HORSESHOES NEAR HOMOCLINIC ORBITS FOR PIECEWISE LINEAR DIFFERENTIAL SYSTEMS IN $\mathbb{R}^3$

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For a three-parametric family of continuous piecewise linear differential systems introduced by Arneodo et al. [Arneodo *et al.*, 1981] and considering a situation which is reminiscent of the Hopf-Zero bifurcation, an analytical proof on the existence of a two–parametric family of homoclinic orbits is provided. These homoclinic orbits exist both under Shil'nikov ($0 < \delta < 1$) and non-Shil'nikov assumptions ($\delta \geq 1$). As it is well known for the case of differentiable systems, under Shil'nikov assumptions there exist infinitely many periodic orbits accumulating to the homoclinic loop. We also prove that this behaviour persists at $\delta = 1$. Moreover, for $\delta > 1$ and suficiently close to 1 we show that these periodic orbits persist but then they do not accumulate to the homoclinic orbit.

1.   Introduction

It is well known that three dimensional differential systems can exhibit chaotic dynamics. In some specific cases, homoclinic loops (invariant closed curves with exactly one singular point) act as organizing centers of such complex dynamical behavior. In fact, the celebrated paper of Shil'nikov [Shil'nikov, 1965] guarantees the existence of infinitely many unstable periodic orbits in every neighborhood of a homoclinic orbit associated to a saddle–focus equilibrium point under certain hypotheses on the eigenvalues of its linearization. More precisely, if λ and $-\lambda\delta \pm i\omega$ are the eigenvalues of the saddle–focus point, the Shil'nikov case requires that $0 < \delta < 1$. The ratio δ is strongly related to the saddle quantity σ quoted in Shil'nikov's and Belyakov's works, see [Kuznetsov, 2004], [Shil'nikov *et al.*, 2001], and references therein.

Later on, in [Shil'nikov, 1970] the same author shows that under the same hypotheses the dynamics associated to the existence of the homoclinic orbit is that of a Birkhoff–Morse system (conjugated to a shift with infinitely many symbols). The richness of the structure of periodic orbits around a homoclinic orbit of Shil'nikov type was analyzed by Belyakov [Belyakov, 1974, 1980, 1984], Glendinning and Sparrow [Glendinning & Sparrow, 1984] and Gaspard, Kapral and Nicolis [Gaspard *et al.*, 1984]. In any case, the application of Shil'nikov theorems needs firstly to show that such homoclinic orbit does exist, what in general is not a trivial task.

Several authors have paid attention to the problem of finding concrete systems having homoclinic orbits to a saddle–focus. For instance Arneodo, Coullet and Tresser introduce in [Arneodo *et al.*, 1981, Coullet *et al.*, 1979] the class of forced oscillators $x'' + \beta x' + x = \eta(x)$, where $\beta > 0$ is a dissipative term and $d\eta/dt = f_{a,\mu}(x)$ is the two–parametric family of continuous piecewise linear functions

$$f_{a,\mu}(x) = \begin{cases} 1 + ax & \text{if } x \le 0, \\ 1 - \mu x & \text{if } x > 0. \end{cases}$$

They show the existence of certain parameter values for which the above system has a Shil'nikov homoclinic orbit. Their proof is based on continuity arguments starting from numerical computations. Also Gribov and Krishchenko in [Gribov & Krishchenko, 2002] require numerical arguments to ensure the existence of homoclinic orbits in the Chua equations. In [Rodriguez, 1986], Rodriguez builds some systems with homoclinic orbits of saddle–focus type. In all these cases homoclinic orbits are under the Shil'nikov assumptions.

A natural question is whether the Shil'nikov condition for the saddle–focus $(0 < \delta < 1)$ is strictly necessary to get such a rich periodic behavior around the homoclinic orbit. This question has been analized in [Belyakov, 1974, 1984], always under the hypothesis of persistence of the homoclinic orbit in some curve of a two–parametric neighbourhood. Also, taking $\delta = 1$ and imposing some extra conditions Pumariño and Rodriguez [Pumariño & Rodriguez, 2001] give some results about the complexity that can be found in some classes of three dimensional vector fields.

In this paper we will revisit the piecewise linear differential system introduced by Arneodo et al. in a concrete region of its parametric space, giving sufficient conditions for the existence of homoclinic orbits both in Shil'nikov and non–Shil'nikov cases (that is, we will work on both sides of $\delta = 1$), and obtaining information about the involved dynamics in each case.

Note that the lack of differentiability of these systems makes that we cannot take advantage of the generic results included in the previous quoted papers, where not only the existence of homoclinic orbits is supposed as the starting point for the analysis but also smoothness up to certain high order is assumed. Thus, we are enforced to derive a specific analysis which does not intend to be generic; nevertheless, it can be very useful in studying more general piecewise linear systems.

An equivalent formulation for the three–parametric family of continuous piecewise linear differential systems introduced by Arneodo et al. [Arneodo *et al.*, 1981] is

$$\mathbf{x}' = \begin{cases} A_-\mathbf{x} + \mathbf{b} & \text{if } x \le 0, \\ A_+\mathbf{x} + \mathbf{b} & \text{if } x > 0, \end{cases} \tag{1}$$

where $\mathbf{x} = (x, y, z)^T$,

$$A_- = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & -1 & -\beta \end{pmatrix}, A_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\mu & -1 & -\beta \end{pmatrix}$$

and $\mathbf{b} = (0, 0, 1)^T$.

Assuming $a > 0$ and $\mu > 0$, the piecewise linear differential system (1) has exactly two singular points: $\mathbf{e}_+ = (1/\mu, 0, 0)^T$ which belongs to the half–space $\{x \geq 0\}$, and $\mathbf{e}_- = (-1/a, 0, 0)^T$ which belongs to the half–space $\{x \leq 0\}$.

Consider the change of parameters given by

$$\begin{aligned} \beta &= \lambda\,(2\delta - 1), \\ a &= \lambda\,(1 + 2\lambda^2\delta), \hspace{2cm} (2) \\ \mu &= (1 + 4R^2 + 4\lambda\delta R - 2\lambda R)\,(2R + 2\lambda\delta - \lambda), \end{aligned}$$

and defined in the parameter region $\delta > 0$, $R > 0$ and $\lambda > 0$, but taking λ sufficiently small. Note that the case $\beta \leq 0$ is included, and so β will belong to a neighbourhood of zero. The above change is chosen in order to make explicit the eigenvalues of the matrices A_- and A_+, namely λ and $-\lambda\delta \pm i\omega$ and $-L$ and $R \pm i\Omega$, respectively, where

$$\begin{aligned} \omega^2 &= 1 + \lambda^2\,(2 - \delta)\,\delta, \\ L &= 2R + \lambda\,(2\delta - 1) > 0, \hspace{1cm} (3) \\ \Omega^2 &= 1 + R\,(4\lambda\delta - 2\lambda + 3R), \end{aligned}$$

so that $\mu = (1 + 2LR)L$. Therefore, $\mathbf{e}_-$ and $\mathbf{e}_+$ are saddle–focus points.

The following two theorems summarize the main results in this paper and have an asymptotic character in the sense that they give valuable information only for λ sufficiently small. In Theorem 1.1 we provide for the class of differential systems (1) an analytical proof on the existence of a two–parametric family of homoclinic orbits. This family of homoclinic orbits exists under Shil'nikov and non–Shil'nikov assumptions; its existence could be also analytically proved for non-small values of λ by following a different, non-asymptotic approach, which is out of the scope of this paper.

Theorem 1.1. *In the (λ, δ, R)–parameter space there exists a two–dimensional continuous surface G such that if $(\lambda, \delta, R) \in G$, then the piecewise linear differential systems (1) has a homoclinic orbit $\Gamma_{\lambda,\delta}$ to the singular point $\mathbf{e}_-$. Moreover, if $\lambda > 0$ is small enough and $\delta \in (0, 1.3]$ the surface G is defined by the equation*

$$R\,(\lambda, \delta) = \frac{\sqrt{3}}{4e^{\theta^*}\sin\left(\sqrt{3}\theta^*\right)}\frac{1}{\lambda} + O\,(\lambda),$$

where θ^* is the unique zero in $\left(0, \pi/\sqrt{3}\right)$ of the function $f(\theta) = 2e^{3\theta}\cos\left(\sqrt{3}\theta - \pi/3\right) - 1$.

We remark that the first term in the equation of the surface G given in the above result does not depend on the eigenvalues real part ratio δ. This dependence will be explicit in higher order terms. On the other hand, the maximum allowed value of $\delta = 1.3$ is a consequence of the method used to prove the theorem and has not dynamic implications.

The singular point $\mathbf{e}_-$ goes to infinity as λ tends to zero, and consequently the associated homoclinic orbit $\Gamma_{\lambda,\delta}$ also does so. Thus, we are studying a family of homoclinic orbits which bifurcate from the infinity. It must be also noticed that for $\lambda = 0$ we have a sort of a piecewise linear version of the Hopf–Zero bifurcation so that one equilibrium goes to (or comes from) infinity with one zero plus one complex pair of pure imaginary eigenvalues. In this sense Theorem 1.1 represents partial information regarding the unfolding of such bifurcation point. Notice that this situation is more degenerate than the considered one in Belyakov [Belyakov, 1974].

In Theorem 1.2 we show the existence of infinitely many periodic orbits in a neighbourhood of the homoclinic orbit $\Gamma_{\lambda,\delta}$. The accumulation of these periodic orbits to the homoclinic orbit is proved under the Shil'nikov assumptions and in the boundary of these assumptions. The result is obtained from a carefully study of the Poincaré map defined on the plane $\{x = 0\}$ in a vicinity of one of the two intersection points of the homoclinic orbit with such plane, namely near its intersection point with the half-plane $\{x = 0, y < 0\}$. Again, it is remarked that, due to the lack of differentiability of piecewise linear systems, such kind of results cannot be derived from known generic results for smooth systems, and so a specific analysis is needed.

Theorem 1.2. *For $(\lambda, \delta, R) \in G$ and λ sufficiently small, we consider the piecewise linear differential system (1).*

If $\delta \leq 1$ then the Poincaré map defined in the intersection of every neighbourhood of the homoclinic orbit $\Gamma_{\lambda,\delta}$ with the half-plane $\{x = 0, y < 0\}$ has infinitely many shifts of two symbols as a subsystem. Consequently, there exist infinitely many periodic orbits accumulating to the homoclinic orbit.

Given a neighbourhood U of the homoclinic orbit $\Gamma_{\lambda,\delta}$, there exists a value $\varepsilon(\lambda) > 0$ such that if

$1 < \delta < 1 + \varepsilon(\lambda)$, *then the Poincaré map defined in* $U \cap \{x = 0, y < 0\}$ *has finitely many shifts of two symbols as a subsystem.*

It must be emphazised that Theorem 1.2 analyzes the richness of periodic behaviour near the homoclinic orbit $\Gamma_{\lambda,\delta}$ for parameter values on the biparametric surface G without leaving it, that is, without breaking the homoclinic orbit by perturbations.

The existence of the infinitely many periodic orbits is shown by proving the existence of Smale horseshoes for the Poincaré map defined in the half-plane $\{x = 0, y < 0\}$ near the intersection of the homoclinic orbit. At the end of the paper, we describe the mechanism which explains why the nearest horsehoes are destroyed for $\delta > 1$ and consequently how the associated shifts of two symbols disappear.

The rest of the paper is organized as follows. In Section 2, we give explicit expressions for the flow of system (1). In Section 3, we describe the geometry of the problem. In Section 4, we prove Theorem 1.1, and in Section 5 we prove Theorem 1.2.

2.   The flow in $\{x \geq 0\}$ and $\{x \leq 0\}$

For any point $\mathbf{p} = (x_\mathbf{p}, y_\mathbf{p}, z_\mathbf{p})^T$ we denote by $\gamma_\mathbf{p}$ the orbit through $\mathbf{p}$. If $\mathbf{p}$ is in the half-space $\{x \geq 0\}$, let $\mathbf{x}_\mathbf{p}^+ (s) = \left(x_\mathbf{p}^+ (s), y_\mathbf{p}^+ (s), z_\mathbf{p}^+ (s)\right)^T$ be the solution of system (1) with initial condition $\mathbf{x}_\mathbf{p}^+ (0) = \mathbf{p}$. While $x_\mathbf{p}^+ (s) \geq 0$ we have

$$
\begin{aligned}
x_\mathbf{p}^+ (s) =&\, C_\mathbf{p}^1 e^{Rs} \cos(\Omega s) + C_\mathbf{p}^2 e^{Rs} \sin(\Omega s) \\
&+ C_\mathbf{p}^3 e^{-Ls} + \frac{1}{\mu}, \\
y_\mathbf{p}^+ (s) =&\, \left(C_\mathbf{p}^1 R + C_\mathbf{p}^2 \Omega\right) e^{Rs} \cos(\Omega s) \\
&+ \left(C_\mathbf{p}^2 R - C_\mathbf{p}^1 \Omega\right) e^{Rs} \sin(\Omega s) \\
&- C_\mathbf{p}^3 L e^{-Ls}, \\
z_\mathbf{p}^+ (s) =&\, \left[C_\mathbf{p}^1 \left(R^2 - \Omega^2\right) + 2C_\mathbf{p}^2 \Omega R\right] e^{Rs} \cos(\Omega s) \\
&+ \left[C_\mathbf{p}^2 \left(R^2 - \Omega^2\right) - 2C_\mathbf{p}^1 \Omega R\right] e^{Rs} \sin(\Omega s) \\
&+ C_\mathbf{p}^3 L^2 e^{-Ls},
\end{aligned}
\tag{4}
$$

where $\mathbf{C}_\mathbf{p} = (C_\mathbf{p}^1, C_\mathbf{p}^2, C_\mathbf{p}^3)^T$ is obtained from

$$
\mathbf{C}_\mathbf{p} = \frac{1}{(L + R)^2 + \Omega^2} M^+ (\mathbf{p} - \mathbf{e}_+),
$$

and M^+ is the following matrix

$$
\begin{pmatrix}
L(2R + L) & 2R & -1 \\[2mm]
-\dfrac{L(R^2 + RL - \Omega^2)}{\Omega} & \dfrac{L^2 - R^2 + \Omega^2}{\Omega} & \dfrac{R + L}{\Omega} \\[2mm]
R^2 + \Omega^2 & -2R & 1
\end{pmatrix}.
\tag{5}
$$

If $x_\mathbf{p} = 0$ and $y_\mathbf{p} > 0$, then $\mathbf{e}_1^T \dot{\mathbf{p}} > 0$, where $\mathbf{e}_1 = (1, 0, 0)^T$ and $\dot{\mathbf{p}}$ is the value of the vector field associated to system (1) at the point $\mathbf{p}$. Therefore, the orbit $\gamma_\mathbf{p}$ through $\mathbf{p}$ crosses the plane $\{x = 0\}$ from the half-space $\{x < 0\}$ to the half-space $\{x > 0\}$.

Differential system (1) is linear in the half-space $\{x \geq 0\}$ and the stable and unstable manifolds of the saddle-focus $\mathbf{e}_+$ intersect the plane $\{x = 0\}$ (see Section 3). Then, if $\mathbf{p}$ does not belong to the stable manifold of $\mathbf{e}_+$, there exists $s_\mathbf{p}^+ > 0$ such that $x_\mathbf{p}^+ (s_\mathbf{p}^+) = 0$ and $x_\mathbf{p}^+ (s) > 0$ for $s \in (0, s_\mathbf{p}^+)$.

In short, if $x_\mathbf{p} = 0$ and $y_\mathbf{p} > 0$, then we define the Poincaré map Π_+ as $\Pi_+ (\mathbf{p}) = \left(0, y_\mathbf{p}^+ (s_\mathbf{p}^+), z_\mathbf{p}^+ (s_\mathbf{p}^+)\right)^T$.

For any point $\mathbf{p} = (x_\mathbf{p}, y_\mathbf{p}, z_\mathbf{p})^T$ in the half-space $\{x \leq 0\}$, let $\mathbf{x}_\mathbf{p}^- (s) = \left(x_\mathbf{p}^- (s), y_\mathbf{p}^- (s), z_\mathbf{p}^- (s)\right)^T$ be the solution of system (1) with initial condition $\mathbf{x}_\mathbf{p}^- (0) = \mathbf{p}$. While $x_\mathbf{p}^- (s) \leq 0$ we have

$$
\begin{aligned}
x_\mathbf{p}^- (s) =&\, D_\mathbf{p}^1 e^{-\lambda\delta s} \cos(\omega s) + D_\mathbf{p}^2 e^{-\lambda\delta s} \sin(\omega s) \\
&+ D_\mathbf{p}^3 e^{\lambda s} - \frac{1}{a}, \\
y_\mathbf{p}^- (s) =&\, \left(D_\mathbf{p}^2 \omega - D_\mathbf{p}^1 \lambda\delta\right) e^{-\lambda\delta s} \cos(\omega s) \\
&- \left(D_\mathbf{p}^1 \omega + D_\mathbf{p}^2 \lambda\delta\right) e^{-\lambda\delta s} \sin(\omega s) \\
&+ D_\mathbf{p}^3 \lambda e^{\lambda s}, \\
z_\mathbf{p}^- (s) =&\, D_\mathbf{p}^1 \left(\lambda^2\delta^2 - \omega^2\right) e^{-\lambda\delta s} \cos(\omega s) \\
&- 2D_\mathbf{p}^2 \lambda\delta\omega e^{-\lambda\delta s} \cos(\omega s) \\
&+ D_\mathbf{p}^2 \left(\lambda^2\delta^2 - \omega^2\right) e^{-\lambda\delta s} \sin(\omega s) \\
&+ 2D_\mathbf{p}^1 \lambda\delta\omega e^{-\lambda\delta s} \sin(\omega s) \\
&+ D_\mathbf{p}^3 \lambda^2 e^{\lambda s},
\end{aligned}
\tag{6}
$$

where $\mathbf{D}_\mathbf{p} = (D_\mathbf{p}^1, D_\mathbf{p}^2, D_\mathbf{p}^3)^T$ is obtained from

$$
\mathbf{D}_\mathbf{p} = \frac{1}{\lambda^2 (1 + \delta)^2 + \omega^2} M^- (\mathbf{p} - \mathbf{e}_-)
$$

and M^- is the following matrix

$$\begin{pmatrix} \lambda^2\left(1+2\delta\right) & -2\lambda\delta & -1 \\[2mm] \frac{\lambda\left(\lambda^2\delta+\lambda^2\delta^2-\omega^2\right)}{\omega} & \frac{\lambda^2-\lambda^2\delta^2+\omega^2}{\omega} & -\frac{\lambda(1+\delta)}{\omega} \\[2mm] \lambda^2\delta^2+\omega^2 & 2\lambda\delta & 1 \end{pmatrix}.$$

If $x_{\mathbf{p}} = 0$ and $y_{\mathbf{p}} < 0$, then $\mathbf{e}_1^T\dot{\mathbf{p}} < 0$. Therefore, the orbit $\gamma_{\mathbf{p}}$ through $\mathbf{p}$ crosses the plane $\{x = 0\}$ from the half–space $\{x > 0\}$ to the half–space $\{x < 0\}$. Differential system (1) is linear in the half–space $\{x \le 0\}$ and the stable and unstable manifolds of the saddle–focus $\mathbf{e}_-$ intersect the plane $\{x = 0\}$. Then, if $\mathbf{p}$ does not belong to the stable manifold of $\mathbf{e}_-$, there exists $s_{\mathbf{p}}^- > 0$ such that $x_{\mathbf{p}}^-\left(s_{\mathbf{p}}^-\right) = 0$ and $x_{\mathbf{p}}^-(s) > 0$ for $s \in \left(0, s_{\mathbf{p}}^-\right)$. Thus, if $x_{\mathbf{p}} = 0$ and $y_{\mathbf{p}} < 0$, then we define the Poincaré map Π_- as $\Pi_-\left(\mathbf{p}\right) = \left(0, y_{\mathbf{p}}^-\left(s_{\mathbf{p}}^-\right), z_{\mathbf{p}}^-\left(s_{\mathbf{p}}^-\right)\right)^T$.

If $\mathbf{p}$ is on the z–axis; i.e. $x_{\mathbf{p}} = 0$ and $y_{\mathbf{p}} = 0$, then $\mathbf{e}_1^T\dot{\mathbf{p}} = 0$ and $\mathbf{p}$ is called a contact point of the flow of system (1) with the plane $\{x = 0\}$, for more information about contact points see [Llibre & Teruel, 2004]. For such a point $\mathbf{p}$ we denote by $\mathbf{x}_{\mathbf{p}}(s)$ the solution of system (1) having $\mathbf{x}_{\mathbf{p}}(0) = \mathbf{p}$. Expanding in Taylor series $\mathbf{x}_{\mathbf{p}}(s)$ at $s = 0$ up to fourth order in s, passing the constant term from the right hand part to the left one, and taking its first coordinate, we obtain

$$\mathbf{e}_1^T\left(\mathbf{x}_{\mathbf{p}}(s) - \mathbf{p}\right) = z_{\mathbf{p}}\frac{s^2}{2} + \left(1 - \beta z_{\mathbf{p}}\right)\frac{s^3}{3!} + \mathbf{e}_1^T\mathbf{x}_{\mathbf{p}}^{(4)}(\xi)\frac{s^4}{4!}.$$

Hence, if $z_{\mathbf{p}} < 0$, then the orbit $\gamma_{\mathbf{p}}$ is locally contained in the half–space $\{x \le 0\}$; if $z_{\mathbf{p}} > 0$, then $\gamma_{\mathbf{p}}$ is locally contained in the half–space $\{x \ge 0\}$; and if $z_{\mathbf{p}} = 0$, then $\gamma_{\mathbf{p}}$ crosses the plane $\{x = 0\}$ from the half–space $\{x \le 0\}$ to the half–space $\{x \ge 0\}$.

3. Stable and unstable manifolds of equilibria

We note that the invariant manifolds of the singular points $\mathbf{e}_+$ and $\mathbf{e}_-$ are linear manifolds in a neighbourhood of the singular points $\mathbf{e}_+$ and $\mathbf{e}_-$. Thus, the unstable manifold $W^u\left(\mathbf{e}_-\right)$ of $\mathbf{e}_-$ contains the half–line

$$\mathcal{L}_- = \left\{x \le 0,\ y = \lambda x + \frac{1}{1+2\lambda^2\delta},\ z = \lambda y\right\}$$

generated by the eigenvector $\left(1, \lambda, \lambda^2\right)^T$ associated to the eigenvalue λ of A_-. This half–line intersects the plane $\{x = 0\}$ at the point

$$\mathbf{m}_- = \left(0, \frac{1}{1+2\lambda^2\delta}, \frac{\lambda}{1+2\lambda^2\delta}\right)^T, \qquad (7)$$

see Figure 1.

The stable manifold $W^s\left(\mathbf{e}_-\right)$ of $\mathbf{e}_-$ contains a piece of the half–plane

$$\mathcal{P}_- = \left\{\lambda\left(1+2\lambda^2\delta\right)x + 2\lambda^2\delta y + \lambda z = -1 : x \le 0\right\}$$

generated by the eigenvectors associated to the eigenvalues $-\lambda\delta \pm i\omega$ of A_-, see the shadowed region in Figure 1. The intersection of the planes $\mathcal{P}_-$ and $\{x = 0\}$ is the straight line

$$\mathcal{D}_- = \left\{(0, y, z) \in \mathbb{R}^3 :\ z = -2\lambda\delta y - \frac{1}{\lambda}\right\}. \qquad (8)$$

We emphasize that not every point in $\mathcal{D}_-$ belongs to $W^s\left(\mathbf{e}_-\right)$.
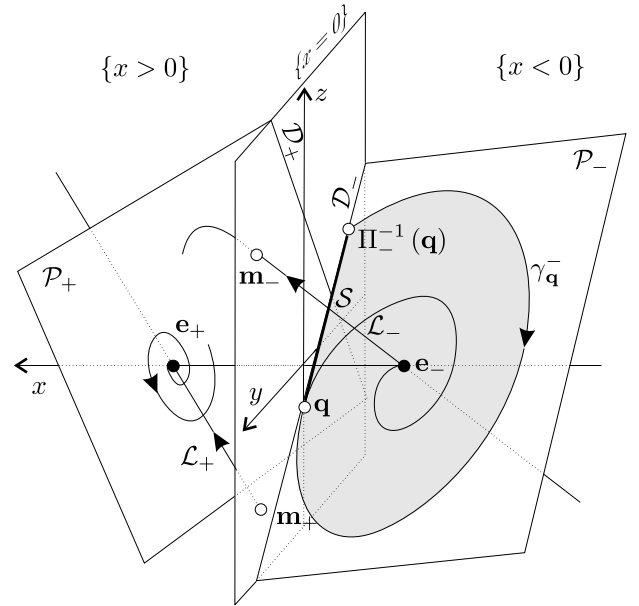

Fig. 1. Invariants manifolds of $\mathbf{e}_+$ and $\mathbf{e}_-$.

The stable manifold $W^s\left(\mathbf{e}_+\right)$ of $\mathbf{e}_+$ contains the half–line

$$\mathcal{L}_+ = \left\{x \ge 0,\ y = -Lx + \frac{1}{1+2LR},\ z = -Ly\right\}$$

generated by the eigenvector $\left(1, -L, L^2\right)^T$ associated to the eigenvalue $-L$ of A_+. This half–line

reaches the plane $\{x = 0\}$ at the point

$$\mathbf{m}_+ = \left(0, \frac{1}{1 + 2LR}, -\frac{L}{1 + 2LR}\right)^T. \qquad (9)$$

Finally, the unstable manifold $W^u(\mathbf{e}_+)$ of $\mathbf{e}_+$ contains a piece of the half–plane

$$\mathcal{P}_+ = \{(1 + 2LR)x - 2Ry + z = \frac{1}{L} : x \geq 0\}$$

generated by the eigenvectors associated to the eigenvalues $R \pm i\Omega$ of A_+. The intersection of the planes $\mathcal{P}_+$ and $\{x = 0\}$ is the straight line

$$\mathcal{D}_+ = \left\{(0, y, z) \in \mathbb{R}^3 : z = 2Ry + \frac{1}{L}\right\}. \qquad (10)$$

See in Figure 1 the points $\mathbf{m}_+$, $\mathbf{m}_-$, the straight lines $\mathcal{D}_+$, $\mathcal{D}_-$ and the half–planes $\mathcal{P}_+$ and $\mathcal{P}_-$.

4.   Existence of the homoclinic orbit $\Gamma_{\lambda,\delta}$

In this section we prove Theorem 1.1. We emphasize that we only look for homoclinic orbits with exactly two intersection points with the plane $\{x = 0\}$, namely $\mathbf{m}_-$ and $\Pi_+(\mathbf{m}_-)$. The way to look for this homoclinic orbit is to follow the orbit through $\mathbf{m}_-$ (which belongs to the unstable manifold of $\mathbf{e}_-$) and to move the parameters λ, δ and R so that this orbit intersects the stable manifold of $\mathbf{e}_-$.

Consider the point $\mathbf{q} = (0, 0, -1/\lambda)^T$ on the z–axis. Let $\mathcal{S}$ be the segment with endpoints $\mathbf{q}$ and $\Pi_-^{-1}(\mathbf{q})$, where Π_-^{-1} denotes the inverse of the Poincaré map Π_-, see Figure 1. It is clear that $\mathcal{S} \subset W^s(\mathbf{e}_-) \cap \mathcal{D}_-$. Note that the existence of the homoclinic orbit that we are looking for is characterized by the condition $\Pi_+(\mathbf{m}_-) \in \mathcal{S}$.

From (6), the solution of system (1) with initial condition $\mathbf{x}(0) = \mathbf{q}$ satisfies

$$x_{\mathbf{q}}^-(-s) = \frac{1}{a}\left[e^{s\delta\lambda}\cos(\omega s) - \frac{\lambda\delta}{\omega}e^{s\delta\lambda}\sin(\omega s) - 1\right],$$

$$y_{\mathbf{q}}^-(-s) = \frac{1}{\omega\lambda}e^{s\delta\lambda}\sin(\omega s), \qquad (11)$$

$$z_{\mathbf{q}}^-(-s) = -\frac{1}{\lambda}e^{s\delta\lambda}\cos(\omega s) - \frac{\delta}{\omega}e^{s\delta\lambda}\sin(\omega s).$$

The change of variables $\theta = \omega s$ and $\rho = \lambda\delta/\omega$ transforms equation $x_{\mathbf{q}}^-(-s) = 0$ in expression (11) into equation $\cos(\theta) - \rho\sin(\theta) = e^{-\rho\theta}$, which has a

unique zero θ_0 in $(\pi, 2\pi)$. Therefore, the flying time $s_{\Pi_-^{-1}(\mathbf{q})}^-$ to go from point $\Pi_-^{-1}(\mathbf{q})$ to point $\mathbf{q}$ satisfies $s_{\Pi_-^{-1}(\mathbf{q})}^- = \theta_0/\omega \in (\pi/\omega, 2\pi/\omega)$.

In the following result we give asymptotical expressions in λ of the flying time $s_{\Pi_-^{-1}(\mathbf{q})}^-$ and of the second coordinate of the point $\Pi_-^{-1}(\mathbf{q})$, which will be used later on.

Lemma 4.1. *If $\lambda > 0$ is sufficiently small and $\delta \in (0, 1.3]$, then the flying time $s_{\Pi_-^{-1}(\mathbf{q})}$ of the orbit through $\Pi_-^{-1}(\mathbf{q})$ to go from $\Pi_-^{-1}(\mathbf{q})$ to $\mathbf{q}$ satisfies*

$$s_{\Pi_-^{-1}(\mathbf{q})}^- = 2\pi - 2\sqrt{\pi\delta\lambda} + \frac{2}{3}(\pi\delta\lambda)^{3/2} + O(\lambda^2).$$

Moreover, the second coordinate of the point $\Pi_-^{-1}(\mathbf{q})$ is

$$y_{\mathbf{q}}^-\left(-s_{\Pi_-^{-1}(\mathbf{q})}^-\right) = -2\sqrt{\frac{\pi\delta}{\lambda}} - 2\pi\delta\sqrt{\pi\delta\lambda} + O(\lambda).$$

Proof: Thinking about the orbit through the point $\mathbf{q}$ in backward time, we obtain $\omega s_{\Pi_-^{-1}(\mathbf{q})}^- \in (\pi, 2\pi)$. Since $\omega = 1 - \delta(\delta - 2)\lambda^2/2 + O(\lambda^4)$, see (3), when λ is small enough it follows that $s_{\Pi_-^{-1}(\mathbf{q})}^- \in (\pi, 2\pi)$. From (11), expanding $x_{\mathbf{q}}^-(-s)$ at $s = 2\pi$ we have $x_{\mathbf{q}}^-(-s) = x_0 + x_1(s - 2\pi) + x_2(s - 2\pi)^2 + O\left((s - 2\pi)^3\right)$, where

$$x_0 = \frac{1}{1 + 2\lambda^2\delta}\left[-\frac{1}{\lambda} + e^{2\pi\lambda\delta}\cos(2\pi\omega)\right.$$
$$\left. - e^{2\pi\lambda\delta}\frac{\delta}{\omega}\sin(2\pi\omega)\right],$$

$$x_1 = -\frac{1}{\lambda\omega}e^{2\pi\lambda\delta}\sin(2\pi\omega),$$

$$x_2 = \frac{1}{2}e^{2\pi\lambda\delta}\left[\frac{1}{\lambda}\cos(2\pi\omega) + \frac{\delta}{\omega}\sin(2\pi\omega)\right].$$

Solving $x_0 + x_1(s - 2\pi) + x_2(s - 2\pi)^2 = 0$ for s, expanding the solution in power series of λ, and neglecting the terms of order 2 in λ, we obtain the following approximation to $s_{\Pi_-^{-1}(\mathbf{q})}^-$

$$\widetilde{s}_{\Pi_-^{-1}(\mathbf{q})}^- = 2\pi - 2\sqrt{\pi\delta\lambda} + \frac{2}{3}(\pi\delta\lambda)^{3/2}.$$

It can be shown that

$$x_{\mathbf{q}}^-\left(-\widetilde{s}_{\Pi_-^{-1}(\mathbf{q})}^- + 10\lambda^2\right)x_{\mathbf{q}}^-\left(-\widetilde{s}_{\Pi_-^{-1}(\mathbf{q})}^-\right)$$
$$= \frac{4}{9}(\pi\delta)^2(\pi\delta^2 + 6\pi\delta - 30)\lambda^3 + O(\lambda^4),$$

where the factor $\pi\delta^2 + 6\pi\delta - 30$ is negative in the interval $0 < \delta \leq 1.3$. Hence, we conclude that $\widetilde{s}^{-}_{\Pi_-^{-1}(\mathbf{q})} - 10\lambda^2 < s^{-}_{\Pi_-^{-1}(\mathbf{q})} < \widetilde{s}^{-}_{\Pi_-^{-1}(\mathbf{q})}$ when λ small enough, which proves the first part of the lemma.

Now, we compute the second coordinate of the point $\Pi_-^{-1}(\mathbf{q})$. From (11) it follows that

$$y^{-}_{\mathbf{q}}\left(-\widetilde{s}^{-}_{\Pi_-^{-1}(\mathbf{q})}\right) = -2\sqrt{\frac{\pi\delta}{\lambda}} - 2\pi\delta\sqrt{\pi\delta\lambda} + O\left(\lambda\right).$$

Since $dy^{-}_{\mathbf{q}}/ds\big|_{-s} = z^{-}_{\mathbf{q}}(-s) < 0$, for every s in the interval $\left(-\widetilde{s}^{-}_{\Pi_-^{-1}(\mathbf{q})}, -\widetilde{s}^{-}_{\Pi_-^{-1}(\mathbf{q})} + 10\lambda^2\right)$ we obtain that

$$\left|\frac{dy^{-}_{\mathbf{q}}}{ds}\right| < \left|z^{-}_{\mathbf{q}}\left(-\widetilde{s}^{-}_{\Pi_-^{-1}(\mathbf{q})} + 10\lambda^2\right)\right| = O\left(\lambda^{-1}\right).$$

Therefore and by the Mean Value Theorem, the error in the second component of $\Pi_-^{-1}(\mathbf{q})$ is $O(\lambda)$; that is, $y^{-}_{\mathbf{q}}\left(-s^{-}_{\Pi_-^{-1}(\mathbf{q})}\right) = y^{-}_{\mathbf{q}}\left(-\widetilde{s}^{-}_{\Pi_-^{-1}(\mathbf{q})}\right) + O(\lambda)$, and the lemma follows. $\qquad\square$

We remark that the hypothesis $\delta \leq 1.3$ in Lemma 4.1 is only required to assure the error order in the approximation of $s^{-}_{\Pi_-^{-1}(\mathbf{q})}$.

Denote by $\mathcal{P}^{*}_+$ the parallel plane to $\mathcal{P}_+$ through the point $\mathbf{m}_-$; i.e.

$$\mathcal{P}^{*}_+ = \left\{(1 + 2LR)x - 2Ry + z = \frac{\lambda - 2R}{1 + 2\lambda^2\delta}\right\}.$$

Let $\mathcal{D}^{*}_+$ be the intersection of $\mathcal{P}^{*}_+$ with the plane $\{x = 0\}$ and let $\mathcal{B}$ be the region in the half–plane $\{x = 0, \ y < 0\}$ limited by the straight lines $\mathcal{D}_+$ and $\mathcal{D}^{*}_+$, see the shadowed region Figure 2. Using the projection of the flow of the linear system in the half–space $\{x > 0\}$ onto the two invariant manifolds of the saddle–focus $\mathbf{e}_+$, we get that the orbit through $\mathbf{m}_-$ in $\{x > 0\}$ remains between the planes $\mathcal{P}_+$ and $\mathcal{P}^{*}_+$, so that $\Pi_+(\mathbf{m}_-) \in \mathcal{B}$. Since $\mathcal{D}_+$ has positive slope and $\mathcal{D}_-$ passes through the point $\mathbf{q}$ with negative slope, the straight line $\mathcal{D}_-$ splits $\mathcal{B}$ into the two regions $\mathcal{B}_1$ and $\mathcal{B}_2$, being $\mathcal{B}_1$ the bounded one, see Figure 2.

Lemma 4.2. *If λ is sufficiently small, $\delta \in (0, 1.3]$ and*

$$R \geq \frac{1}{4\sqrt{\pi\delta\lambda}} + \frac{1 - 2\delta}{2}\lambda,$$

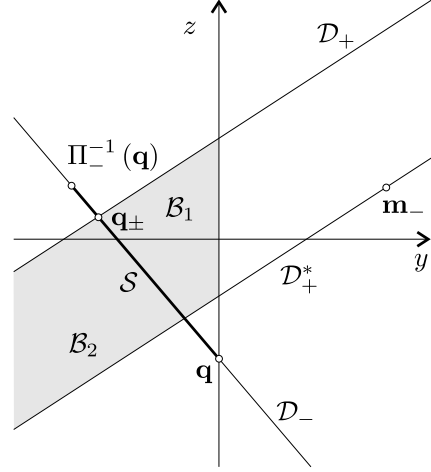*then $\mathcal{B} \cap \mathcal{D}_- \subset \mathcal{S}$.*

Fig. 2. Region $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ on the plane $\{x = 0\}$.

Proof: Denote by $\mathbf{q}_{\pm}$ the intersection point of the straight lines $\mathcal{D}_+$ and $\mathcal{D}_-$, see Figure 2; that is

$$\mathbf{q}_{\pm} = \left(0, \frac{1}{L\lambda}, \frac{\lambda - 2R}{L\lambda}\right)^T. \qquad (12)$$

If $||\mathbf{q} - \mathbf{q}_{\pm}|| \leq ||\mathbf{q} - \Pi_-^{-1}(\mathbf{q})||$, then $\mathcal{B} \cap \mathcal{D}_- \subset \mathcal{S}$ and the lemma holds. Now we shall prove this inequality.

Since the point $\Pi_-^{-1}(\mathbf{q})$ is on the straight line $\mathcal{D}_-$ its coordinates are $(0, y^*, -2\lambda\delta y^* - 1/\lambda)$ for an adequate y^*. Then, $||\mathbf{q} - \Pi_-^{-1}(\mathbf{q})|| = |y^*|\sqrt{1 + 4\lambda^2\delta^2}$. By Lemma 4.1, if λ is sufficiently small we have that $|y^*| > 2\sqrt{\pi\delta/\lambda}$. Therefore, we obtain that

$$||\mathbf{q} - \Pi_-^{-1}(\mathbf{q})|| > 2\sqrt{\frac{\pi\delta}{\lambda}}\sqrt{1 + 4\lambda^2\delta^2}.$$

On the other hand, from (12) we get that

$$||\mathbf{q} - \mathbf{q}_{\pm}|| = \frac{\sqrt{1 + 4\lambda^2\delta^2}}{\lambda(2R - \lambda + 2\delta\lambda)}.$$

The lemma follows by using the condition on R. $\square$

From now on, we consider the three–parametric family in λ, δ and $k \in \mathbb{R}$ of piecewise linear differential systems (1) with

$$R = \frac{1}{K^*\lambda} + k\lambda,$$

where

$$K^* = \frac{4}{\sqrt{3}}e^{\theta^*}\sin\left(\sqrt{3}\theta^*\right),$$

and θ^* is the unique zero of $\widetilde{f}(\theta) = e^{-2\theta} f(\theta) = e^{\theta} \cos\left(\sqrt{3}\theta\right) + \sqrt{3}e^{\theta} \sin\left(\sqrt{3}\theta\right) - e^{-2\theta}$ in $\left(0, \pi/\sqrt{3}\right)$. This choice for K^* will be clarified in the light of next two results, which look for accurate estimates of the flying time corresponding to certain distinguished orbits.

We are going to expand in power series of λ the coordinates of the point $\Pi_+(\mathbf{m}_-)$ in order to distinguish the values of the parameter k for which $\Pi_+(\mathbf{m}_-) \in \mathcal{B}_1$ from those for which $\Pi_+(\mathbf{m}_-) \in \mathcal{B}_2$. First of all we give a result to control the time $s^+_{\mathbf{m}_-}$ spent by the orbit $\gamma_{\mathbf{m}_-}$ to go from $\mathbf{m}_-$ to $\Pi_+(\mathbf{m}_-)$.

Lemma 4.3. *If* $\lambda > 0$ *is sufficiently small,* $\delta \in (0, 1.3]$ *and* $R = (K^*\lambda)^{-1} + k\lambda$, *then the flying time* $s^+_{\mathbf{m}_-}$ *satisfies*

$$s^+_{\mathbf{m}_-} < \frac{\pi}{\sqrt{3}} K^*\lambda + O\left(\lambda^3\right).$$

Proof: It is clear that $\mathbf{m}_+ = \mathbf{e}_+ + \sigma_1 \left(1, -L, L^2\right)^T$ with $\sigma_1 = -1/\mu$. Let $\mathbf{m}^*_+ = \mathbf{e}_+ + \sigma_0 \left(1, -L, L^2\right)^T$ be the intersection point of the straight line containing $\mathcal{L}_+$ and the plane $\mathcal{P}^*_+$, hence

$$\sigma_0 = \frac{1}{L^2 + 4LR + 1} \left(\frac{\lambda - 2R}{1 + 2\lambda^2\delta} - \frac{1}{L}\right).$$

Expanding σ_1 and σ_0 in power series of λ, it follows that $\sigma_1 = -K^{*3}\lambda^3/8 + O\left(\lambda^5\right) < 0$ and $\sigma_0 = -K^*\lambda/6 + O\left(\lambda^3\right) < 0$. Since $\lambda > 0$ is sufficiently small $|\sigma_1| < |\sigma_0|$, and consequently the point $\mathbf{m}^*_+$ is located in the half–space $\{x < 0\}$.

Now, we assume that the linear system $\mathbf{x}' = A_+\mathbf{x} + \mathbf{b}$ is defined in the whole space $\mathbb{R}^3$. Then, the time s_L to go from the point $\mathbf{m}^*_+$ to $\mathbf{m}_+$ following the stable manifold of $\mathbf{e}_+$ satisfies the equation $e^{-Ls_L}\sigma_0 = \sigma_1$. Therefore,

$$s_L = -\frac{1}{L} \ln \left|\frac{\sigma_1}{\sigma_0}\right| = O\left(\lambda \ln \lambda\right),$$

because, from (3), we know that $L = O\left(\lambda^{-1}\right)$.

In order to prove $s^+_{\mathbf{m}_-} < \pi/\Omega$ we assume the converse: $s^+_{\mathbf{m}_-} \geq \pi/\Omega$. Then, from the definition of $s^+_{\mathbf{m}_-}$ the point $\mathbf{x}^+_{\mathbf{m}_-}(\pi/\Omega)$ is in the half–space $\{x > 0\}$. Starting from $\mathbf{m}_-$, the orbit $\gamma_{\mathbf{m}_-}$ spirals around the stable manifold $\mathcal{L}_+$ of $\mathbf{e}_+$ in such a way that, after the time π/Ω, it has completed exactly a half–turn. In fact, using (4) one can check that the point $\mathbf{x}^+_{\mathbf{m}_-}(\pi/\Omega)$ belongs to the plane containing the straight line through $\mathcal{L}_+$ and the point $\mathbf{m}_-$.

Therefore, the segment with endpoints at $\mathbf{m}_-$ and $\mathbf{x}^+_{\mathbf{m}_-}(\pi/\Omega)$ intersects the straight line $\mathcal{L}_+$ at one point $\widetilde{\mathbf{m}}$. Since the two endpoints of the above segment are in the half–space $\{x \geq 0\}$, the point $\widetilde{\mathbf{m}}$ also belongs to this half–space. Next, we arrive to a contradiction with this last statement.

From (3) and the choice made for R, the asymptotic expansion of Ω in powers of λ is $\Omega = \sqrt{3}/(K^*\lambda) + O\left(\lambda\right)$, and then $\pi/\Omega = O\left(\lambda\right)$. For $\lambda > 0$ small enough, we have $\pi/\Omega < s_L$. Now, we note that $\mathbf{m}^*_+$ is the projection of the point $\mathbf{m}_-$ on the straight line through $\mathcal{L}_+$ following the parallel plane to the piece of the plane of the unstable manifold of $\mathbf{e}_+$. We denote by $\mathbf{m}^{**}_+$ the projection of the point $\mathbf{x}^+_{\mathbf{m}_-}(\pi/\Omega)$ on the straight line through $\mathcal{L}_+$ following the corresponding parallel plane to the piece of the plane of the unstable manifold of $\mathbf{e}_+$. Note that the arc of the orbit from $\mathbf{m}_-$ to $\mathbf{x}^+_{\mathbf{m}_-}(\pi/\Omega)$ projects on the straight line through $\mathcal{L}_+$ into the segment $\mathcal{S}_{\mathcal{L}_+}$ with endpoints $\mathbf{m}^*_+$ and $\mathbf{m}^{**}_+$. Since $\pi/\Omega < s_L$, the segment $\mathcal{S}_{\mathcal{L}_+}$ is contained in the half–plane $\{x < 0\}$. On the other hand the segment with endpoints $\mathbf{m}_-$ and $\mathbf{x}^+_{\mathbf{m}_-}(\pi/\Omega)$ also projects onto $\mathcal{S}_{\mathcal{L}_+}$. Consequently, the point $\widetilde{\mathbf{m}}$ must be contained into $\mathcal{S}_{\mathcal{L}_+}$, in contradiction with the fact that this point is contained into the half–space $\{x > 0\}$. We conclude that $s^+_{\mathbf{m}_-} < \pi/\Omega$.

Finally, from $\pi/\Omega \leq \pi K^*\lambda/\sqrt{3} + O\left(\lambda^3\right)$ we get that $s^+_{\mathbf{m}_-} \leq K^*\lambda\pi/\sqrt{3} + O\left(\lambda^3\right)$. $\square$

Proposition 4.4. *If* $\lambda > 0$ *is sufficiently small,* $\delta \in (0, 1.3]$ *and* $R = (K^*\lambda)^{-1} + k\lambda$, *the flying time* $s^+_{\mathbf{m}_-}$ *to go from* $\mathbf{m}_-$ *to* $\Pi_+(\mathbf{m}_-)$ *is*

$$s^+_{\mathbf{m}_-} = \theta^* K^*\lambda + O\left(\lambda^3\right).$$

Moreover, the coordinates of the point $\Pi_+(\mathbf{m}_-)$ *are*

$$x^+_{\mathbf{m}_-}\left(s^+_{\mathbf{m}_-}\right) = 0,$$
$$y^+_{\mathbf{m}_-}\left(s^+_{\mathbf{m}_-}\right) = \frac{2}{3}e^{\theta^*} \cos\left(\sqrt{3}\theta^*\right) + \frac{1}{3}e^{-2\theta^*} + O\left(\lambda^2\right),$$
$$z^+_{\mathbf{m}_-}\left(s^+_{\mathbf{m}_-}\right) = -\frac{1}{\lambda} + [mk + b(\delta)]\lambda + O\left(\lambda^2\right),$$

where $m < 0$ *and* $b(\delta)$ *is a linear function in* δ.

Proof: Since $R = (K^*\lambda)^{-1} + k\lambda$, from (3) it follows that $L = 2(K^*\lambda)^{-1} - (1 - 2\delta + 2k)\lambda$ and $\Omega = \sqrt{3}(K^*\lambda)^{-1} + \sqrt{3}/6(K^* - 2 + 4\delta + 6k)\lambda + O\left(\lambda s\right)$.

Therefore,

$$Rs = \frac{s}{K^*\lambda} + k\lambda s,$$

$$e^{Rs} = e^{\frac{s}{K^*\lambda}}(1 + k\lambda s) + O\left(\lambda^2 s^2\right),$$

$$Ls = \frac{2s}{K^*\lambda} - (1 - 2\delta + 2k)\lambda s,$$

$$e^{-Ls} = e^{-\frac{2s}{K^*\lambda}}\left[1 + (1 - 2\delta + 2k)\lambda s\right] + O\left(\lambda^2 s^2\right),$$

$$\Omega s = \frac{\sqrt{3}s}{K^*\lambda} + \frac{\sqrt{3}}{6}(K^* - 2 + 4\delta + 6k)\lambda s + O\left(\lambda^2 s^2\right).$$

From Lemma 4.3 it follows that $s_{\mathbf{m}_-}^+ = O\left(\lambda\right)$. Hence, for every $0 \leq s \leq s_{\mathbf{m}_-}^+$ we conclude that

$$\cos\left(\Omega s\right) = \cos\left(\frac{\sqrt{3}s}{K^*\lambda}\right)$$
$$- \frac{\sqrt{3}}{6}(K^* - 2 + 4\delta + 6k)\sin\left(\frac{\sqrt{3}s}{K^*\lambda}\right)\lambda s$$
$$+ O\left(\lambda^4\right),$$

$$\sin\left(\Omega s\right) = \sin\left(\frac{\sqrt{3}s}{K^*\lambda}\right)$$
$$+ \frac{\sqrt{3}}{6}(K^* - 2 + 4\delta + 6k)\cos\left(\frac{\sqrt{3}s}{K^*\lambda}\right)\lambda s$$
$$+ O\left(\lambda^4\right).$$

Substituting these expressions in (4), the first coordinate of the solution with initial condition at $\mathbf{m}_-$ is

$$x_{\mathbf{m}_-}^+(s) = \frac{K^*\lambda}{6}\left[e^{\frac{s}{K^*\lambda}}\cos\left(\frac{\sqrt{3}s}{K^*\lambda}\right)\right.$$
$$+ \sqrt{3}e^{\frac{s}{K^*\lambda}}\sin\left(\frac{\sqrt{3}s}{K^*\lambda}\right) - e^{-2\frac{s}{K^*\lambda}}\bigg]$$
$$+ O\left(\lambda^3\right).$$

Since θ^* is the unique zero of $\widetilde{f}(\theta) = e^{-2\theta}f(\theta) = e^\theta\cos\left(\sqrt{3}\theta\right) + \sqrt{3}e^\theta\sin\left(\sqrt{3}\theta\right) - e^{-2\theta}$ in $\left(0, \pi/\sqrt{3}\right)$, a quite good approximation for $s_{\mathbf{m}_-}^+$ is $\widetilde{s}_{\mathbf{m}_-}^+ = \theta^*K^*\lambda$. To assess the quality of this approximation, by the Mean Value Theorem we can write

$$s_{\mathbf{m}_-}^+ - \widetilde{s}_{\mathbf{m}_-}^+ = \frac{x_{\mathbf{m}_-}^+\left(s_{\mathbf{m}_-}^+\right) - x_{\mathbf{m}_-}^+\left(\widetilde{s}_{\mathbf{m}_-}^+\right)}{y_{\mathbf{m}_-}^+(\xi)},$$

with ξ in the interval with endpoints $s_{\mathbf{m}_-}^+$ and $\widetilde{s}_{\mathbf{m}_-}^+$. Since

$$y_{\mathbf{m}_-}^+\left(\widetilde{s}_{\mathbf{m}_-}^+\right) = \frac{2}{3}e^{\theta^*}\cos\left(\sqrt{3}\theta^*\right) + \frac{1}{3}e^{-2\theta^*} + O\left(\lambda^2\right),$$

the value of $y_{\mathbf{m}_-}^+(\xi)$ tends to a non–zero constant as λ tends to zero. Thus, the error order in λ of $s_{\mathbf{m}_-}^+ - \widetilde{s}_{\mathbf{m}_-}^+$, is equal to the error order of the difference $x_{\mathbf{m}_-}^+\left(s_{\mathbf{m}_-}^+\right) - x_{\mathbf{m}_-}^+\left(\widetilde{s}_{\mathbf{m}_-}^+\right)$; that is, $s_{\mathbf{m}_-}^+ = \widetilde{s}_{\mathbf{m}_-}^+ + O\left(\lambda^3\right)$.

Once controlled the time error, we study the error in the coordinates of the point $\Pi_+\left(\mathbf{m}_-\right)$. Since

$$z_{\mathbf{m}_-}^+\left(\widetilde{s}_{\mathbf{m}_-}^+\right) = -\frac{1}{\lambda} + [mk + b(\delta)]\lambda + O\left(\lambda^2\right),$$

with $m = -4\left(8\theta^*e^{-2\theta^*} + 3K^*\right)$ and $b(\delta) = (2\theta^* - 1)K^{*2} - 2(2\theta^*e^{-2\theta^*} + 3\theta^* + 1)K^* + 12e^{-2\theta^*} + (36\theta^*K^* - 8K^* + 24)\delta$, using again the Mean Value Theorem and

$$\left.\frac{dy_{\mathbf{m}_-}^+}{ds}\right|_{s=\widetilde{s}_{\mathbf{m}_-}^+} = z_{\mathbf{m}_-}^+\left(\widetilde{s}_{\mathbf{m}_-}^+\right) = O\left(\lambda^{-1}\right),$$

$$\left.\frac{dz_{\mathbf{m}_-}^+}{ds}\right|_{s=\widetilde{s}_{\mathbf{m}_-}^+} = -\mu x_{\mathbf{m}_-}^+\left(\widetilde{s}_{\mathbf{m}_-}^+\right) - y_{\mathbf{m}_-}^+\left(\widetilde{s}_{\mathbf{m}_-}^+\right)$$
$$- \beta z_{\mathbf{m}_-}^+\left(\widetilde{s}_{\mathbf{m}_-}^+\right) + 1 = O\left(\lambda^0\right),$$

we conclude that $y_{\mathbf{m}_-}^+\left(s_{\mathbf{m}_-}^+\right) - y_{\mathbf{m}_-}^+\left(\widetilde{s}_{\mathbf{m}_-}^+\right) = O\left(\lambda^2\right)$ and $z_{\mathbf{m}_-}^+\left(s_{\mathbf{m}_-}^+\right) - z_{\mathbf{m}_-}^+\left(\widetilde{s}_{\mathbf{m}_-}^+\right) = O\left(\lambda^3\right)$, which completes the proof. $\square$

From Proposition 4.4 it follows that

$$2\lambda\delta y_{\mathbf{m}_-}^+\left(s_{\mathbf{m}_-}^+\right) + z_{\mathbf{m}_-}^+\left(s_{\mathbf{m}_-}^+\right) = -\frac{1}{\lambda}$$
$$+ \lambda\left[mk + b(\delta) + \frac{2\delta}{3}\left(2e^{\theta^*}cos(\sqrt{3}\theta^*) + e^{-2\theta^*}\right)\right]$$
$$+ O\left(\lambda^2\right).$$

Thus, if

$$k^* = -\frac{b(\delta) - \frac{2\delta}{3}\left(2e^{\theta^*}cos(\sqrt{3}\theta^*) + e^{-2\theta^*}\right)}{m},$$

$k_1 < k^*$, $R_1 = (K^*\lambda)^{-1} + k_1\lambda$ and λ sufficiently small, then $\Pi_+\left(\mathbf{m}_-\right) \in \mathcal{B}_1$, see (8) for the expression of $\mathcal{D}_-$. Similarly, if $k_2 > k^*$ and $R_2 = (K^*\lambda)^{-1} + k_2\lambda$, then $\Pi_+\left(\mathbf{m}_-\right) \in \mathcal{B}_2$. Hence, by the Continuity Theorem of the solutions of a differential system with respect to initial conditions and parameters, we conclude that if λ is small enough and $\delta \in (0, 1.3]$, then there exists a value of the parameter $R = R\left(\lambda, \delta\right)$ between R_1 and R_2 for which

system (1) has a homoclinic orbit $\Gamma_{\lambda,\delta}$ to the equilibrium point $\mathbf{e}_-$. Therefore,

$$R(\lambda,\delta) = \frac{\sqrt{3}}{4e^{\theta^*}\sin\left(\sqrt{3}\theta^*\right)}\frac{1}{\lambda} + O(\lambda),$$

which completes the proof of Theorem 1.1.

5.   Existence of horseshoes

In this section we prove Theorem 1.2. Thus, for the piecewise linear system (1) with parameters $(\lambda,\delta,R) \in G$ we shall see that the Poincaré map defined in a convenient neighbourhood of the homoclinic orbit $\Gamma_{\lambda,\delta}$ has infinitely many periodic points, and that the homoclinic orbit is an accumulation point of these periodic points, when $\delta \leq 1$.

For every $0 < h < 1/\lambda$ we consider the point $\mathbf{q}_h = (0,0,-1/\lambda+h)^T$ on the z–axis and the segment $T_h = \{\mathbf{q}_{h,t} = (1-t)\,\mathbf{q}_h + t\Pi_-^{-1}(\mathbf{q}_h) : t \in [0,1)\}$, see Figure 3. By the continuity of the flow, the image $\Pi_-(T_h)$ of the segment T_h is homeomorphic to $\mathbb{S}^1$. Let $\Sigma_{\Pi_-(T_h)}$ be the bounded region in the plane $\{x=0\}$ limited by $\Pi_-(T_h)$. Since the segment T_h tends to $\mathcal{S}$ as h tends to zero, the time $s_{\mathbf{q}_{h,t}}^-$ tends to infinity as h tends to zero, and hence, if h is small enough, the orbit through $\mathbf{q}_{h,t}$ spirals around the unstable manifold of $\mathbf{e}_-$ as many times as we want. From this, we conclude that the point $\mathbf{m}_-$ is contained in $\Sigma_{\Pi_-(T_h)}$. Also, the set $\Pi_+\Pi_-(T_h)$ is homeomorphic to $\mathbb{S}^1$ and the point $\Pi_+(\mathbf{m}_-)$ is contained in the bounded region limited by $\Pi_+\Pi_-(T_h)$.

We are looking for conditions on h in order to conclude that the segment T_h and its image $\Pi_+\Pi_-(T_h)$ intersect transversally, see Figure 3. In Lemmas 5.1 and 5.2 we derive expressions as power series in h for the coordinates of $\Pi_-(T_h)$ and $\Pi_+\Pi_-(T_h)$, respectively.

Lemma 5.1. *Consider a piecewise linear differential system (1) with parameters $\lambda > 0$ small enough, $\delta \in (0,1.3]$ and $R = R(\lambda,\delta)$. If h is sufficiently small, then the topological circle $\Pi_-(T_h)$ is contained in the annular region centered at the point $\mathbf{m}_-$ with radii*

$$\begin{aligned}\rho_1 &= h^\delta\left(\lambda^{\delta-1} - 4\sqrt{\pi\delta}\lambda^{\delta-\frac{1}{2}} + O\left(\lambda^\delta\right)\right),\\ \rho_2 &= h^\delta\left(\lambda^{\delta-1} + 4\sqrt{\pi\delta}\lambda^{\delta-\frac{1}{2}} + O\left(\lambda^\delta\right)\right).\end{aligned}$$
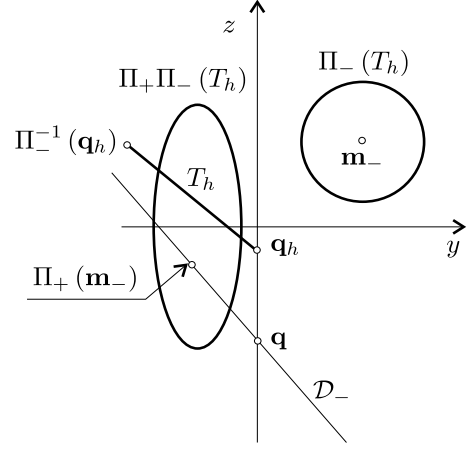

Fig. 3. Images of the segment T_h by the maps Π_- and $\Pi_+\Pi_-$

Proof: First of all we compute the point $\Pi_-^{-1}(\mathbf{q}_h)$ and the segment T_h for any $0 < h < 1/\lambda$. Since the point $\mathbf{q}_h$ tends to the point $\mathbf{q}$ as h tends to zero, the flying time $s_{\Pi_-^{-1}(\mathbf{q}_h)}^-$ tends to $s_0 = s_{\Pi_-^{-1}(\mathbf{q})}^-$ as h tends to zero. Therefore, we can write $s_{\Pi_-^{-1}(\mathbf{q}_h)}^- = s_0 + O(h)$. In Lemma 4.1 we have already obtained an expression for $s_0 = s_{\Pi_-^{-1}(\mathbf{q})}^- \in (\pi/\omega, 2\pi/\omega)$ in power series of λ. However, we present the following computations in terms of s_0 and use the implicit equation $x_{\mathbf{q}}^-(-s_0) = 0$, or equivalently

$$\omega e^{s_0\delta\lambda}\cos\left(\omega s_0\right) - \lambda\delta e^{s_0\delta\lambda}\sin\left(\omega s_0\right) - \omega = 0,$$

to simplify the obtained expressions.

Expanding $x_{\mathbf{q}_h}^-(-s)$ as power series of h at the approximate time $s = \tilde{s}_{\Pi_-^{-1}(\mathbf{q}_h)}^- = s_0 + s_1 h$, we get

$$x_{\mathbf{q}_h}^-\left(-\tilde{s}_{\Pi_-^{-1}(\mathbf{q}_h)}^-\right) = x_1 h + O(h^2),$$

where

$$\begin{aligned}x_1 &= -\frac{e^{s_0\delta\lambda}\sin\left(\omega s_0\right)}{\lambda\omega}s_1\\ &+ \frac{\left(e^{-s_0\lambda}-1\right)\omega + e^{s_0\delta\lambda}\sin\left(\omega s_0\right)\lambda}{\omega\left(\lambda^2 + 4\lambda^2\delta + 1\right)}.\end{aligned}$$

Hence, setting

$$s_1 = \lambda\frac{\omega\left(e^{-s_0\lambda}-1\right) + \lambda e^{s_0\delta\lambda}\sin\left(\omega s_0\right)}{e^{s_0\delta\lambda}\sin\left(\omega s_0\right)\left(\lambda^2 + 4\lambda^2\delta + 1\right)},$$

we obtain $x_1 = 0$. From this, it follows that

$$x_{\mathbf{q}_h}^- \left(-\tilde{s}_{\Pi_-^{-1}(\mathbf{q}_h)}^-\right) = O\left(h^2\right),$$

$$y_{\mathbf{q}_h}^- \left(-\tilde{s}_{\Pi_-^{-1}(\mathbf{q}_h)}^-\right) = y_0 + y_1 h + O\left(h^2\right),$$

$$z_{\mathbf{q}_h}^- \left(-\tilde{s}_{\Pi_-^{-1}(\mathbf{q}_h)}^-\right) = z_0 + z_1 h + O\left(h^2\right),$$

where

$$y_0 = \frac{e^{\lambda\delta s_0} \sin(\omega s_0)}{\lambda\omega},$$

$$y_1 = \frac{1}{1 + \lambda^2 + 4\lambda^2\delta} \left[\frac{\left(e^{-\lambda s_0} - 1\right)\omega}{e^{\lambda\delta s_0} \sin(\omega s_0)} + \lambda e^{-\lambda s_0}\right.$$

$$\left. + 2\lambda\delta(e^{-\lambda s_0} - 1) - \frac{\left(1 + 2\lambda^2\delta\right) e^{\lambda\delta s_0} \sin(\omega s_0)}{\omega}\right],$$

$$z_0 = -\frac{1}{\lambda} - 2\lambda\delta y_0,$$

$$z_1 = e^{-\lambda s_0} - 2\lambda\delta y_1.$$

We now study the error order in h in the approximate time $\tilde{s}_{\Pi_-^{-1}(\mathbf{q}_h)}^-$ and consequently in the coordinates of $\Pi_-^{-1}(\mathbf{q}_h)$. From the expansion of s_0 in power series of λ which appears in Lemma 4.1, we have $y_{\mathbf{q}_h}^- \left(-\tilde{s}_{\Pi_-^{-1}(\mathbf{q}_h)}^-\right) = O\left(\lambda^{-1/2}\right)$ and $z_{\mathbf{q}_h}^- \left(-\tilde{s}_{\Pi_-^{-1}(\mathbf{q}_h)}^-\right) = O\left(\lambda^{-1}\right)$. By the Mean Value Theorem, it follows that $s_{\Pi_-^{-1}(\mathbf{q}_h)}^- = \tilde{s}_{\Pi_-^{-1}(\mathbf{q}_h)}^- + O(h^2)$ and the error order in h in the coordinates of $\Pi_-^{-1}(\mathbf{q}_h)$ is $O\left(h^2\right)$. Then,

$$x_{\mathbf{q}_h}^- \left(-s_{\Pi_-^{-1}(\mathbf{q}_h)}^-\right) = 0,$$

$$y_{\mathbf{q}_h}^- \left(-s_{\Pi_-^{-1}(\mathbf{q}_h)}^-\right) = y_0 + y_1 h + O\left(h^2\right), \quad (13)$$

$$z_{\mathbf{q}_h}^- \left(-s_{\Pi_-^{-1}(\mathbf{q}_h)}^-\right) = z_0 + z_1 h + O\left(h^2\right).$$

Therefore, the points $\mathbf{q}_{h,t}$ of the segment T_h can be written as

$$\mathbf{q}_{h,t} = \begin{pmatrix} 0 \\ ty_0 + ty_1 h + O\left(h^2\right) \\ -\frac{1}{\lambda} - 2\lambda\delta ty_0 + vh + O\left(h^2\right) \end{pmatrix},$$

where $v = 1 - t + t\left(e^{-\lambda s_0} - 2\lambda\delta y_1\right)$ and $t \in [0, 1)$.

To obtain $\Pi_-\left(T_h\right)$, we first compute the flying time $s_{\mathbf{q}_{h,t}}^-$ of the orbit $\gamma_{\mathbf{q}_{h,t}}$ through the point $\mathbf{q}_{h,t}$,

for every t in $[0, 1)$. Since $s_{\mathbf{q}_{h,t}}^-$ is the first positive zero of the following equation

$$x_{\mathbf{q}_{h,t}}^-(s) = e^{-\lambda\delta s} \left[D_{\mathbf{q}_{h,t}}^1 \cos(\omega s) + D_{\mathbf{q}_{h,t}}^2 \sin(\omega s)\right]$$

$$+ D_{\mathbf{q}_{h,t}}^3 e^{\lambda s} - \frac{1}{a},$$

see (6), we consider the approximate value $\tilde{s}_{\mathbf{q}_{h,t}}^- = \tilde{s}_0(h, t) + \tilde{s}_1(h, t)h^\delta$ to $s_{\mathbf{q}_{h,t}}^-$, where the first term $\tilde{s}_0(h, t)$ is chosen to satisfy

$$D_{\mathbf{q}_{h,t}}^3 e^{\lambda\tilde{s}_0(h,t)} = \frac{1}{a}, \quad (14)$$

and the coefficient $\tilde{s}_1(h, t)$ in the second term will be selected appropriately to cancel other terms in the expression of $x_{\mathbf{q}_{h,t}}^-(\tilde{s}_{\mathbf{q}_{h,t}}^-)$. We note that $\tilde{s}_0(h, t)$ is the flying time spent by the orbit $\gamma_{\mathbf{q}_{h,t}}$ to go from $\mathbf{q}_{h,t}$ to the parallel plane to $\mathcal{P}_-$ passing through the point $\mathbf{m}_-$. Therefore $\tilde{s}_{\mathbf{q}_{h,t}}^-$ tends to $\tilde{s}_0(h, t)$ as h tends to zero. We conclude that if h tends to zero, then $\tilde{s}_1(h, t)h^\delta$ tends to zero. Moreover, we will prove that it is enough to take $\tilde{s}_1(h, t) = O\left(\lambda^{\delta-1}\right)$.

Substituting

$$D_{\mathbf{q}_{h,t}}^3 = \frac{1 - t + te^{-\lambda s_0}}{1 + \lambda^2 + 4\lambda^2\delta}h + tO\left(h^2\right)$$

in (14), we obtain

$$\tilde{s}_0(h, t) = \frac{1}{\lambda} \ln\left(\frac{1 + \lambda^2 + 4\lambda^2\delta}{h\lambda(1 + 2\lambda^2\delta)(1 - t - te^{-\lambda s_0})}\right)$$

$$+ tO(h). \quad (15)$$

The last term in the above two expressions means that both are exact for $t = 0$. To shorten expressions, from now on we write $\tilde{s}_0$ instead of $\tilde{s}_0(h, t)$.

Expanding in Taylor series $x_{\mathbf{q}_{h,t}}^-(s)$ at $s = \tilde{s}_0$, it follows that

$$x_{\mathbf{q}_{h,t}}^-(\tilde{s}_{\mathbf{q}_{h,t}}^-) = x_{\mathbf{q}_{h,t}}^-(\tilde{s}_0) + y_{\mathbf{q}_{h,t}}^-(\tilde{s}_0)\tilde{s}_1(h, t)h^\delta$$

$$+ O(\tilde{s}_1^2(h, t)h^{2\delta}),$$

where

$$x_{\mathbf{q}_{h,t}}^-(\tilde{s}_0) = h^\delta \left[D_{\mathbf{q}_{h,t}}^1 \cos(\omega\tilde{s}_0) + D_{\mathbf{q}_{h,t}}^2 \sin(\omega\tilde{s}_0)\right]$$

$$\times \left[\frac{\lambda(1 + 2\lambda^2\delta)(1 - t - te^{-\lambda s_0})}{1 + \lambda^2 + 4\lambda^2\delta}\right]^\delta$$

$$+ O(h^{1+\delta})$$

and $y_{\mathbf{q}_{h,t}}^-(\tilde{s}_0)$ tends to the second component of $\mathbf{m}_-$ as h tends to zero.

Hence, if we take

$$\tilde{s}_1(h,t) = -\frac{D^1_{\mathbf{q}_{h,t}}\cos(\omega\tilde{s}_0) + D^2_{\mathbf{q}_{h,t}}\sin(\omega\tilde{s}_0)}{y^-_{\mathbf{q}_{h,t}}(\tilde{s}_0)}$$

$$\times \left[\frac{\lambda(1+2\lambda^2\delta)(1-t-te^{-\lambda s_0})}{1+\lambda^2+4\lambda^2\delta}\right]^\delta$$

the solution at the approximate time $\tilde{s}^-_{\mathbf{q}_{h,t}}$ satisfies

$$x^-_{\mathbf{q}_{h,t}}(\tilde{s}^-_{\mathbf{q}_{h,t}}) = O\left(h^{1+\delta}\right),$$

$$y^-_{\mathbf{q}_{h,t}}(\tilde{s}^-_{\mathbf{q}_{h,t}}) = h^\delta\left[-\lambda^{\delta-1}\sin(\omega\tilde{s}_0)\right.$$

$$-\lambda^\delta\left(1+\frac{2t\sqrt{\pi\delta}}{\sqrt{\lambda}}\right)\cos(\omega\tilde{s}_0)$$

$$\left.+O\left(\lambda^{\delta+\frac{1}{2}}\right)\right] + \frac{\lambda}{a} + O\left(h^{1+\delta}\right),$$

$$z^-_{\mathbf{q}_{h,t}}(\tilde{s}^-_{\mathbf{q}_{h,t}}) = h^\delta\left[-\lambda^{\delta-1}\cos(\omega\tilde{s}_0)\right.$$

$$+\lambda^\delta\left(\delta-\frac{2t\sqrt{\pi\delta}}{\sqrt{\lambda}}\right)\sin(\omega\tilde{s}_0)$$

$$\left.+O\left(\lambda^{\delta+\frac{1}{2}}\right)\right] + \frac{\lambda^2}{a} + O\left(h^{1+\delta}\right).$$

By the Mean Value Theorem, the error order in h of $s^-_{\mathbf{q}_{h,t}} - \tilde{s}^-_{\mathbf{q}_{h,t}}$ is equal to the error order of the difference $x^-_{\mathbf{q}_{h,t}}(s^-_{\mathbf{q}_{h,t}}) - x^-_{\mathbf{q}_{h,t}}(\tilde{s}^-_{\mathbf{q}_{h,t}})$; that is, $s^-_{\mathbf{q}_{h,t}} = \tilde{s}^-_{\mathbf{q}_{h,t}} + O\left(h^{1+\delta}\right)$. Therefore, we conclude that

$$x^-_{\mathbf{q}_{h,t}}(s^-_{\mathbf{q}_{h,t}}) = 0,$$

$$y^-_{\mathbf{q}_{h,t}}(s^-_{\mathbf{q}_{h,t}}) - \frac{\lambda}{a} = h^\delta\left[-\lambda^{\delta-1}\sin(\omega\tilde{s}_0)\right.$$

$$-2t\sqrt{\pi\delta}\lambda^{\delta-\frac{1}{2}}\cos(\omega\tilde{s}_0) + O\left(\lambda^\delta\right)\bigg]$$

$$+ O\left(h^{1+\delta}\right),$$

$$z^-_{\mathbf{q}_{h,t}}(s^-_{\mathbf{q}_{h,t}}) - \frac{\lambda^2}{a} = h^\delta\left[-\lambda^{\delta-1}\cos(\omega\tilde{s}_0)\right.$$

$$-2t\sqrt{\pi\delta}\lambda^{\delta-\frac{1}{2}}\sin(\omega\tilde{s}_0) + O\left(\lambda^\delta\right)\bigg]$$

$$+ O\left(h^{1+\delta}\right).$$

From this the lemma follows straightforward. $\square$

For every $\rho > 0$ we consider the circle $\mathcal{C}_\rho = \left\{\mathbf{m}_- + \rho\left(0, \cos(\theta), \sin(\theta)\right)^T : \theta \in [0, 2\pi)\right\}$ centered at the point $\mathbf{m}_-$ and with radius ρ. In the next result, we compute the coordinates of the topological circle $\Pi_+(\mathcal{C}_\rho)$ in power series of ρ.

Lemma 5.2. *Denote by $\mathbf{m}_{\rho,\theta}$ the points in $\mathcal{C}_\rho$. Then, for the orbits through these points, we have*

$$x^+_{\mathbf{m}_{\rho,\theta}}\left(s^+_{\mathbf{m}_{\rho,\theta}}\right) = 0,$$

$$y^+_{\mathbf{m}_{\rho,\theta}}\left(s^+_{\mathbf{m}_{\rho,\theta}}\right) = y^+_{\mathbf{m}_-}\left(s^+_{\mathbf{m}_-}\right)$$

$$+ \rho\left[y^+_{\mathbf{m}_-}\left(s^+_{\mathbf{m}_-}\right)\cos(\theta) + O\left(\lambda^2\right)\right]$$

$$+ O\left(\rho^2\right),$$

$$z^+_{\mathbf{m}_{\rho,\theta}}\left(s^+_{\mathbf{m}_{\rho,\theta}}\right) = z^+_{\mathbf{m}_-}\left(s^+_{\mathbf{m}_-}\right)$$

$$+ \rho\left[z^+_{\mathbf{m}_-}\left(s^+_{\mathbf{m}_-}\right)\cos(\theta) + O\left(\lambda^0\right)\right]$$

$$+ O\left(\rho^2\right).$$

Proof: Note that

$$\begin{pmatrix} C^1_{\mathbf{m}_{\rho,\theta}} \\ C^2_{\mathbf{m}_{\rho,\theta}} \\ C^3_{\mathbf{m}_{\rho,\theta}} \end{pmatrix} = M^+\begin{pmatrix} -\dfrac{1}{\mu} \\ \dfrac{\lambda}{a} + \rho\cos(\theta) \\ \dfrac{\lambda^2}{a} + \rho\sin(\theta) \end{pmatrix}$$

$$= \begin{pmatrix} C^1_{\mathbf{m}_-} \\ C^2_{\mathbf{m}_-} \\ C^3_{\mathbf{m}_-} \end{pmatrix} + \rho\begin{pmatrix} C^1_\theta \\ C^2_\theta \\ C^3_\theta \end{pmatrix},$$

where M^+ is the matrix with parameters L, R and Ω which appears in (5). With the notation $g_\theta(s) = C^1_\theta e^{Rs}\cos(\Omega s) + C^2_\theta e^{Rs}\sin(\Omega s) + C^3_\theta e^{-Ls}$, it is easy to see that the flow on the circle $\mathcal{C}_\rho$ can be written as follows

$$x^+_{\mathbf{m}_{\rho,\theta}}(s) = x^+_{\mathbf{m}_-}(s) + \rho g_\theta(s),$$

$$y^+_{\mathbf{m}_{\rho,\theta}}(s) = y^+_{\mathbf{m}_-}(s) + \rho g'_\theta(s),$$

$$z^+_{\mathbf{m}_{\rho,\theta}}(s) = z^+_{\mathbf{m}_-}(s) + \rho g''_\theta(s).$$

For a suitable value $\hat{s}_\theta$, it is clear that $\tilde{s}^+_{\mathbf{m}_{\rho,\theta}} = s^+_{\mathbf{m}_-} + \hat{s}_\theta\rho$ approaches $s^+_{\mathbf{m}_{\rho,\theta}}$ as ρ tends to zero.

Moreover, since

$$x_{\mathbf{m}_{\rho,\theta}}^+\left(\tilde{s}_{\mathbf{m}_{\rho,\theta}}^+\right) = \rho\left[g_\theta\left(s_{\mathbf{m}_-}^+\right) + \hat{s}_\theta y_{\mathbf{m}_-}^+\left(s_{\mathbf{m}_-}^+\right)\right]$$
$$+ O\left(\rho^2\right),$$

$$y_{\mathbf{m}_{\rho,\theta}}^+\left(\tilde{s}_{\mathbf{m}_{\rho,\theta}}^+\right) = y_{\mathbf{m}_-}^+\left(s_{\mathbf{m}_-}^+\right)$$
$$+ \rho\left[g_\theta'\left(s_{\mathbf{m}_-}^+\right) + \hat{s}_\theta z_{\mathbf{m}_-}^+\left(s_{\mathbf{m}_-}^+\right)\right]$$
$$+ O\left(\rho^2\right),$$

$$z_{\mathbf{m}_{\rho,\theta}}^+\left(\tilde{s}_{\mathbf{m}_{\rho,\theta}}^+\right) = z_{\mathbf{m}_-}^+\left(s_{\mathbf{m}_-}^+\right)$$
$$+ \rho\left[g_\theta''\left(s_{\mathbf{m}_-}^+\right) + \hat{s}_\theta - \hat{s}_\theta y_{\mathbf{m}_-}^+\left(s_{\mathbf{m}_-}^+\right)\right.$$
$$\left.- \hat{s}_\theta\beta z_{\mathbf{m}_-}^+\left(s_{\mathbf{m}_-}^+\right)\right] + O\left(\rho^2\right),$$

setting $\hat{s}_\theta = -g_\theta\left(s_{\mathbf{m}_-}^+\right)/y_{\mathbf{m}_-}^+\left(s_{\mathbf{m}_-}^+\right)$ we have $x_{\mathbf{m}_{\rho,\theta}}^+\left(\tilde{s}_{\mathbf{m}_{\rho,\theta}}^+\right) = O\left(\rho^2\right)$. By the Mean Value Theorem, it follows that $s_{\mathbf{m}_{\rho,\theta}}^+ = s_{\mathbf{m}_-}^+ + \hat{s}_\theta\rho + O\left(\rho^2\right)$, $y_{\mathbf{m}_{\rho,\theta}}^+\left(s_{\mathbf{m}_{\rho,\theta}}^+\right) = y_{\mathbf{m}_{\rho,\theta}}^+\left(\tilde{s}_{\mathbf{m}_{\rho,\theta}}^+\right) + O\left(\rho^2\right)$ and $z_{\mathbf{m}_{\rho,\theta}}^+\left(s_{\mathbf{m}_{\rho,\theta}}^+\right) = z_{\mathbf{m}_{\rho,\theta}}^+\left(\tilde{s}_{\mathbf{m}_{\rho,\theta}}^+\right) + O\left(\rho^2\right)$.

From Proposition 4.4, $s_{\mathbf{m}_-}^+ = \theta^* K^*\lambda + O\left(\lambda^3\right)$. Then, taking into account that

$$C_\theta^1 = \frac{1}{6}K^*\lambda\cos(\theta) - \frac{1}{12}K^{*2}\lambda^2\sin(\theta) + O\left(\lambda^3\right),$$

$$C_\theta^2 = \frac{\sqrt{3}}{6}K^*\lambda\cos(\theta) + \frac{\sqrt{3}}{12}K^{*2}\lambda^2\sin(\theta) + O\left(\lambda^3\right),$$

$$C_\theta^3 = -\frac{1}{6}K^*\lambda\cos(\theta) + \frac{1}{12}K^{*2}\lambda^2\sin(\theta) + O\left(\lambda^3\right),$$

we have

$$g_\theta\left(s_{\mathbf{m}_-}^+\right) = \frac{\sqrt{3}}{6}K^{*2}e^{\theta^*}\sin\left(\sqrt{3}\theta^*\right)\sin(\theta)\lambda^2$$
$$+ O\left(\lambda^3\right),$$

$$g_\theta'\left(s_{\mathbf{m}_-}^+\right) = \frac{1}{3}\left[2e^{\theta^*}\cos\left(\sqrt{3}\theta^*\right) + e^{-2\theta^*}\right]\cos(\theta)$$
$$+ O\left(\lambda^2\right),$$

$$g_\theta''\left(s_{\mathbf{m}_-}^+\right) = -\frac{4e^{\theta^*}\sin\left(\sqrt{3}\theta^*\right)}{\sqrt{3}K^*\lambda}\cos(\theta) + O\left(\lambda^0\right)$$
$$= -\frac{1}{\lambda}\cos(\theta) + O\left(\lambda^0\right).$$

Therefore, $\hat{s}_\theta = O\left(\lambda^2\right)$. By using the expansion in power series of λ of the coordinates $y_{\mathbf{m}_-}^+\left(s_{\mathbf{m}_-}^+\right)$ and $z_{\mathbf{m}_-}^+\left(s_{\mathbf{m}_-}^+\right)$, which appear in

Proposition 4.4, the lemma follows.   □

Consider now two segments T_h and $T_{h''}$, where $0 < h'' < h$ and h small enough. The topological circle $\Pi_-\left(T_{h''}\right)$ is contained in an annular region with center at $\mathbf{m}_-$ and radii $\rho_1 = (h'')^\delta\left(\lambda^{\delta-1} - 4\sqrt{\pi\delta}\lambda^{\delta-1/2} + O(\lambda^\delta)\right)$ and $\rho_2 = (h'')^\delta\left(\lambda^{\delta-1} + 4\sqrt{\pi\delta}\lambda^{\delta-1/2} + O(\lambda^\delta)\right)$, see Lemma 5.1. Let $\mathcal{C}_{\rho_1}$ the inner boundary of such annular region. The points on the topological circle $\Pi_+\left(\mathcal{C}_{\rho_1}\right)$ satisfy

$$z_{\mathbf{m}_{\rho_1,\theta}}^+\left(s_{\mathbf{m}_{\rho_1,\theta}}^+\right) + 2\lambda\delta y_{\mathbf{m}_{\rho_1,\theta}}^+\left(s_{\mathbf{m}_{\rho_1,\theta}}^+\right)$$
$$= -\frac{1}{\lambda} + \rho_1\left[-\frac{1}{\lambda}\cos(\theta) + O\left(\lambda^0\right)\right] + O\left(\rho_1^2\right),$$

see Lemma 5.2. Since $z + 2\lambda\delta y = h - 1/\lambda$ is the expression of the straight line parallel to $\mathcal{D}_-$ through the point $\mathbf{q}_h$, a sufficient condition on h and h'' in order to conclude that T_h and $\Pi_+\Pi_-\left(T_{h''}\right)$ intersect transversally is

$$h < (h'')^\delta\left[-\lambda^{\delta-2}\cos(\theta) + O\left(\lambda^{\delta-\frac{3}{2}}\right)\right], \quad (16)$$

for some θ. Moreover, $z + 2\lambda\delta y = e^{-\lambda s_0}h - 1/\lambda$ is the expression of the line parallel to $\mathcal{D}_-$ through the point $\Pi_-^{-1}\left(\mathbf{q}_h\right)$, see (13). Hence, a sufficient condition on h and h'' in order to assure that T_h and $\Pi_+\Pi_-\left(T_{h''}\right)$ do not intersect is

$$e^{-\lambda s_0}h > (h'')^\delta\left[-\lambda^{\delta-2}\cos(\theta) + O\left(\lambda^{\delta-\frac{3}{2}}\right)\right], \quad (17)$$

for every θ.

According to [Robinson, 1999], to prove that the Poincaré map $\Pi_+\Pi_-$ define a Smale horseshoe in a neighbourhood of the homoclinic orbit it is enough to find a rectangle $\mathcal{R}$ on the cross section $\{x = 0\}$ in such a way that its image $\Pi_+\Pi_-\left(\mathcal{R}\right)$ can be obtained by compressing in the y direction, stretching in the z direction, folding the resulting rectangle and fitting it back onto the original rectangle $\mathcal{R}$.

For $h > 0$ small enough, we construct a rectangle $\mathcal{R}_h$ as follows. Consider the point $\mathbf{q}_h$. As we have proved in Lemma 5.1,

$$\Pi_-\left(\mathbf{q}_h\right) - \mathbf{m}_-$$
$$= h^\delta\lambda^{\delta-1}\begin{pmatrix} 0 \\ -\sin(\omega\tilde{s}_0(h,0)) + O(\lambda^{1/2}) \\ -\cos(\omega\tilde{s}_0(h,0)) + O(\lambda^{1/2}) \end{pmatrix}.$$

Therefore, $\Pi_-(\mathbf{q}_h)$ spirals around $\mathbf{m}_-$ as h tends to zero, see (15). Moreover, the angular coordinate η_h of $\Pi_-(\mathbf{q}_h)$ with respect to $\mathbf{m}_-$ satisfies

$$\tan(\eta_h) = \tan(\omega\tilde{s}_0(h,0)) + O(\lambda^{1/2}).$$

Hence $\eta_h = \omega\tilde{s}_0(h,0) + O(\lambda^{1/2})$.

We define the points $\mathbf{q}_{h'}$ and $\mathbf{q}_{h''}$ in such a way that $0 < h'' < h' < h$ and $\eta_{h'} = \eta_h + 2\pi$ and $\eta_{h''} = \eta_h + 4\pi$, see Figure 4. Therefore, from (15) it follows that

$$\frac{h}{h'} = e^{\frac{2\pi\lambda}{\omega}} + O(\lambda^{3/2}), \quad \frac{h}{h''} = e^{\frac{4\pi\lambda}{\omega}} + O(\lambda^{3/2}).$$

Define the rectangle $\mathcal{R}_h$ with vertices at $\mathbf{q}_h$, $\mathbf{q}_{h'}$, $\Pi_-^{-1}(\mathbf{q}_{h''})$ and $\Pi_-^{-1}(\mathbf{q}_{h'})$.
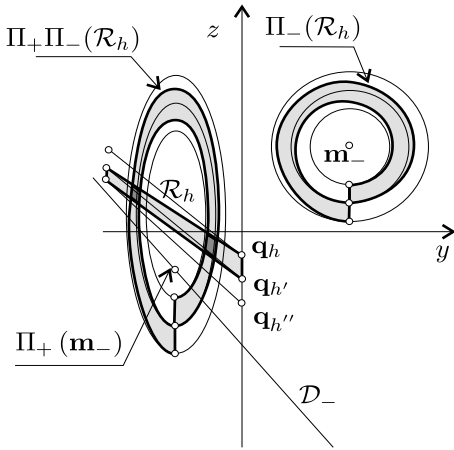

Fig. 4. Geometric construction of a horseshoe

From (16) a sufficient condition on h in order to assure that $\Pi_+\Pi_-(\mathcal{R}_h)$ and $\mathcal{R}_h$ intersect transversally is

$$h < h^\delta\left[-e^{-\frac{4\pi\lambda\delta}{\omega}}\lambda^{\delta-2}\cos(\theta) + O\left(\lambda^{\delta-3/2}\right)\right] \quad (18)$$

for some θ.

Since $\omega = O(\lambda^0)$, see (3), the exponential in (18) tends to 1 as λ tends to zero. Hence, for a fixed δ in $(0,1]$ and $\theta = \pi$, it easy to see that there exists $h_1 > 0$ such that inequality (18) holds for every $h \leq h_1$. In short, for $0 < \delta \leq 1$ we have proved the existence of horseshoes as close as we want to the homoclinic orbit, see Figure 5(a). Consequently, the Poincaré map $\Pi_+\Pi_-$ has the shift of two symbols as a subsystem, see [Robinson, 1999] for more details.

Since (18) holds for $\delta = 1$ and $h = h_1$, there exists a function $\varepsilon(\lambda) > 0$ such that for every $\delta \in (1, 1 + \varepsilon(\lambda))$ and $h < h_1$ but close to h_1, inequality (18) holds.

From (17), a sufficient condition on h in order to conclude that $\Pi_+\Pi_-(\mathcal{R}_h)$ and $\mathcal{R}_h$ do not intersect is

$$h > h^\delta e^{\lambda s_0}\left[-e^{-\frac{4\pi\lambda\delta}{\omega}}\lambda^{\delta-2}\cos(\theta) + O\left(\lambda^{\delta-3/2}\right)\right],$$

for every θ in $[0, 2\pi]$. We note that for $\theta = \pi$, the right hand side of the previous inequality takes its maximum value. Moreover, both exponentials tends to 1 as λ tends to zero. Therefore, if λ is small enough, when $\delta > 1$ there exists a value $h_2 > 0$ in $(0, h_1)$ such that for every $0 < h < h_2$ the inequality holds for every θ in $[0, 2\pi]$, which implies that $\Pi_+\Pi_-(\mathcal{R}_h)$ and $\mathcal{R}_h$ do not intersect.

In short, for $\delta \in (1, 1 + \varepsilon(\lambda))$ and $h < h_1$, but close to it, $\Pi_+\Pi_-(\mathcal{R}_h)$ and $\mathcal{R}_h$ intersect transversally. Therefore, some horseshoes persist but far from the homoclinic orbit. However, if $h \in (0, h_2)$, then $\Pi_+\Pi_-(\mathcal{R}_h)$ and $\mathcal{R}_h$ do not intersect, so those horseshoes closer to the homoclinic orbit are destroyed, as Figure 5(b) illustrates. This proves Theorem 1.2.
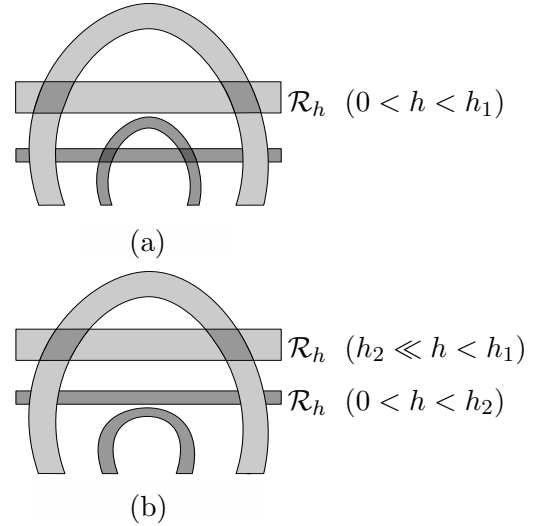

Fig. 5. (a) Existence of horseshoes for $\delta \leq 1$ and $0 < h < h_1$. (b) For $\delta \in (1, 1 + \varepsilon(\lambda))$, persistence of the horseshoes for $h_2 \ll h < h_1$ and destruction for $h < h_2$.

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