9.5 Experimental results: probabilistic/possibilistic Markov chains

In this section we will show the experiments carried out to compare the number of steps required to converge to a stationary state using probabilistic and possibilistic Markov chains induced from the indistinguishability operators given in (9.4) and (9.6).

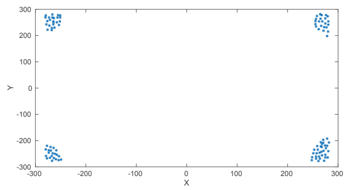
The robots must perform the task according to the stimulus defined in Section 9.4 under different configurations of the system: different position of the objects, parameters of the possibility response functions (θ and the power n) and number of tasks. All the experiments have been carried out using MATLAB with different synthetic environments. Figure 9.1 shows an example of the 3 types of environment used during the experiments depending on the position of the tasks: randomly placed (Figure 9.1(a)), tasks grouped into 2 clusters (Figure 9.1(b)) and grouped into 4 clusters (Figure 9.1(c)). This section extends the previous work given in [39] in order to considering clustered tasks.



(a) Tasks placed randomly.



(b) Tasks arranged into 2 clusters.



(c) Tasks arranged into 4 clusters.

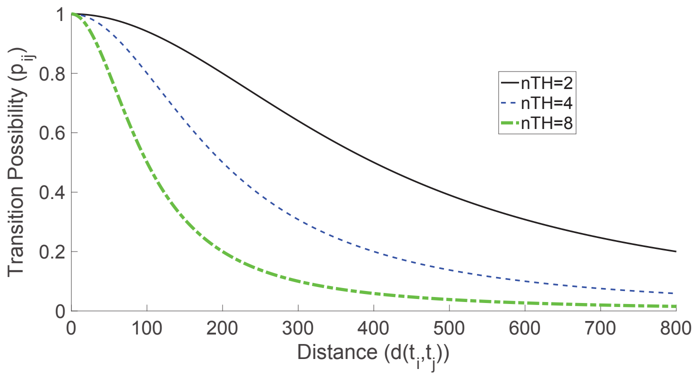
Figure 9.1: Environments with 100 tasks used for the experiments. Blue dots represent the position of the tasks or objects.

As pointed out in Section 9.1, the threshold value θ must depend on the position of the tasks. During the performed experiments the θ will depend on the maximum distance between tasks as follows:

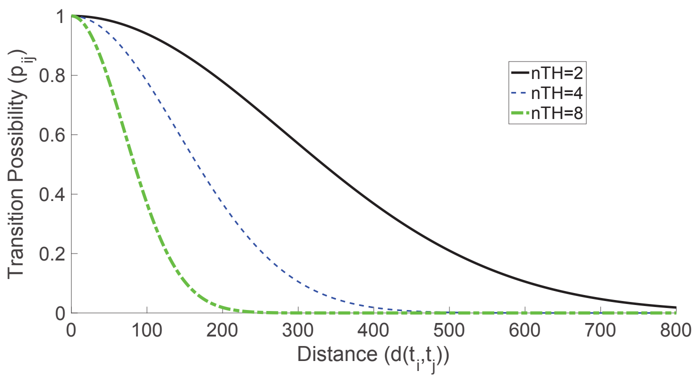
$$\theta = \frac{d_{max}}{nTH}, \tag{9.11}$$

where d_{max} is the maximum distance between two objects and nTH is a parameter of the system which allows us to generate thresholds. In our simulations d_{max} is constant and equals to 800.5 units. In order to see the impact of the parameter nTH on the transition possibility (p_{ij}) from a task t_i

to the task t_j, Figure 9.2 shows the values of p_{ij} using the indistinguishability operators SLPRF (Figure 9.2(a)) and EPRF (Figure 9.2(b)) with $nTH = 2, 4, 8$ and the power value $n = 2$. It should be noted that, if the distance is equal to 0 ($d(t_i, t_j) = 0$) then $t_j = t_i$ and $p_{ij} = p_{ii}$ is the possibility of remain in the current task.



(a) p_{ij} with the the indistinguishability operator SLPRF.



(b) p_{ij} with the the indistinguishability operator EPRF.

Figure 9.2: Transition possibility p_{ij} with $nTH = 2, 4, 8$ and power value $n = 2$.

Whichever possibility response function is used, (9.9) or (9.10), the possibilistic transition matrix for each robot, M, must be transformed into a probabilistic matrix in order to be able to compare the possibilistic and probabilistic Markov chains results. To make this conversion we use the transformation proposed in [96], where each element of M is normalized (divided by the sum of all the elements in its row) meeting the conditions of a probability distribution.

9.5.1 Experiments with randomly placed objects

This section focuses on experiments with tasks placed randomly, as can be seen in Figure 9.1(a). All the experiments have been performed with 500 different environments, with different number of tasks ($m = 50, 100$) and different values of the power n in the expression of the indistinguishability operators (9.9) and (9.10). The threshold θ values under consideration are obtained from (9.11) setting $nTH = 2, 4, 8$.

In [39], it was shown that SLPRF and EPRF in these randomly generated environments needed the same number of steps to converge. This number of steps does not depend on either θ or the power value n. These simulations also show that only a 50% of the 500 environments could converge to a stationary state when probabilistic Markov chains are used. In contrast, with fuzzy Markov chains all the experiments converge and they required less than 25 steps. This shows that fuzzy Markov chains with indistinguishability operators always outperform their probability counterpart. Figure 9.3 shows the mentioned percentage of experiments that, using probabilistic Markov chains, do converge with 100 randomly placed tasks.

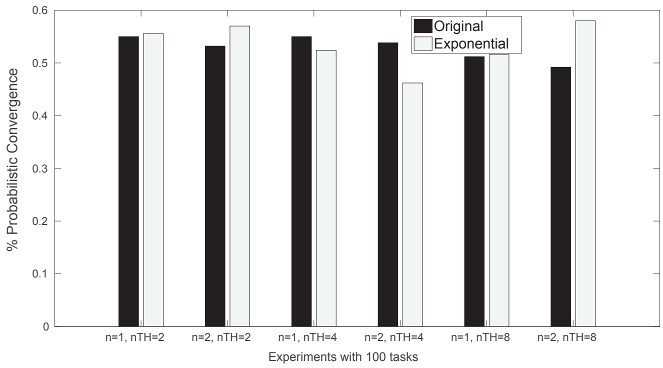


Figure 9.3: Percentage of experiments that converge with 100 tasks using probabilistic Markov process.

9.5.2 Clustered environments

This section shows the results obtained using environments with the tasks arranged in the groups or clusters, shown in Figures 9.1(b) and 9.1(c). As occurred in Subsection 9.5.1 the obtained results are very similar whichever indistinguishability operator is applied, SLPRF or EPRF. Therefore, even if the tasks are arranged into clusters, both indistinguishability operators present an equivalent behavior and they are not affected by its parameters.

Figure 9.4 shows the number of iterations required to converge with fuzzy Markov chains with 2 and 4 clusters of tasks and different number of tasks ($m = 20, 40, 60, 80, 100, 120$). As can be observed, the number of clusters have a great impact on the system. For all cases, the environment with 4 groups needs a lower number of iterations to converge compared to the environment with 2 clusters. From these results, we can see that, with fuzzy Markov chains, the number of steps to converge to stable state depends only on the placement of tasks and not on the parameters of possibility transition

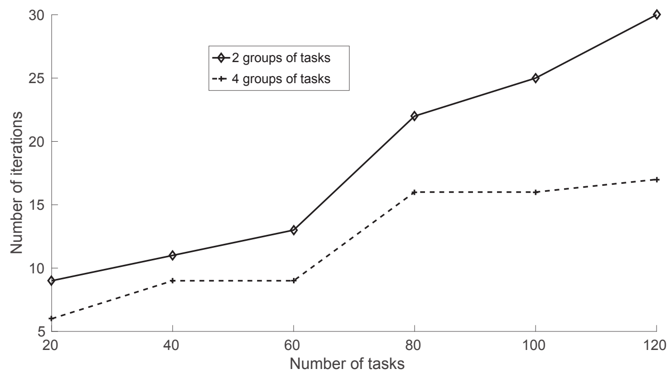
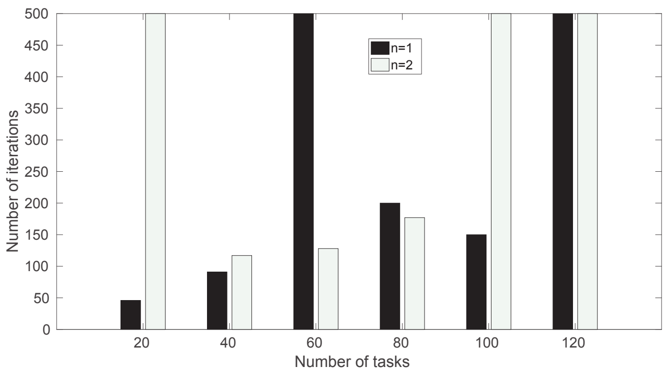


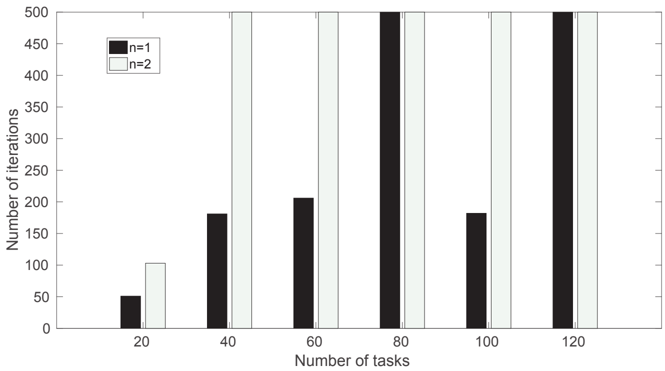
Figure 9.4: Number of iteration required to converge with fuzzy Markov chains for environments with 2 and 4 cluster (groups) of tasks.

function. Recall that the number of iterations required to converge is the same whichever indistinguishability operators is under consideration.

Figure 9.5 shows the number of steps required to converge with several number of tasks ($m = 20, 40, 60, 80, 100, 120$), a single environment with 2 clusters of tasks, $nTH = 2$ and probabilistic Markov chains. When the number of iterations is equal to 500 means that the chain does not converge. Figure 9.5(a) shows this number of steps with the indistinguishability operator SLPRF and Figure 9.5(b) with the indistinguishability operator EPRF, when the evolution of the process is modelled as a probbilistic Markov chain. These results show that the value n has a great impact on the results and that, in general, the exponential transition requires a greater number of steps to converge compared to its original counterpart. Furthermore, the indistinguishability operator has a great impact on the number of steps required to converge when probabilistic Markov chains are considered. It must be recalled that with possibilistic Markov chains both indistinguishability operators provide very similar results and it must be stressed that with possibilistic Markov chains, in general, the convergence is not guaranteed in a finite amount of steps.



(a) Results obtained from SLPRF indistinguishability operator.



(b) Results obtained from EPRF indistinguishability operator.

Figure 9.5: Number of iteration required to converge with probabilistic Markov chain with different values of n power ($n = 1, 2$), $nTH = 2$, several number of tasks and 2 clusters of tasks. 500 iterations means no convergence.

Chapter 10

On the use of fuzzy preorders in multi-robot task allocation problem

In Chapter 9 we have introduced the use of indistinguishability operators and possibility theory to model response functions in response threshold RTM for allocation problem. Concretely, we have proposed this kind of operators to represent response functions when the stimulus of the agent only depends on the distance between the agent and a determined task. In this chapter we extend the previous work in order to be able to model response functions when the stimulus under consideration depends on the distance between tasks and the utility of them. Thus, the resulting response functions that model transitions in the Markov chains must be asymmetric. In the light of this asymmetry, it seems natural to use fuzzy preorders in order to model the aforementioned response functions and, thus, the system's behaviour. The results of the simulations executed in MATLAB validate our approach and they show again how the possibilistic Markov chains outperform their probabilistic counterpart.

10.0.1 The asymmetric response function and fuzzy preorders

As pointed out before, in the modeling of the task allocation problem, the probability of executing the next task (referenced as response function) depends strongly on the current task (state). Therefore, from the classical viewpoint, the decision about the next task to be executed is a memoryless process that holds the conditions of a probabilistic Markov chain. Such classical probabilistic approach presents a huge number of inconveniences, such as problems with the selection of the probability response function when more than two tasks are under consideration, asymptotic converge, and so on (see [38]). In order to overcome the aforementioned problems, we have proposed a new possibilistic theoretical formalism for implementing the RTM algorithms based on the use of indistinguisability operators in Chapter 9, which allows us to introduce a formal method for generating possibilistic response functions.

Let us recall, with the aim of incorporating asymmetric response functions in the RTM algorithm, that one of the main advantages of the possibilistic Markov chains (incorporating indistinguisability operators as response functions) with respect to their probabilistic counterpart is given by the fact that under certain conditions, Duan's convergence requirements provided in [17], the system converges to a stationary distribution in a finite number of steps. Contrarily, the convergence of probabilistic Markov chains is, in general, only guaranteed asymptotically.

Next we construct a new response function in order to model those situations in which the stimulus of a robot depends on two factors, the distance between tasks and the utility associated to them. Thus we will show that this new possibilistic response function will be able to reflect that, in general, the robot will perceive as more attractive those tasks with better associated utility. First, we recall the following possibilistic response function, which has been provided in Chapter 9.

$$p(r_k, ij) = \frac{\theta^n}{\theta^n + d^n(r_i, t_j)}. \tag{10.1}$$

Notice that the response function (10.1) can be modeled by the next indistinguishability operator:

$$E(x, y) = \frac{\theta^n}{\theta^n + d^n(x, y)} \text{ for each } x, y \in \mathbb{R}^2 \tag{10.2}$$

where d denotes the Euclidean metric on $\mathbb{R}^2$ and x and y denotes the coordinates of the allocation t_i and t_j, respectively.

To achieve the target, let us fix, for the shake of simplicity, a few aspect of the mission under consideration. From now on, we will assume that the tasks are randomly placed in an environment and the robots are initially randomly placed too. Besides, each robot is always assigned to a task and only one robot per task can be assigned at the same time. Moreover, each task t_j has associated an utility, $U_j \in \mathbb{R}_+$ which indicates how useful is the task for that robot. Hence each task t_j can be identified with a triple (U_j, x_j, y_j), where the first coordinate represents the utility task and, in addition, the remainder two coordinates denotes the allocation of the task. Furthermore, each robot stimulus depends on both, the distance between the robot (current task allocation) and the task to perform and the improvement made in utility.

In order to construct the response function the tasks distance, will be measured via the Euclidean distance $d((x_i, y_i), (x_j, y_j))$, where (x_i, y_i) and (x_j, y_j) denotes the allocation coordinates of tasks t_i and t_j respectivaly. In addition, the utility improvement will be measure via the upper quasi-metric $q_u(U_i, U_j)$, where $q_u(U_i, U_j) = \max\{U_i - U_j, 0\}$.

Clearly $q_u(U_i, U_j) = 0$ provides that the task t_i is more attractive than the task t_j. However, positive values of $q_u(U_i, U_j)$ (which means $U_i \leq U_j$) measures the improvement in utility made when the task t_i is leaved by the robot and it starts to perform the task t_j.

Since the stimulus must depend on the Euclidean distance and the upper quasi-metric we merge both information in oder to obtain a global measure between tasks that incorporates the information coming from both different sources. Such an information fusion is provided by the function $\Phi : (\mathbb{R}_+)^2 \to \mathbb{R}_+$ given by $\Phi(x, y) = \alpha_u \cdot x + y$, where α_u will be a system's parameter that, on the one hand, makes that the utility value has the same dimension and scale as the distance and, on the other hand, indicates how important is the utility with respect to the distance (see Subsection 10.1.3 for a detailed discussion). Since Φ is a quasi-metric preserving function we have, by Theorem 1.1.12, that the non-negative real valued function Q_Φ, given by $Q_\Phi((U_i, x_i, y_i), (U_j, x_j, y_j)) = \alpha_u \cdot q_u(U_i, U_j) + d(x_i, x_j)$, is a quasi-metric.

Under this considerations, the decision process of a robot r_k will follow a possibilistic Markov chain in such a way that it will leave the task t_i (where is allocated) in order to perform the task t_j according to the transition $p(r_k, ij)$ given by

$$p(r_k, ij) = \frac{\theta^n}{\theta^n + Q^n_\Phi((U_i, x_i, y_i), (U_j, x_j, y_j))} \tag{10.3}$$

Notice that the expression (10.3) is obtained by replacing in the expression of (10.1) the Euclidean distance by the quasi-metric Q_Φ.

Next we prove that the following fuzzy set E^n_{Dom} is a fuzzy preorder on $\mathbb{R}^3$, where

$$E^n_{Q_\Phi,Dom}(x, y) = \frac{\theta^n}{\theta^n + Q^n_\Phi(x, y)} \text{ for each } x, y \in \mathbb{R}^3. \tag{10.4}$$

Clearly the possibility $p(r_k, ij)$, given by (10.3), matches up with the value $E^n_{Dom}(x, y)$, where x and y denotes the coordinates (utility and allocation) of tasks t_i and t_j respectively.

Let us recall that the family of Dombi t-norms $\{*^\lambda_{Dom}\}_\lambda$ is given as follows:

$$*^\lambda_{Dom}(a, b) = \begin{cases} 0, & \text{if } a = 0 \text{ or } b = 0 \\ \frac{1}{1+\left(\left(\frac{1-a}{a}\right)^\lambda + (\left(\frac{1-b}{b}\right)^\lambda\right)^{\frac{1}{\lambda}}}, & \text{elsewhere} \end{cases}. \tag{10.5}$$

Taking into account the exposed information we have the next result.

Proposition 10.0.1. *Let $n \in \mathbb{N}$ and let q be a quasi-metric on a non-empty set X. The fuzzy set $E^n_{q,Dom}$ on $X \times X$, defined by $E^n_{q,Dom}(x, y) = \frac{1}{1+(q(x,y))^n}$, is a $*^{\frac{1}{n}}_{Dom}$-preorder that separates points.*

Proof. Next we show that E^n_q satisfies $(E1), (E2)$ and $(E3)$ when the t-norm $*^{\frac{1}{n}}_{Dom}$ is considered.

(i) $E^n_{q,Dom}(x,x) = \frac{1}{1+(q(x,x))^n} = 1$, since q is a quasi-metric on X and so $q(x,x) = 0$. Moreover, $E^n_{q,Dom}$ also satisfies $(E1')$, since $q(x,y) = q(y,x) = 0 \Leftrightarrow x = y$ and, thus, $E^n_{q,Dom}(x,y) = E^n_{q,Dom}(y,x) = 1 \Leftrightarrow x = y$.

(iii) Let $x,y,z \in X$. We will show that

$$E^n_{q,Dom}(x,z) \geq *^{\frac{1}{n}}_{Dom}\left(E^n_{q,Dom}(x,y), E^n_{q,Dom}(y,z)\right).$$

First, observe that, for each $x,y \in X$, we have that

$$\frac{1-E^n_{q,Dom}(x,y)}{E^n_{q,Dom}(x,y)} = \frac{1-\frac{1}{1+(q(x,y))^n}}{\frac{1}{1+(q(x,y))^n}} = \frac{\frac{(q(x,y))^n}{1+(q(x,y))^n}}{\frac{1}{1+(q(x,y))^n}} = (q(x,y))^n.$$

It follows that

$*^{\frac{1}{n}}_{Dom}\left(E^n_{q,Dom}(x,y), E^n_{q,Dom}(y,z)\right) = \frac{1}{1+(q(x,y)+q(y,z))^n} \leq \frac{1}{1+q(x,z)^n} = E^n_{q,Dom}(x,z)$,

since q is a quasi-metric on X and, so, it satisfies $q(x,z) \leq q(x,y) + q(y,z)$.

∎

In the light of the preceding result we obtain immediately that $E^n_{Q_\Phi,Dom}$ is a $*^{\frac{1}{n}}_{Dom}$-preorder and, thus, that the transition value $p(r_k, ij)$, given by (10.3), can be understood as a fuzzy preorder. Clearly, the response function provided by (10.3) is asymmetric. Moreover it must be stressed that if we take $\alpha_u = 0$ in (10.3), then indistinguishability operator given by (10.2) is retrieved as a particular case of the fuzzy preorder given by (10.4). So this new framework allows to model the new situations and those explored in [38, 40]. Furthermore, observe that the obtained transition possibilities does not fulfill equality $\sum_{j=1}^{m} p(r_k, ij) = 1$ that is assumed for probability distributions. So, as claimed, the new transitions does not meet the axioms of the probability theory.

Finally it must be pointed out that if the asymmetric response function given by (10.3) is incorporated in the RTM algorithm which implement a possibilistic Markov chain in order to describe the evolution of the decision process, then the conditions that ensure the finite convergence of the chain, Duan's convergence requirements given in [17], are hold. Therefore, the possibilistic Markov chain whose transition possibilities are given by (10.3) converges to a stationary distribution in at most $m-1$ steps, where m is the number of tasks.

10.1 Experimental Results

In this section we will analyze the results of experiments executed to study the number of steps needed to converge to stationary possibilistic distribution with the fuzzy Markov chains induced from the transitions possibilities (10.3), or equivalently by the fuzzy preorder (10.4).

10.1.1 Experimental framework

The experiments have been carried out with different positions of the objects in the environment (placement of tasks). We assume that the power value n will always be equal to 2. All the experiments have been carried out using MATLAB with different synthetic environments following a uniform distribution to generate the position of the tasks. Figure 10.1 represents one of these environments, where each blue dot represents a task. Furthermore, all the environments have the same dimension (width=600 units and high=600 units), the threshold value θ_{rk} will always be equal to $\frac{d_{max}}{4}$, where d_{max} is the maximum distance between two tasks. In our case d_{max} is equal to 800.5 units of distance. Moreover, all the experiments have been performed with 500 different environments, all of them with 100 tasks ($m = 100$).

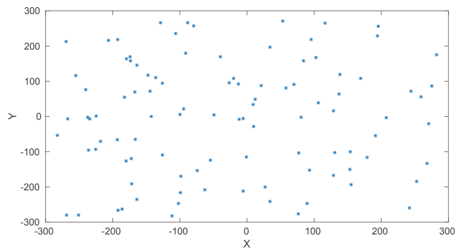


Figure 10.1: An example of environment.

Finally, each task has a randomly utility generated following an uniform distribution between 100 and 200. In order to aggregate the distance and the utility in the possibilistic response function (10.3) (that is to obtain the asymmetric distance Q_Φ), the parameter α_u has been split into two components as follows:

$$\alpha_u = \alpha_c \cdot \alpha_w.$$

In the preceding expression, α_c is a weighting factor that makes the utility component of Q_Φ has the same dimension and scale as the Euclidean distance component. Thus, α_c is the same for all the experiments and is equal to

$$\alpha_c = \frac{d_{max}}{U_{max}},$$

where U_{max} is the maximum value of the utility, i.e. 200, and d_{max} is the aforesaid maximum distance (800.5 units of distance). The second parameter, α_w, is a weighting factor that indicates how important is the improvement made in utility with respect to the distance during the allocation process.

The impact of this value on the system's performance will be evaluated in the following sections.

10.1.2 Results: probabilistic/possibilistic Markov chains

This section analyses the number of steps required, during the allocation process, to converge to a stationary distribution when possibilistic and probabilistic Markov chains are under consideration. In order to transform the possibilitic transition matrix M (see Section 9.3) obtained from the transitions possibilities given by (10.3), we again make use of the transformation proposed in [96], where each element of the matrix (M) is normalized (divided by the sum of all the elements in its row). Obviously, the resulting matrix meets all the conditions of a probability distribution.

Along the experiments we assume that if the convergence is not reached after 500 steps, the system does not converge.

Table 10.1 shows some results related to the number of steps that the possibilistic and probabilistic Markov chains required to converge to stationary distribution for several values of the parameter α_w. The first table's column shows the percentage of experiments that does not converge in the probabilistic case. The number of steps required to converge, when the probabilistic system converges, are shown in the second column. As can be seen, the percentage of experiments that do converge stays stable (it is similar in all considered cases) when the value of α_w is low ($\alpha_w \leq 10$) and dramatically decreases when α_w is high. Actually, if α_w is greater than 400 there are not any experiment that converge for the probabilistic approach. Likewise, when the system converges, the number of steps is clearly increases. Thus, we can conclude that, for the experiments carried out in this paper, the parameter

α_w (the importance of the improvement made in utility with respect to the distance between tasks) has a great impact on the probabilistic Markov chain behaviour. The third column of Table 10.1 shows the percentage of simulations that do converge with possibilistic Markov chain. As was pointed out before, the convergence of these chains is always guaranteed and, therefore, in all cases this percentage is 100%. The last column of Table 10.1 shows the number of steps that possibilistic Markov chains need to converge to stationary distribution. As can be seen, contrary to its probabilistic counterpart, possibilistic Markov chains require a lower number of steps to converge when the parameter α_w is higher. Moreover, in all cases, possibilistic Markov chains need much lower number of steps to converge than the probabilistic ones.

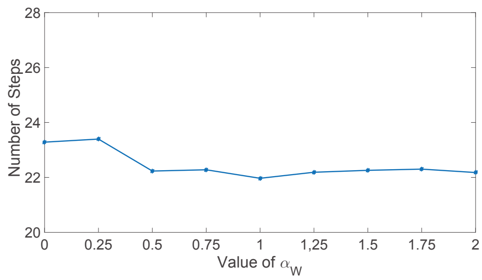
Table 10.1: Steps required to converge for probabilistic/possibilistic Markov chains.

α_w	% Conv. Prob.	Steps Prob.	% Conv. Fuzzy	Steps Fuzzy
0	32,6%	274,30	100%	23,28
2	41,6%	319,03	100%	22,17
10	30,04%	313,52	100%	25,34
40	13,8%	402,72	100%	27,56
100	0,2%	499	100%	28
400	0%	-	100%	11,17

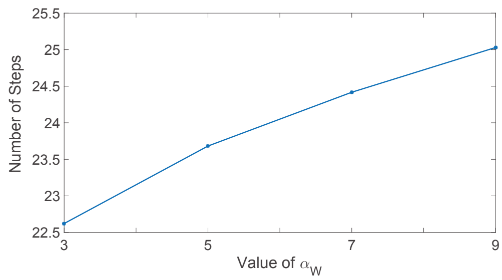
10.1.3 More on possibilistic Markov chains results

In this section we analyze in a more detailed way the impact of the parameter α_w on the system's behaviour for the possibilistic case.

Figures 10.2 and 10.3 show the mean number of steps required to converge



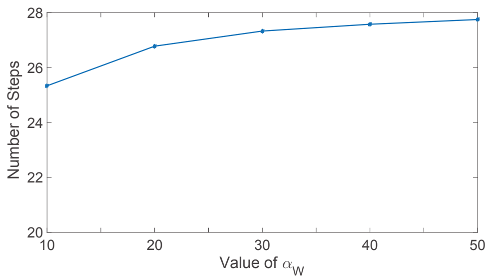
(a) $0 \leq \alpha_w \leq 2$.



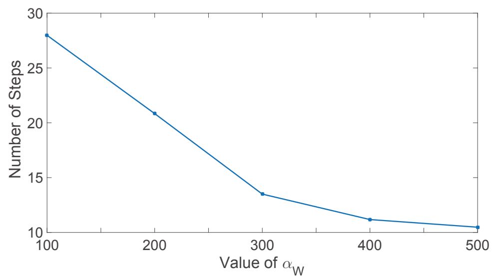
(b) $3 \leq \alpha_w \leq 9$.

Figure 10.2: Number of steps to converge to stationary possibilistic distribution for $0 \leq \alpha_w \leq 9$.

to stationary distribution for possibilistic Markov chains with respect to the parameter α_w. As this number of steps clearly depends on the value of the parameter α_w, the results have been split into 4 figures: Figure 10.2(a) shows the results when $0 \leq \alpha_w \leq 2$; in Figure 10.2(b) can be seen the number of steps if $3 \leq \alpha_w \leq 9$; Figure 10.3(a) shows the values when $10 \leq \alpha_w \leq 50$; and, finally, Figure 10.3(b) shows the number of steps required to converge when $100 \leq \alpha_w \leq 500$. In the shake of improving the visualization of the results, the minimum value of the Figures 10.2(a), 10.2(b) and 10.3(a) is 20 and the minimum value for the Figure 10.3(b) is 10 steps. As can be seen, if the value of the parameter α_w is lower than 50, the number of steps required to converge increases slightly with respect to the increments of the value



(a) $10 \leq \alpha_w \leq 50$.



(b) $100 \leq \alpha_w \leq 500$.

Figure 10.3: Steps to converge to stationary possibilistic distribution for $10 \leq \alpha_w \leq 500$.

α_w. When α_w is greater than 100, the number of steps start to dramatically decrease. It must be stressed that, as was said in Subsection 10.1.2, the probabilistic Markov chain show the opposite behaviour.

Chapter 11

Discussion of the obtained results and Conlusions

In this dissertation we have tackled some topics in the study of those functions that transform a finite family of generalized metrics, the same class of different, (including a single one) into a new generalized metric. Below we discuss the results obtained in each of the chapters and the conclusions extracted from them.

In Chapter 2 it has been proved a new characterization of those functions that transform a partial metric into another partial metric, the so-called partial metric preserving functions. Such a characterization shows that the aforementioned functions coincide with the functions which are strictly monotone and concave. It makes easier to check if a function is a partial metric preserving, since monotony or concavity are properties widely studied in the literature. In addition, it has been established the conditions that a partial metric preserving function must satisfy to preserve the topology, completeness and contractivity. Taking into account the lack of examples of partial metrics in the literature, the results obtained in this chapter allow to construct new examples of partial metrics from the known ones. It can extend the applicability of partial metrics to new problems arising in applied sciences.

In Chapter 3, we have continued the study of the problem of aggregation of distances. Concretely, we have introduced the notion of quasi-metric aggregation function and we have provided a characterization of such a notion in terms of (triangle) triplets. Besides, the relationship with metric aggregation functions has been also discussed and a few differences have been shown. Moreover, we have analyzed some properties fulfilled by the functions under consideration which play a central role in order to discard those functions that are useless as quasi-metric aggregation functions. Appropriate and illustrative examples have been given. Finally, two possible fields where the developed theory can be useful have been exposed.

Chapter 4 has been devoted to generalize Matthews' methods to construct a partial metric from a quasi-metric and vice-versa, by means of real functions. Besides, it has been characterized, in both cases, when the topology and the order are preserved. According to the method introduced by Matthews in [56], given a partial metric space (X, p), the function q_p is a quasi-metric on X where $q_p(x, y) = p(x, y) - p(x, x)$ for each $x, y \in X$. Clearly, the induced quasi-metric q_p is weighted with weight function w_{q_p} given by $w_{q_p}(x) = p(x, x)$. Of course, the preceding method has been generalized in Section 4.1 by means of *qmg*-functions and, in addition, a characterization of such functions has been given in the same section. However, such a general method does not produce in general weighted quasi-metrics. Indeed if we consider the mapping Ψ_2 introduced in Proposition 4.1.7 and the partial metric space $(\mathbb{R}_+, p_m)$, then it is not hard to verify that the induced quasi-metric space $(\mathbb{R}_+, q_{\Phi_2,p_m})$ is not weighted. Therefore, it remains, as an open question, to characterize those *qmg*-functions that generate always weighted quasi-metrics.

Partial metrics and quasi-metrics have been shown to be useful to develop quantitative mathematical models in denotational semantics and in asymptotic complexity analysis of algorithms, respectively. The aforesaid models are implemented independently and they are not related. A first natural attempt to develop a framework which remains valid, at the same time, for modeling in denotational semantics and in complexity analysis of algorithms suggests to construct a generalized metric by means of the aggregation of a partial metric and a quasi-metric. Inspired by the preceding fact, in Chapter 5 we have studied the way of merging, by means of a function, the aforementioned generalized metrics into a new one. We have shown that the induced generalized metric matches up with a partial quasi-metric. Thus, we have characterized those functions that allow to generate partial quasi-metrics from the combination of a partial metric and a quasi-metric. Moreover, we have explored the relationship between the problem under consideration and the problems of merging partial metrics and quasi-metrics.

In Chapter 6, we have provided a technique to construct fuzzy metric spaces, in the sense of George and Veeramani, from a classical metric space. The fuzzy metrics are obtained by means of preserving metric functions and they are defined for the Lukasievicz t-norm. We have proved some properties of the fuzzy metrics constructed, as they are strong and completable. As pointed out in Subsection 1.1.3, fuzzy metrics have shown to be useful in engineering problems. So, the provided technique allows to construct new examples of fuzzy metrics to increase the applicability of fuzzy metrics to more fields.

In Chapter 7 we have addressed the problem of establishing whether there is a relationship between pseudo-metrics and fuzzy (pseudo-)metrics, inspired by the duality relationship between indistinguishability operators and (pseudo-)metrics. Thus, we have yielded a method for generating fuzzy (pseudo-)metrics from (pseudo)-metrics and vice-versa. In such methods we have made use of the pseudo-inverse of the additive generator of a continuous Archimedean t-norm. From our new methods we have derived a new technique to generate non-strong fuzzy (pseudo-)metrics from (pseudo-)metrics. We have illustrated the aforementioned methods by means of appropriate examples. Finally, we have shown that the classical duality relationship between indistinguishability operators and (pseudo)-metrics can be retrieved as a particular case of our results when continuous Archimedean t-norms are under consideration.

In the literature there are mainly two tools that allow to measure the degree of similarity between objects. They are the so-called indistinguishability operators and fuzzy metrics. The former provide the degree up to which two objects are equivalent when there is a limitation on the accuracy of measurement between the objects being compared. The fuzzy metrics provide the degree up to which two objects are equivalent when the measurement is relative to a parameter. Motivated by the fact that none of these type of similarity measurements generalizes the other one, in Chapter 8 we have introduced a new notion of indistinguishability operator which unifies both notions, fuzzy metrics and indistinguishability operators, under a new one. Moreover, we have explored the metric behavior of this new kind of operators in such a way that the new results extend the classical results to the new framework and, in addition, allow to explore also the aforesaid duality relationship when fuzzy metrics are considered instead of indistinguishability operators. The fact that new notion of indistinguishability operator does not involve the continuity on the t-norm in their axiomatic presents an advantage with respect to the fuzzy metrics. The assumption of continuity could be limiting the range of applications of fuzzy metrics in those case where (classical) indistinguishability operators work well. As a future work remains open to study which properties of classical indistinguishability operators are also verified in the new framework. Besides, the utility of the new operators in applied problems must be explored.

In Chapter 9 we have shown that the two most famous response functions given in literature, are retrieved as a particular cases from appropriate indistinguishability operators. This fact opens a wide range of potential applications from a mixed framework based on indistinguishability operators and distances to task allocation problems in multi-agent systems. We have applied the aforementioned indistinguishability operators to allocate tasks to a group of robots according to a fuzzy Markov chain. We have shown that the results are very similar whichever indistinguishability operator is applied and, thus, that both present an equivalent behaviour. The simulations extend the results previously obtained in [39] to analyze environments where the tasks are arranged in groups or clusters. The results show that the number of iterations to converge with fuzzy Markov chains only depend on the placement of tasks in the environment and that they are not affected by the remainder of parameters of the system, whichever indistinguishability operator is applied. In contrast, when probabilistic Markov chains were used, this number of steps also depends on the indistinguishability operator. The theoretical and empirical obtained results in this chapter open a wide range of potential applications from a mixed framework based on indistinguishability operators and distances to task allocation problems in multi-agent systems when fuzzy Markov chains are under consideration.

As a future work, we plan to propose several families of indistinguishability operators to perform a large number of experiments in order to compare the new results with those provided by the task allocation methods that implement the two aforesaid response functions. An implementation of these methods on real robots is also under consideration.

For the first time, Chapter 10 applies the concept of fuzzy preorders to implement a multi-robot task allocation method. The task allocation algorithm implemented in this work is based on the Response Threshold method (RTM), where a robot selects the next task to execute according to a probabilistic Markov chain. Chapter 9 shows that the possibilistic Markov chains outperform their probabilistic counterparts when the possibility of transition only depends on the inverse of the Euclidean distance between tasks. This chapter also considers the utility of a task as a criteria to make the task allocation. Therefore, the utility value of the tasks has been included into the possibility transition function. The new resulting function is asymmetric and has been shown to be a fuzzy preorder with respect to a t-norm belonging to the Dombi family. The results of the simulations show that, in all cases, the possibilistic Markov chains converge in a lower number of steps than its probabilistic counterpart. Moreover, the weight of the utility (parameter α_w) has critical impact on the system's behavior. In the light of these first results new challenges and questions arise. For example, a deeper study of the impact of the utility on the system, different ways of aggregating the utility and distance component in the construction of the response function and the implantation of these methods on a physical multi-robot simulator are under consideration for a future work.

The results presented in this dissertation have been published (or submitted) either as papers in international journals or as papers in proceedings of peer-reviewed conferences. Below, we list them:

1. J.J. Miñana, *An Overview on Transformations on Generalized Metrics*, Proceedings of the Workshop on Applied Topological Structures 2017, WATS'17; Valencia; Spain; 11-12 July 2017, pages 95-102, Editorial Universitat Politècnica de València

2. J. Guerrero, J.J. Miñana, O. Valero, *A Comparative Analysis of Indistinguishability Operators Applied to Swarm Multi-Robot Task Allocation Problem*, Lecture Notes in Computer Science **10451** (2017), 21-28.

   14th International Conference on Cooperative Design, Visualization, and Engineering, CDVE 2017; Mallorca; Spain; 17-20 September 2017 (Listed in CORE Conference Ranking 2017: CORE C)

3. J.J. Miñana, O. Valero, *On Indistinguishability Operators, Fuzzy Metrics and Modular Metrics*, Axioms **6:4** (2017) 1-18 (Listed in Emerging Sources Citation Index - Clarivate Analytics)

4. J. Guerrero, J.J. Miñana, O. Valero, G. Oliver, *Indistinguishability Operators Applied to Task Allocation Problems in Multi-Agent Systems*, Applied Sciences-Basel **7:10** (2017) 1-16 (Listed in Journal Citation Reports 2017-Clarivate Analytics: Q3)

5. V. Gregori, J.J. Miñana, O. Valero, *A technique for fuzzifying metric spaces via metric preserving mappings*, Fuzzy Sets and Systems **330** (2018) 1-15 (Listed in Journal Citation Reports 2018-Clarivate Analytics: Q1)

6. J.J. Miñana, O. Valero, *A duality relationship between fuzzy metrics and metrics*, International Journal of General Systems **47:6** (2018) 593-612 (Listed in Journal Citation Reports 2018-Clarivate Analytics: Q2)

7. J.J. Miñana, O. Valero, *What is the aggregation of a partial metric*

and a quasi-metric?, Communications in Computer and Information Science Volume 853, 2018, pages 231-243. Springer

17th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems, IPMU 2018; Cadiz; Spain; 11-15 June 2018 (Listed in CORE Conference Ranking 2018: CORE C)

8. J. Guerrero, J.J. Miñana, O. Valero, *On the use of fuzzy preorders in multi-robot task allocation problem*, Communications in Computer and Information Science Volume 853, 2018, pages 195-206. Springer

   17th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems, IPMU 2018; Cadiz; Spain; 11-15 June 2018 (Listed in CORE Conference Ranking 2018: CORE C)

9. J.J. Miñana, O. Valero, *On partial metric preserving functions and their characterization*, submitted to FILOMAT

10. J.J. Miñana, O. Valero, *Characterizing quasi-metric aggregation functions*, submitted to International Journal of General Systems

11. J.J. Miñana, O. Valero, *On Matthews' relationship between quasi-metrics and partial metrics: an aggregation perspective*, submitted to Results in Mathematics

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