EXISTENCE OF POINCARÉ MAPS IN PIECEWISE LINEAR DIFFERENTIAL SYSTEMS IN $\mathbb{R}^N$.

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In this paper we present a relationship between the algebraic notion of proper system, the geometric notion of contact point and the dynamic notion of Poincaré map for piecewise linear differential systems. This allows to present sufficient conditions (which are also necessary under additional hypotheses) for the existence of Poincaré maps in piecewise linear differential systems. Moreover, an adequate parametrization of the Poincaré maps make such maps invariant under linear transformations.

1. Introduction and main results

Due to the facility with they arise in the applications (control theory [Lefschetz, 1965] and [Narendra & Taylor, 1973], design of electric circuits [Chua & Lin, 1990], neurobiology [FitzHugh, 1961] and [Nagumo *et al.*, 1962], etc...) piecewise linear differential systems were early studied from the point of view of the qualitative theory of the ordinary differential equations [Andronov *et al.*, 1987]. Nowadays, a lot of papers are devoted to these differential systems.

Also in mathematics piecewise linear differential systems appear in a natural way between linear differential systems (whose qualitative behavior is "well known") and non–linear differential systems (whose study is very difficult and the knowledge about them is poor, mainly in high dimension). With the advantage that, "the richness of dynamical behavior found in piecewise linear differential systems seems to be almost the same of general non–linear systems", (see [Freire *et al.*, 1998], [Llibre & Sotomayor, 1996] and [Teruel, 2000] for dimension 2 and [Carmona, 2002] for dimension 3) while some dynamical conclusions can easily be obtained from their linear parts. Nevertheless, the analysis of the corresponding dynamics is far from being trivial.

In this paper we emphasize a deep relationship existing in piecewise linear differential systems between the algebraic notion of proper system, the geometric existence of contact points and the dynamical existence of Poincaré maps.

Consider the n–dimensional piecewise linear (differential) systems in the *Lure's form*

$$\frac{d\mathbf{x}}{ds} = \dot{\mathbf{x}} = A\mathbf{x} + \varphi\left(\mathbf{k}^T\mathbf{x}\right)\mathbf{u} + \mathbf{v}, \tag{1}$$

where A is an $n \times n$ real matrix, $\mathbf{k}$, $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{k}$ and $\mathbf{u}$ different from $\mathbf{0}$ and

$$\varphi(\sigma) = \begin{cases} 1 & \text{if } \sigma > 1, \\ \sigma & \text{if } |\sigma| \leq 1, \\ -1 & \text{if } \sigma < -1. \end{cases}$$

As it is proved in Lemma 7 of [Carmona *et al.*, 2002], a long class of piecewise linear systems can be written in this form. In fact, the assumption that φ is symmetric with respect to the origin is irrelevant for the results of this paper. We assume it because the details of the proofs are simpler and easier to write.

Function φ splits the phase space into the three regions $S_+ = \left\{\mathbf{k}^T\mathbf{x} > 1\right\}$, $S_- = \left\{\mathbf{k}^T\mathbf{x} < -1\right\}$ and $S_0 = \left\{\left|\mathbf{k}^T\mathbf{x}\right| < 1\right\}$ separated by the hyperplanes $L_+ = \left\{\mathbf{k}^T\mathbf{x} = 1\right\}$ and $L_- = \left\{\mathbf{k}^T\mathbf{x} = -1\right\}$, in such away that the differential system becomes linear in each of these regions. More precisely $\dot{\mathbf{x}} = A\mathbf{x}+\mathbf{v}+\mathbf{u}$ if $\mathbf{x} \in S_+ \cup L_+$, $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{v} - \mathbf{u}$ if $\mathbf{x} \in S_- \cup L_-$ and $\dot{\mathbf{x}} = B\mathbf{x} + \mathbf{v}$ if $\mathbf{x} \in L_- \cup S_0 \cup L_+$, where

$$B = A + \mathbf{u}\mathbf{k}^T. \tag{2}$$

Given $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^n$, we denote by $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ the matrix whose columns are the components of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Following [Komuro, 1988] and [Wu & Chua, 1996], a differential system (1) is said to be *proper* if the $n \times n$ matrix

$$O_A = \left(\mathbf{k}, A^T\mathbf{k}, \left(A^2\right)^T\mathbf{k}, \ldots, \left(A^{n-1}\right)^T\mathbf{k}\right)^T,$$

has rank n. Other authors (see for instance [Carmona *et al.*, 2002] and [Llibre & Ponce, 1999]) call such systems *observable* ones.

Take $\mathbf{p} \in L_+$ (respectively, L_-), the point $\mathbf{p}$ is said to be a *contact point of order* k of the flow of system (1) with L_+ (respectively, L_-) if $\mathbf{k}^T B^{j-1}\left(B\mathbf{p} + \mathbf{v}\right) = 0$ for $j = 1, 2, \ldots, k$ and $\mathbf{k}^T B^k \left(B\mathbf{p} + \mathbf{v}\right) \neq 0$, where B^0 denotes the identity matrix. When $\mathbf{k}^T B^j \left(B\mathbf{p} + \mathbf{v}\right) = 0$ for any $j \geq 0$, the point $\mathbf{p}$ is said to be a *contact point of order* ∞.

Under the assumption of the existence of at least a zero $\mathbf{e}$ of $B\mathbf{x} + \mathbf{v} = 0$ we transform system (1) into the following one

$$\dot{\mathbf{x}} = A\mathbf{x} + \varphi^*\left(\mathbf{k}^T\mathbf{x}\right)\mathbf{u}, \tag{3}$$

where

$$\varphi^*(\sigma) = \begin{cases} 1 - \mathbf{z} & \text{if } \sigma > 1 - \mathbf{z}, \\ \sigma & \text{if } -1 - \mathbf{z} \leq \sigma \leq 1 - \mathbf{z}, \\ -1 - \mathbf{z} & \text{if } \sigma < -1 - \mathbf{z}, \end{cases}$$

and $\mathbf{z} = \mathbf{k}^T\mathbf{e}$.

If the flow of differential system (3) defines a Poincaré map, Π_{++}, when we take as a transversal section the hyperplane L_+, then Π_{++} is defined either by the flow of the linear system $\dot{\mathbf{x}} = B\mathbf{x}$ and we refer it by Π^B_{++}, or by the flow of the linear system

$\dot{\mathbf{x}} = A\mathbf{x} + \left(1 - \mathbf{k}^T\mathbf{e}\right)\mathbf{u}$ and we refer it by Π^A_{++}. In a similar way we consider the Poincaré maps Π^A_{--} and Π^B_{--}.

When the flow of differential system (3) defines a Poincaré map Π_{+-} taking as a transversal sections the hyperplanes L_+ and L_-, we refer it by Π^B_{+-}. In a similar way we consider the Poincaré map Π^B_{-+}.

In our main results we obtain dynamic properties of the flow from the existence of a contact point of order $n-1$.

Theorem 1.1. *Consider a piecewise linear differential system* (3) *without singular points in* $L_+ \cup L_-$.

(a) *The differential system is proper if and only if there exists exactly one contact point of order* $n-1$ *of the flow with* L_+ *(respectively,* L_-*).*

(b) *If the differential system is proper, then the Poincaré maps* Π^A_{++}, Π^A_{--} *(or* Π^B_{++} *and* Π^B_{--}*),* Π^B_{-+} *and* Π^B_{+-} *are defined.*

(c) *If the Poincaré maps are defined, then there exists a* $(n-2)$*–dimensional vector subspace of* L_+ *(respectively,* L_-*) formed by the contact points of order greater than or equal to* 1.

When $n = 2$, Theorem 1.1 characterizes the existence of the Poincaré maps by using the existence of exactly one contact point. This result can be found in [Teruel, 2000].

If $\mathbf{p}_+$ and $\mathbf{p}_-$ are contact points of order $n-1$ of the flow of differential system (3) with L_+ and L_-, respectively, then $\left\{B^j\mathbf{p}_+\right\}_{j=1}^{n-1}$ and $\left\{B^j\mathbf{p}_-\right\}_{j=1}^{n-1}$ are bases of L_+ and L_-, see Lemma 3.6. Let π^M_{jk} be the parametrization in such bases of the Poincaré maps Π^M_{jk}, for $j,k \in \{+,-\}$ and $M \in \{A,B\}$.

Theorem 1.2. *Under the assumptions of Theorem 1.1, if system* (3) *is proper, then the Poincaré maps* π^M_{jk} *for* $j,k \in \{+,-\}$ *and* $M \in \{A,B\}$ *are invariant by linear changes of coordinates.*

As a consequence of Theorem 1.2, for studying the behavior of the maps π^M_{jk} for $j,k \in \{+,-\}$ and $M \in \{A,B\}$ it is enough to consider matrices A and B in the canonical Jordan form. These arguments have been used to study completely the Poincaré maps of differential system (3) when $n = 2$, see [Teruel, 2000] and in particular cases when $n = 3$, see [Carmona *et al.*, 2002].

The paper is divided in three sections. In Section 2 we present some results about the differentiability of the flow in a neighborhood of a contact point. Section 3 contains a discussion of the relationship between contact points and proper systems. In Section 4 we prove Theorems 1.1 and 1.2.

2. Differentiability of the flow at contact points

In the next result we present a characterization of the contact points of the flow of system (3) with L_+ or with L_- in terms of the matrices A and B.

Lemma 2.1. **(a)** *Let* $\mathbf{p}$ *be a point in* L_+, $\mathbf{p}$ *is a contact point of order* k *of the flow with* L_+ *if and only if* $B^j\left(B\mathbf{p}+\mathbf{v}\right) = A^j\left(A\mathbf{p}+\mathbf{u}+\mathbf{v}\right)$ *for* $j = 0,1,...,k$ *and* $B^{k+1}\left(B\mathbf{p}+\mathbf{v}\right) \neq A^{k+1}\left(A\mathbf{p}+\mathbf{u}+\mathbf{v}\right)$.

(b) *Let* $\mathbf{p}$ *be a point in* L_-, $\mathbf{p}$ *is a contact point of order* k *of the flow with* L_- *if and only if* $B^j\left(B\mathbf{p}+\mathbf{v}\right) = A^j\left(A\mathbf{p}-\mathbf{u}+\mathbf{v}\right)$ *for* $j = 0,1,...,k$ *and* $B^{k+1}\left(B\mathbf{p}+\mathbf{v}\right) \neq A^{k+1}\left(A\mathbf{p}-\mathbf{u}+\mathbf{v}\right)$.

Proof: (a) From expression (2) and since $\mathbf{k}^T\mathbf{p} = 1$ we obtain $B\mathbf{p}+\mathbf{v} = A\mathbf{p}+\mathbf{u}+\mathbf{v}$ and

$$B\left(B\mathbf{p}+\mathbf{v}\right) = A\left(A\mathbf{p}+\mathbf{u}+\mathbf{v}\right) + \mathbf{u}\mathbf{k}^T\left(B\mathbf{p}+\mathbf{v}\right).$$

Thus, if $\mathbf{p}$ is a contact point of order k, then $B\left(B\mathbf{p}+\mathbf{v}\right) = A\left(A\mathbf{p}+\mathbf{u}+\mathbf{v}\right)$. Assuming $B^j\left(B\mathbf{p}+\mathbf{v}\right) = A^j\left(A\mathbf{p}+\mathbf{u}+\mathbf{v}\right)$ to hold for $0 \leq j < r$ where $r \leq k$, we will prove it for r. Since $B^r\left(B\mathbf{p}+\mathbf{v}\right) = BB^{r-1}\left(B\mathbf{p}+\mathbf{v}\right) = BA^{r-1}\left(A\mathbf{p}+\mathbf{u}+\mathbf{v}\right)$, from (2) it follows that

$$\begin{aligned} B^r\left(B\mathbf{p}+\mathbf{v}\right) &= A^r\left(A\mathbf{p}+\mathbf{u}+\mathbf{v}\right) \\ &\quad + \mathbf{u}\mathbf{k}^T B^{r-1}\left(B\mathbf{p}+\mathbf{v}\right), \end{aligned}$$

where $\mathbf{k}^T B^{r-1}\left(B\mathbf{p}+\mathbf{v}\right) = 0$ and $\mathbf{k}^T B^k\left(B\mathbf{p}+\mathbf{v}\right) \neq 0$. Therefore, $B^j\left(B\mathbf{p}+\mathbf{v}\right) = A^j\left(A\mathbf{p}+\mathbf{u}+\mathbf{v}\right)$ for $j = 0,1,...,k$ and $B^{k+1}\left(B\mathbf{p}+\mathbf{v}\right) \neq A^{k+1}\left(A\mathbf{p}+\mathbf{u}+\mathbf{v}\right)$. Reciprocally, if $B^j\left(B\mathbf{p}+\mathbf{v}\right) = A^j\left(A\mathbf{p}+\mathbf{u}+\mathbf{v}\right)$ for $j = 0,1,...,k$ and $B^{k+1}\left(B\mathbf{p}+\mathbf{v}\right) \neq A^{k+1}\left(A\mathbf{p}+\mathbf{u}+\mathbf{v}\right)$ using expression (2) we have

$$\begin{aligned} B^j(B\mathbf{p}+\mathbf{v}) &= A^j(A\mathbf{p}+\mathbf{u}+\mathbf{v}) \\ &\quad +\mathbf{u}\mathbf{k}^T B^{j-1}(B\mathbf{p}+\mathbf{v}), \end{aligned}$$

for $j=1,2,...,k+1$. Then, $\mathbf{k}^T B^{j-1}(B\mathbf{p}+\mathbf{v})=0$ for $j=1,2,...,k$ and $\mathbf{k}^T B^k(B\mathbf{p}+\mathbf{v})\neq 0$; that is, $\mathbf{p}$ is a contact point of order k.

Statement (b) follows in a similar way. ■

Using the characterization of a contact point of order k showed in Lemma 2.1, in the next result we establish the relation between contact point and differentiability.

Lemma 2.2. *Let* $\mathbf{p}$ *be a point in* L_+ *(respectively,* L_-*) and* $\mathbf{x}(s)$ *be the solution of the differential system* (1) *through* $\mathbf{p}$ *at* $s=0$*. If* $\mathbf{p}$ *is a contact point of order* k*, then* $\mathbf{x}(s)$ *is* $k+1$ *times continuously differentiable at* $s=0$*.*

Proof: If $\mathbf{x}(s)$ is locally contained in one of the regions limited by L_+, then $\mathbf{x}(s)$ is infinitely many times continuously differentiable at $s=0$.

Suppose now that $\mathbf{x}(s)$ crosses L_+ at $\mathbf{p}$. In this case there exits $\varepsilon>0$ such that $\mathbf{x}(s)$ is infinitely many times continuously differentiable in $(-\varepsilon,0)$ and infinitely many times continuously differentiable in $(0,\varepsilon)$. From Lemma 2.1 we have $\lim_{s\nearrow 0}\mathbf{x}^{(j)}(s)=\lim_{s\searrow 0}\mathbf{x}^{(j)}(s)$ for $j=0,1,...,k+1$ and $\lim_{s\nearrow 0}\mathbf{x}^{(k+2)}(s)\neq\lim_{s\searrow 0}\mathbf{x}^{(k+2)}(s)$, therefore, $\mathbf{x}(s)$ is $k+1$ times continuously differentiable at $s=0$. ■

Proposition 2.3. *Let* $\mathbf{p}$ *be a point in* L_+ *(respectively,* L_-*) and* $\mathbf{x}(s)$ *be the solution of the differential system* (1) *through* $\mathbf{p}$ *at* $s=0$*.*

- **(a)** *The point* $\mathbf{p}$ *is a contact point of order* $k=2r+1$ *if and only if* $\mathbf{x}(s)$ *is locally contained in* S_+ *(respectively,* S_-*) or in* S_0*, in such a case* $\mathbf{x}(s)$ *is infinitely many times continuously differentiable at* $s=0$*.*
- **(b)** *The point* $\mathbf{p}$ *is a contact point of order* $k=2r$ *if and only if* $\mathbf{x}(s)$ *crosses* L_+ *(respectively,* L_-*) at* $s=0$*, in such a case* $\mathbf{x}(s)$ *is* $k+1$ *(but not* $k+2$*) times continuously differentiable at* $s=0$*.*

Proof: From Lemma 2.2 if $\mathbf{p}$ is a contact point of order k, then $\mathbf{x}(s)$ is $k+1$ times continuously differentiable. Expanding $\mathbf{x}(s)$ at $s=0$ we have

$$\mathbf{x}(s)-\mathbf{p}=\sum_{j=1}^{k}\mathbf{x}^{(j)}(0)\frac{s^j}{j!}+\mathbf{x}^{(k+1)}(\xi)\frac{s^{k+1}}{(k+1)!}$$

with $|\xi|<|s|$. From this and noting that $\mathbf{x}^{(j)}(0)=B^{j-1}(B\mathbf{p}+\mathbf{v})$ for $j=1,2,...,k+1$ it follows that

$$\mathbf{k}^T(\mathbf{x}(s)-\mathbf{p})=\mathbf{k}^T\mathbf{x}^{(k+1)}(\xi)\frac{s^{k+1}}{(k+1)!}. \quad (4)$$

Since $\mathbf{k}^T B^k(B\mathbf{p}+\mathbf{v})\neq 0$, for s small enough we obtain that $\mathbf{k}^T\mathbf{x}^{(k+1)}(\xi)\neq 0$ and hence the sign of $\mathbf{k}^T(\mathbf{x}(s)-\mathbf{p})$ depends on k is even or not. Therefore, if k even then $\mathbf{x}(s)$ crosses the hyperplane at $s=0$ and if k odd, then $\mathbf{x}(s)$ is locally contained in the regions limited by the hyperplane.

Respectively, if $\mathbf{x}(s)$ is locally contained in one of the regions limited by L_+, where the system is linear, then $\mathbf{k}^T(\mathbf{x}(s)-\mathbf{p})=\mathbf{k}^T\mathbf{x}^{(1)}(\xi)s$ does not change the sign in a neighborhood of $s=0$. This implies that $\mathbf{k}^T(B\mathbf{p}+\mathbf{v})=0$ and $\mathbf{p}$ is a contact point of order k greater than or equal to 1. Therefore, we obtain again the expression (4), which shows that k has to be a odd number. Similar arguments apply when $\mathbf{x}(s)$ crosses the hyperplane L_+. ■

3. Contact points and proper systems

Proposition 3.1. *Consider a piecewise linear differential system* (1)*.*

- **(a)** *The order of any contact point is a number in the set* $\{1,2,\ldots,n-1,\infty\}$*.*
- **(b)** *If the differential system is proper, then* $\mathbf{p}$ *is a contact point of order* ∞ *if and only if* $\mathbf{p}$ *is a singular point.*

Proof: (a) Let $p_B(x)=d_0+d_1x+\cdots+d_{n-1}x^{n-1}+x^n$ be the characteristic polynomial of B. By the Cayley–Hamilton Theorem we have

$$\begin{aligned} B^n(B\mathbf{p}+\mathbf{v}) &= -d_0(B\mathbf{p}+\mathbf{v})-d_1B(B\mathbf{p}+\mathbf{v}) \\ &\quad -\cdots-d_{n-1}B^{n-1}(B\mathbf{p}+\mathbf{v}). \end{aligned}$$

Thus, if $\mathbf{k}^T B^{j-1}(B\mathbf{p}+\mathbf{v})=0$ for $j=1,\ldots,n$, then $\mathbf{p}$ is a contact point of order ∞.

(b) Singular points belonging to L_+ or to L_- are clearly contact points of order ∞ with L_+ or L_-, respectively. Reciprocally, if $\mathbf{p}$ is a contact point of order ∞ we have

$$O_A\left(A\mathbf{p}+\mathbf{u}+\mathbf{v}\right) = \begin{pmatrix} \mathbf{k}^T \\ \mathbf{k}^T A \\ \vdots \\ \mathbf{k}^T A^{n-1} \end{pmatrix}\left(A\mathbf{p}+\mathbf{u}+\mathbf{v}\right) = \begin{pmatrix} \mathbf{k}^T\left(B\mathbf{p}+\mathbf{v}\right) \\ \mathbf{k}^T B\left(B\mathbf{p}+\mathbf{v}\right) \\ \vdots \\ \mathbf{k}^T B^{n-1}\left(B\mathbf{p}+\mathbf{v}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (5)$$

Since O_A has rank n it follows that $A\mathbf{p}+\mathbf{u}+\mathbf{v}=\mathbf{0}$ and $\mathbf{p}$ is a singular point. ■

Using the relationship between the rank of the matrix O_A and the order of the contact point $\mathbf{p}$ that appears in expression (5), in the next result we characterize the proper differential systems.

Lemma 3.2. *Differential system* (1) *is proper if and only if there exists exactly one contact point of the flow with L_+ (respectively, L_-) of order greater than or equal to $n-1$.*

Proof: The existence of exactly one contact point $\mathbf{p}$ of order greater than or equal to $n-1$ is equivalent to the existence of exactly one solution of the linear system $O_A\mathbf{p}=\mathbf{b}$, where

$$\mathbf{b}=\left(1,-\mathbf{k}^T\left(\mathbf{v}+\mathbf{u}\right),\ldots,-\mathbf{k}^T A^{n-2}\left(\mathbf{v}+\mathbf{u}\right)\right)^T.$$

Similar arguments prove the statement when we consider L_-. ■

We remark that non–singular solutions of a proper differential system (1) crossing the hyperplane L_+ or L_- are at most $n-1$ times continuously differentiable, see Lemma 3.2 and Propositions 2.2 and 3.1.

Proposition 3.3. *Differential system* (1) *is proper if and only if the $n\times n$ matrix*

$$O_B=\left(\mathbf{k},B^T\mathbf{k},\left(B^2\right)^T\mathbf{k},\ldots,\left(B^{n-1}\right)^T\mathbf{k}\right)^T,$$

has rank n.

Proof: From Lemma 3.2, if system (1) is proper, then there exists exactly one contact point $\mathbf{p}$ of order greater than or equal to $n-1$ with L_+; that is, $\mathbf{k}^T B^{j-1}\left(B\mathbf{p}+\mathbf{v}\right)=0$ for $j=1,2,\ldots,n-1$. The linear system $O_B\mathbf{p}=\mathbf{b}$, where $\mathbf{b}=\left(1,-\mathbf{k}^T\mathbf{v},\ldots,-\mathbf{k}^T B^{n-2}\right)^T$ has exactly one solution. Thus, O_B has rank n. Reciprocally, if the matrix O_B has rank n, then system (1) is proper. ■

In [Carmona *et al.*, 2002] the authors prove that proper piecewise linear systems (1) can be transformed by a linear change of coordinates into the canonical form $\dot{\mathbf{x}}$ equal to

$$\begin{pmatrix} -c_0 & 1 & 0 & \cdots & 0 \\ -c_1 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -c_{n-2} & \vdots & & \ddots & 1 \\ -c_{n-1} & 0 & \cdots & \cdots & 0 \end{pmatrix}\mathbf{x}+\varphi\left(\mathbf{e}_1^T\mathbf{x}\right)\mathbf{w}+a\mathbf{e}_n,$$

called the *generalized Liénard's form.* Here $\mathbf{e}_k$ denotes the k–th element in the canonical base of $\mathbb{R}^n$. Clearly, the first column in the matrix of the system is formed by the coefficients of the characteristic polynomial of A and

$$\mathbf{w}=\left(c_0-d_0,c_1-d_1,\ldots,c_{n-1}-d_{n-1}\right)^T,$$

where d_i for $i=0,\ldots,n-1$ are the coefficients of the characteristic polynomial of B.

Proposition 3.4. *A piecewise linear differential system can be written in the generalized Liénard's form if and only if it is proper.*

Proof: Here we prove the direct implication, the reverse one can be found in Proposition 16 of [Carmona *et al.*, 2002]. Let $\mathbf{c}$ be the vector $\left(-c_0,-c_1,\ldots,-c_n\right)^T$. Hence, $A=\left(\mathbf{c},\mathbf{e}_1,\ldots,\mathbf{e}_{n-1}\right)$ and by induction we obtain $A^j=\left(\mathbf{s}_1,\mathbf{s}_2,\ldots,\mathbf{s}_{j-1},\mathbf{c},\mathbf{e}_1,\ldots,\mathbf{e}_{n-j}^T\right)$ for $j=2,\ldots,n-1$, where $\mathbf{s}_k$ are adequate vectors of $\mathbb{R}^n$. Therefore, since $\mathbf{k}=\mathbf{e}_1$, O_A is a lower triangular matrix with 1's on the diagonal. ■

A restricted version of Proposition 3.4 for homogeneous linear system ($\mathbf{v} = \mathbf{0}$) can be found in Theorem 1.19 of [Carmona, 2002].

From now on we suppose that there exists at least a zero $\mathbf{e}$ of $B\mathbf{x} + \mathbf{v} = 0$. The change of coordinates $\mathbf{x} \longrightarrow \mathbf{x} - \mathbf{e}$ transforms system (1) into piecewise linear system (3).

For simplicity of notation, we continue writing L_+ and L_- for the hyperplanes after translation; i.e. $L_+ = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{k}^T\mathbf{x} = 1 - \mathbf{k}^T\mathbf{e}\}$ and $L_- = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{k}^T\mathbf{x} = -1 - \mathbf{k}^T\mathbf{e}\}$, and S_+, S_0 and S_- for the translated regions. In S_0 system (3) becomes the homogeneous linear system $\dot{\mathbf{x}} = B\mathbf{x}$, where B satisfies again equation (2).

Lemma 3.5. **(a)** *Differential systems* (3) *with a contact point of order $n-1$ satisfy that* $\det(B) \neq 0$.

(b) *Proper differential systems* (3) *with no singular points on L_+ (respectively, L_-) has exactly one contact point of order $n-1$ with L_+ (respectively, L_-).*

Proof: (a) Let $\mathbf{p}$ be a contact point of order $n-1$; that is $\mathbf{k}^T B^j \mathbf{p} = 0$ for $j = 1, 2, \ldots, n-1$ and $\mathbf{k}^T B^n \mathbf{p} \neq 0$. Let d_j from $j = 0, 1, \ldots, n-1$ be the coefficients of the characteristic polynomial of B, by the Cayley–Hamilton Theorem it follows that $\mathbf{k}^T B^n \mathbf{p} = (-1)^{n-1} \det(B)\, \mathbf{k}^T \mathbf{p}$. Therefore, $\det(B) \neq 0$.

(b) From Lemma 3.2 it follows that there exists exactly one contact point $\mathbf{p}$ of order greater than or equal to $n-1$. Since the differential system has no singular points in $L_+ \cup L_-$ and singular points are the unique ones with order greater than $n-1$, see Proposition 3.1, the contact point has order equal to $n-1$. ■

Lemma 3.6. *Let $\mathbf{p} \in L_+$ (respectively, L_-) be a contact point of order $n-1$ of the flow of a differential system* (3) *without singular points in $L_+ \cup L_-$. The vector set $\mathcal{B} = \{B^j\mathbf{p}\}_{j=1}^{n-1}$ is a base of L_+ and $\widetilde{\mathcal{B}} = \{\mathbf{p}\} \cup \mathcal{B}$ is a base of $\mathbb{R}^n$.*

Proof: Since $\mathbf{k}^T B^j \mathbf{p} = 0$ for $j = 1, 2, \ldots, n-1$, these vectors are parallel to L_+. Thus, it is enough to prove that all vectors in $\widetilde{\mathcal{B}}$ are independent.

From Lemma 3.5(a), we obtain $\det(B) \neq 0$. Suppose that there exists n real numbers $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ such that $\lambda_0\mathbf{p} + \sum_{j=1}^{n-1} \lambda_j B^j \mathbf{p} = \mathbf{0}$. Multiplying by $\mathbf{k}^T$ we obtain $\lambda_0 \mathbf{k}^T\mathbf{p} = \lambda_0\left(1 - \mathbf{k}^T\mathbf{e}\right)$; i.e $\lambda_0 = 0$, because $\mathbf{k}^T\mathbf{e} \neq 0$, otherwise $\mathbf{0}$ would be a singular point in L_+. Hence, $\mathbf{0} = \sum_{j=1}^{n-1} \lambda_j B^j \mathbf{p}$. Multiplying by $\mathbf{k}^T B^{-1}$ yields $\lambda_1 \mathbf{k}^T \mathbf{p} = 0$ and then $\lambda_1 = 0$. Iterating $n-2$ times this procedure we conclude that $\lambda_j = 0$ for $j = 0, 1, \ldots, n-1$. ■

For a contact point $\mathbf{p} \in L_+$ of order $n-1$ of system (3) we have

$$B^j\mathbf{p} = A^{j-1}\left(A\mathbf{p} + \left(1 - \mathbf{k}^T\mathbf{e}\right)\mathbf{u}\right)$$

for $j = 1, 2, \ldots, n-1$. Hence, a base of L_+ is $\left\{A^{j-1}\left(A\mathbf{p} + \left(1 - \mathbf{k}^T\mathbf{e}\right)\mathbf{u}\right)\right\}_{j=1}^{n-1}$.

One dynamical consequence of the existence of exactly one contact point of order $n-1$ with L_+ is that the hyperplane cannot be parallel to any subspace invariant by the flow. A similar result is proved in [Chen, 1984].

Lemma 3.7. *Let $\mathbf{p} \in L_+$ (respectively, L_-) be a contact point of order $n-1$ of the flow of differential system* (3). *If E is a m–dimensional subspace of $\mathbb{R}^n$ such that $\mathbf{k}^T\mathbf{z} = 0$ for every $\mathbf{z} \in E$, then E is not invariant by the flow.*

Proof: Let $\mathbf{z} \neq \mathbf{0}$ be a vector of E such that $\mathbf{z} \in S_0$ and suppose that E is invariant by the flow. From Lemma 3.5(b) we obtain $\det(B) \neq 0$ and then $B\mathbf{z} \in E$. By Lemma 3.6, since E is orthogonal to $\mathbf{k}^T$ it follows that $\mathbf{z} = \sum_{j=1}^{n-1} \lambda_j B^j \mathbf{p}$ and therefore,

$$B\mathbf{z} = \sum_{j=1}^{n-1} \lambda_j B^{j+1}\mathbf{p}. \tag{6}$$

On other hand, if d_j for $j = 0, 1, \ldots, n-1$ are the coefficients of the characteristic polynomial of B, by the Cayley–Hamilton Theorem yields $B^n\mathbf{p} = -d_0\mathbf{p} - d_1 B\mathbf{p} - \cdots - d_{n-1}B^{n-1}\mathbf{p}$. Substituting $B^n\mathbf{p}$ in expression (6) yields

$$\begin{aligned} B\mathbf{z} &= -d_0\lambda_{n-1}\mathbf{p} - d_1\lambda_{n-1}B\mathbf{p} \\ &\quad + \sum_{j=2}^{n-1} \left(\lambda_{j-1} - \lambda_{n-1}d_j\right) B^j\mathbf{p}. \end{aligned}$$

Taking into account that $d_0 = (-1)^n \det(B)$ we obtain $\lambda_{n-1} = 0$.

Again, $B\mathbf{z}$ belongs to E and E is invariant by the flow, so we get $B^2\mathbf{z} \in E$ and previous arguments can be repeated to prove that $\lambda_j = 0$ for $j = 1, 2, \ldots, n-1$. Therefore $\mathbf{z} = \mathbf{0}$, in contradiction with the assumptions. Thus, E cannot be a subspace invariant by the flow.

If $\mathbf{z} \in S_+$ or $\mathbf{z} \in S_-$, then we consider on E the base $\left\{A^{j-1}\left(A\mathbf{p}+\left(1-\mathbf{k}^T\mathbf{e}\right)\mathbf{u}\right)\right\}_{j=1}^{n-1}$. Similar arguments proves the statement in those cases. ■

4. Contact points and Poincaré maps

Define on L_+ the half–hyperplanes $L_+^I = \left\{\mathbf{q}\in L_+ : \mathbf{k}^T B\mathbf{q} < 0\right\}$ and $L_+^O = \left\{\mathbf{q}\in L_+ : \mathbf{k}^T B\mathbf{q} > 0\right\}$, and on L_- the half–hyperplanes $L_-^I = \left\{\mathbf{q}\in L_- : \mathbf{k}^T B\mathbf{q} > 0\right\}$ and $L_-^O = \left\{\mathbf{q}\in L_+ : \mathbf{k}^T B\mathbf{q} < 0\right\}$. Orbits intersecting with L_+ (respectively, L_-) at a point in L_+^O (respectively, L_-^O) goes from S_0 to S_+ (respectively, S_-), and orbits intersecting with L_+^I (respectively, L_-^I) goes from S_+ (respectively, S_-) to S_0.

Suppose that differential system (3) is proper and has not singular points in $L_+ \cup L_-$. Then, from Lemma 3.5, there exist contact points $\mathbf{p}_+ \in L_+$ and $\mathbf{p}_- \in L_-$ of order $n-1$ and $\det(B) \neq 0$. Furthermore, since there are no singular points in $L_+ \cup L_-$ we have $\mathbf{k}^T\mathbf{e} \neq 1$ and $\mathbf{k}^T\mathbf{e} \neq -1$. In the next result we characterize the half–hyperplanes L_+^I, L_+^O, L_-^I and L_-^O depending on the sign of $\det(B)\left(1-\mathbf{k}^T\mathbf{e}\right)$ and $\det(B)\left(-1-\mathbf{k}^T\mathbf{e}\right)$.

Proposition 4.1. *Consider a proper differential system* (3) *without singular points in* $L_+ \cup L_-$.

(a) *If* $\det(B)\left(1-\mathbf{k}^T\mathbf{e}\right) > 0$, *then* L_+^I *is equal to*
$$\left\{\mathbf{p}_+ + \sum_{j=1}^{n-1} a_j B^j \mathbf{p}_+ : a_j \in \mathbb{R}^n \text{ and } (-1)^n a_{n-1} > 0\right\},$$
and L_+^O *is equal to*
$$\left\{\mathbf{p}_+ + \sum_{j=1}^{n-1} a_j B^j \mathbf{p}_+ : a_j \in \mathbb{R}^n \text{ and } (-1)^n a_{n-1} < 0\right\}.$$

(b) *If* $\det(B)\left(1-\mathbf{k}^T\mathbf{e}\right) < 0$, *then* L_+^I *is equal to*
$$\left\{\mathbf{p}_+ + \sum_{j=1}^{n-1} a_j B^j \mathbf{p}_+ : a_j \in \mathbb{R}^n \text{ and } (-1)^n a_{n-1} < 0\right\},$$
and L_+^O *is equal to*
$$\left\{\mathbf{p}_+ + \sum_{j=1}^{n-1} a_j B^j \mathbf{p}_+ : a_j \in \mathbb{R}^n \text{ and } (-1)^n a_{n-1} > 0\right\}.$$

(c) *If* $\det(B)\left(-1-\mathbf{k}^T\mathbf{e}\right) > 0$, *then* L_-^I *is equal to*
$$\left\{\mathbf{p}_- + \sum_{j=1}^{n-1} a_j B^j \mathbf{p}_- : a_j \in \mathbb{R}^n \text{ and } (-1)^n a_{n-1} > 0\right\},$$
and L_-^O *is equal to*
$$\left\{\mathbf{p}_- + \sum_{j=1}^{n-1} a_j B^j \mathbf{p}_- : a_j \in \mathbb{R}^n \text{ and } (-1)^n a_{n-1} < 0\right\}.$$

(d) *If* $\det(B)\left(-1-\mathbf{k}^T\mathbf{e}\right) < 0$, *then* L_-^I *is equal to*
$$\left\{\mathbf{p}_- + \sum_{j=1}^{n-1} a_j B^j \mathbf{p}_- : a_j \in \mathbb{R}^n \text{ and } (-1)^n a_{n-1} < 0\right\},$$
and L_-^O *is equal to*
$$\left\{\mathbf{p}_- + \sum_{j=1}^{n-1} a_j B^j \mathbf{p}_- : a_j \in \mathbb{R}^n \text{ and } (-1)^n a_{n-1} > 0\right\}.$$

Proof: (a) From Lemma 3.6 it follows that $L_+ = \left\{\mathbf{p}_+ + \sum_{j=1}^{n-1} a_j B^j \mathbf{p}_+ : a_j \in \mathbb{R}\right\}$. Hence, if $\mathbf{q} \in L_+$, then $B\mathbf{q} = B\mathbf{p}_+ + \sum_{j=1}^{n-1} a_j B^{j+1}\mathbf{p}_+$ and $\mathbf{k}^T B\mathbf{q} = a_{n-1}\mathbf{k}^T B^n \mathbf{p}_+$. Applying the Cayley–Hamilton Theorem we have $\mathbf{k}^T B^n \mathbf{p}_+ = (-1)^{n-1}\det(B)\left(1-\mathbf{k}^T\mathbf{e}\right)$, see the proof of Lemma 3.5(a) for more details. Statement follows straightforward.

The remainder statements follows in a similar way. ■

Lemma 4.2. **(a)** *Given a proper differential system* (3) *without singular points in* $L_+ \cup L_-$, *the sets* L_+^I, L_+^O, L_-^I *and* L_-^O *are non–empty.*

(b) *If* L_+^I *and* L_+^O *(respectively,* L_-^I *and* L_-^O*) are non–empty sets, then there exists a* $(n-2)$*–dimensional vector subspace of* L_+ *(respectively,* L_-*) formed by the contact points of the flow with* L_+ *(respectively,* L_-*) of order at least* 1.

Proof: Statement (a) is a consequence of Proposition 4.1.

(b) Take $\mathbf{q}_1 \in L_+^I$ and $\mathbf{q}_2 \in L_+^O$. Function $f(\lambda) = \mathbf{k}^T B\left((1-\lambda)\mathbf{q}_1 + \lambda\mathbf{q}_2\right)$ satisfies that $f(0) < 0$ and $f(1) > 0$. Thus, there exists $\lambda_0 \in (0,1)$ such that $\mathbf{p}_+ = (1-\lambda_0)\mathbf{q}_1 + \lambda_0\mathbf{q}_2$ is a contact point of order greater than or equal to 1; i.e. $\mathbf{k}^T\mathbf{p}_+ = 1 - \mathbf{k}^T\mathbf{e}$ and $\mathbf{k}^T B\mathbf{p}_+ = 0$. Therefore, the hyperplanes L_+ and $\mathbf{k}^T B\mathbf{x} = 0$ intersects at a $(n-2)$–dimensional vector subspace formed by contact points of order greater than or equal than 1. ■

Suppose that the flow of system (3) defines a Poincaré map Π_{++} when we take as a transversal section the hyperplane L_+. There exist two possibilities.

(i) Π_{++} transforms points of L_+^I into points of L_+^O. Thus, Π_{++} is defined by the flow of the homogeneous linear system $\dot{\mathbf{x}} = B\mathbf{x}$ and we refer to it by Π_{++}^B.

(ii) Π_{++} transforms points of L_+^O into points of L_-^I. Thus, Π_{++} is defined by the flow of the non–homogeneous linear system $\dot{\mathbf{x}} = A\mathbf{x} + \left(1 - \mathbf{k}^T\mathbf{e}\right)\mathbf{u}$ and we refer to it by Π_{++}^A.

In a similar way we consider the Poincaré maps Π_{--}^A and Π_{--}^B.

Let Π_{+-} be the Poincaré map which transforms point of L_+^I into points of L_-^O, and Π_{-+} the Poincaré map which transforms points of L_-^I into points of L_+^O. Since both maps are defined by the flow of the linear system $\dot{\mathbf{x}} = B\mathbf{x}$, we refer to them by Π_{+-}^B and Π_{-+}^B.

Proof of Theorem 1.1*:* (a) The statement follows immediately from Lemmas 3.2 and 3.5(b).

(b) From Lemma 3.5(b), there exists exactly one contact point $\mathbf{p}_+ \in L_+$ of order $n-1$. Hence, the orbit $\gamma_{\mathbf{p}_+}$ through $\mathbf{p}_+$ satisfies the following local behavior.

If n even, from Proposition 2.3(a), then $\gamma_{\mathbf{p}_+}$ does not cross the hyperplane L_+, see Figure 1. We can consider a tubular neighborhood U of $\gamma_{\mathbf{p}_+}$ contained in a flux box surrounding a piece of $\gamma_{\mathbf{p}_+}$ in a neighborhood of $\mathbf{p}_+$. According to the Continuous Dependence Theorem of the solutions of a differential equation with respect to the initial conditions U intersects with L_+^I and L_+^O. Take $\mathbf{q}_1 \in L_+^O \cap U$. The orbit through $\mathbf{q}_1$, $\gamma_{\mathbf{q}_1}$, crosses L_+ from S_0 to S_+. Since $\gamma_{\mathbf{p}_+}$ does not cross L_+, $\gamma_{\mathbf{q}_1}$ has to intersect with $L_+^I \cap U$ at a point $\mathbf{q}_2$. Therefore, we can define the Poincaré map Π_{++}^A or Π_{++}^B depending if $\gamma_{\mathbf{p}_+}$ is, locally contained in S_0 or in S_+, respectively.

If n odd, from Proposition 2.3(b), then $\gamma_{\mathbf{p}_+}$ crosses L_+ at $\mathbf{p}_+$, see Figure 2. Define again a tubular neighborhood U of $\gamma_{\mathbf{p}_+}$ contained in a flux box surrounding a piece of $\gamma_{\mathbf{p}_+}$ in a neighborhood of $\mathbf{p}_+$. Clearly U intersects with L_+^O and L_+^I. Let $\mathbf{q}_1$ be a point in $U \cap L_+^O$, the orbit $\gamma_{\mathbf{q}_1}$ through $\mathbf{q}_1$ is contained in U. Thus, after intersecting with L_+^O

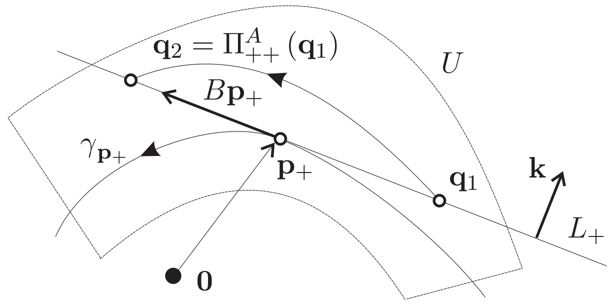

Fig. 1. Existence of the Poincaré map Π_{++}^A in a neighborhood of the contact point $\mathbf{p}_+$ when $n = 2$.

at $\mathbf{q}_1$ the orbit $\gamma_{\mathbf{q}_1}$ has to intersect with L_+^I at $\mathbf{q}_2$, see Figure 2. Therefore, the Poincaré map Π_{++}^A or Π_{++}^B is defined depending on $\gamma_{\mathbf{p}_+}$ crosses L_+ from S_+ to S_0, or from S_0 to S_+, respectively.

Suppose now that no orbit starting at L_+^I intersects with L_-^O. Then, orbits remains inside S_0 when s tends to $+\infty$, this implies the existence of a subspace invariant by the flow contained in S_0, in contradiction with Lemma 3.7. Therefore, Π_{+-}^B is defined. The existence of Π_{-+}^B follows in a similar way.

(c) If the Poincaré maps are defined, then L_+^I, L_+^O, L_-^I and L_-^O are non–empty. The statement follows from Lemma 4.2(b). ■

Take $\mathbf{q}_1 \in L_+^I$ and $\mathbf{q}_2 \in L_+^O$ such that $\mathbf{q}_2 = \Pi_{++}^B(\mathbf{q}_1)$. By Proposition 4.1, $\mathbf{q}_1 = \mathbf{p}_+ + \sum_{j=1}^{n-1} a_j B^j \mathbf{p}_+$ and $\mathbf{q}_2 = \mathbf{p}_+ + \sum_{j=1}^{n-1} a_j^* B^j \mathbf{p}_+$. We denote by π_{++}^B the Poincaré map given by $\pi_{++}^B(a_1, a_2, \ldots, a_{n-1}) = \left(a_1^*, a_2^*, \ldots, a_{n-1}^*\right)$. In a similar way we define the Poincaré maps π_{--}^B, π_{++}^A, π_{--}^A, π_{+-}^B and π_{-+}^B.

Proof of Theorem 1.2: The change of coordinates $\mathbf{y} = M\mathbf{x}$ transforms system (3) into the system

$$\dot{\mathbf{y}} = A^*\mathbf{y} + \varphi^*\left(\mathbf{k}^{*T}\mathbf{y}\right)\mathbf{u}^*, \tag{7}$$

where $A^* = MAM^{-1}$, $\mathbf{k}^{*T} = \mathbf{k}^T M^{-1}$ and $\mathbf{u}^* = M\mathbf{u}$.

From Theorem 1.1(a) there exists exactly one contact point $\mathbf{p}_+$ of order $n-1$ with L_+. It is easy to check that $\mathbf{p}_+^* = M\mathbf{p}_+$ is the contact point of order $n-1$ of the flow of system (7) with the hyperplane $L_+^* = ML_+ =$

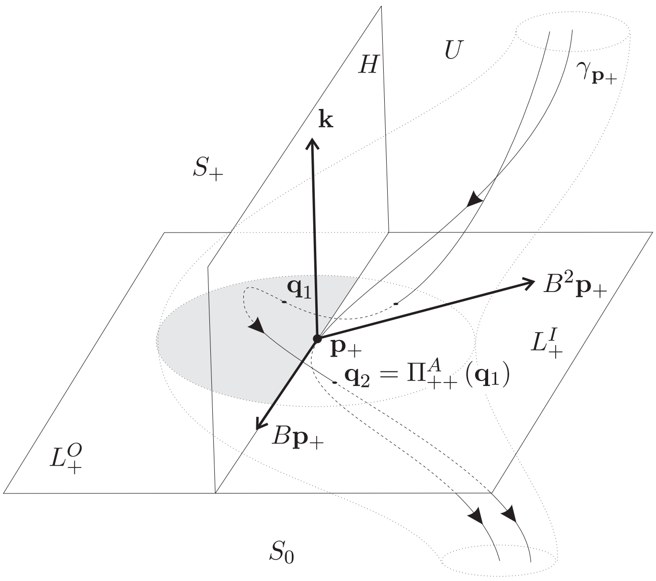

Fig. 2. Existence of the Poincaré map Π_{++}^A in a neighborhood of the contact point $\mathbf{p}_+$ when $n = 3$.

$\{M\mathbf{q} : \mathbf{q} \in L_+\}$. Consider on L_+^* the half–hyperplanes L_+^{*I} and L_+^{*O}. If $\det(B)\left(1 - \mathbf{k}^T\mathbf{e}\right) > 0$ (when $\det(B)\left(1 - \mathbf{k}^T\mathbf{e}\right) < 0$ arguments are similar), then $\det(B^*)\left(\mathbf{1} - \mathbf{k}^{*T}\mathbf{e}^*\right) > 0$, where $B^* = MBM^{-1}$. According to Proposition 4.1(a), $L_+^I = \left\{\mathbf{p}_+ + \sum_{j=1}^{n-1} a_j B^j \mathbf{p}_+ : a_j \in \mathbb{R}^n \text{ and } (-1)^n a_{n-1} > 0\right\}$ and $L_+^{*I} = \{\mathbf{p}_+^* + \sum_{j=1}^{n-1} a_j B^{*j}\mathbf{p}_+^* : a_j^* \in \mathbb{R}^n$ and $(-1)^n a_{n-1} > 0\} = \{M\mathbf{p}_+ + \sum_{j=1}^{n-1} a_j M B^j \mathbf{p}_+ : a_j \in \mathbb{R}^n$ and $(-1)^n a_{n-1} > 0\}$, which implies that $L_+^{*I} = ML_+^I$. Similarly, $L_+^{*O} = ML_+^O$.

Consider the Poincaré map π_{++}^A. The arguments are the same if we consider another Poincaré map. Since Π_{++}^A transforms points of L_+^O into points of L_+^I, the flow of system (7) defines a Poincaré map $\Pi_{++}^{A^*}$ which transforms points of L_+^{*I} into points of L_+^{*O}.

Set $\mathbf{q}_1 \in L_+^O$ and $\mathbf{q}_2 \in L_+^I$ such that $\mathbf{q}_2 = \Pi_{++}^A(\mathbf{q}_1)$. Thus $\mathbf{q}_1 = \mathbf{p}_+ + \sum_{j=1}^{n-1} a_j B^j \mathbf{p}_+$, $\mathbf{q}_2 = \mathbf{p}_+ + \sum_{j=1}^{n-1} b_j B^j \mathbf{p}_+$ and $\pi_{++}^A(a_1, \ldots, a_{n-1}) = (b_1, \ldots, b_{n-1})$. Since $\mathbf{q}_1^* = M\mathbf{q}_1 = M\mathbf{p}_+ + \sum_{j=1}^{n-1} a_j M B^j \mathbf{p}_+ = \mathbf{p}_+^* + \sum_{j=1}^{n-1} a_j B^{*j}\mathbf{p}_+^*$ and $\mathbf{q}_2^* = M\mathbf{q}_2 = \mathbf{p}_+^* + \sum_{j=1}^{n-1} b_j B^{*j}\mathbf{p}_+^*$ we obtain $\pi_{++}^{A^*}(a_1, \ldots, a_{n-1}) = (b_1, \ldots, b_{n-1})$. Therefore, $\pi_{++}^A = \pi_{++}^{A^*}$. ∎

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