Some questions in fuzzy metric spaces

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Abstract

The George and Veeramani's fuzzy metric defined by $M^*(x,y,t) = \frac{min\{x,y\}+t}{max\{x,y\}+t}$ on $[0,\infty[$ (the set of non-negative real numbers) has shown some advantages in front of classical metrics in the process of filtering images. In this paper we study from the mathematical point of view this fuzzy metric and other fuzzy metrics related to it. As a consequence of this study we introduce, throughout the paper, some questions relative to fuzzy metrics. Also, as another practical application, we show that this fuzzy metric is useful for measuring perceptual colour differences between colour samples.

Key words: Fuzzy metric spaces; fuzzy metric completion; strong (non-Archimedean) fuzzy metric; principal fuzzy metric.
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1 Introduction

Kramosil and Michalek, [25], extended the concept of Menger space, [29], to the fuzzy context and they defined the notion of fuzzy metric space. Later, George and Veeramani, [7,9], introduced and studied a notion of fuzzy metric which constitutes a modification of the one due to Kramosil and Michalek. From now on, by fuzzy metric we mean a fuzzy metric in the sense of George and Veeramani. In [8,15] it is proved that the class of topological spaces which are fuzzy metrizable agrees with the class of metrizable spaces, and in [15] several properties of classical metrics were extended to the fuzzy context.

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Nevertheless, the theory of fuzzy metric completion is different from the classical theory of metric spaces or Menger spaces. Indeed, there exist fuzzy metric spaces which are non-completable, [12,16,18].

The concept of fuzzy metric includes in its definition a parameter, t, that allows to introduce novel (fuzzy metric) concepts with respect to the classical metric concepts. For instance, the concepts of principal and strong fuzzy metric were motivated by the study of the p-convergence, [31], and the generalization of non-Archimedean fuzzy metrics, [44], respectively. Moreover, recently, fuzzy metrics have been applied to colour image filtering by replacing classical metrics and some improvements have been achieved [2,3,34–39]. In this context, the presence of the t parameter is indeed a key issue because it allows the fuzzy metric to perform adaptively which is beneficial to improve performance. In particular, a fuzzy metric used frequently in the above cited papers has been the fuzzy metric M^* defined on $[0, \infty[$ (the set of non-negative real numbers) by $M^*(x, y, t) = \frac{min\{x,y\}+t}{max\{x,y\}+t}$.

From the mathematical point of view, the aim of this paper is double. First, to study some aspects of M^* and as well as the well-known fuzzy metric M_0 given by $M_0(x, y) = \frac{min\{x,y\}}{max\{x,y\}}$ on $]0, \infty[$ (the set of positive real numbers). This study is carried out in such a manner (see Remark 24) that it creates an appropriate context to introduce five questions in fuzzy metric spaces (relative to completion, uniform continuity, extension and contractivity) which is the second aim of the paper. In spite of the risk of this proposal, [6] (Preface), we do hope that these problems will provide the basis of much future research. Finally, as practical application, we show that this fuzzy metric is useful for measuring perceptual colour differences between colour samples.

So, the structure of the paper is as follows. After the preliminary section, in Section 3 it is proved that $(]0, \infty[, M_0, \cdot)$ is complete. Also, we construct the completion of $(]0, \infty[, M^*, \cdot)$ where M^* is given by the above expression. In Section 4 we study some aspects on the continuity of M_0. In Section 5 an extension of M^* (defined on $[0, \infty[$) to $\mathbb{R}$ is constructed. In Section 6 we study some aspects about contractivity with respect to M_0, and, finally, in Section 7 we show a new application of these fuzzy metrics.

2 Preliminaries

Let us recall, [47], that a *continuous t-norms* is a binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ such that $([0, 1], \leq, *)$ is an ordered Abelian topological monoid with unit 1.

Definition 1 *(George and Veeramani [7]). A fuzzy metric space is an ordered* *triple* $(X, M, *)$ *such that* X *is a (non-empty) set,* $*$ *is a continuous* t*-norm and* M *is a fuzzy set on* $X \times X \times]0, \infty[$ *satisfying the following conditions, for all* $x, y, z \in X,\ s, t > 0$*:*

(GV1) $M(x, y, t) > 0$;
(GV2) $M(x, y, t) = 1$ *if and only if* $x = y$;
(GV3) $M(x, y, t) = M(y, x, t)$;
(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(GV5) $M(x, y, \_) :]0, \infty[\to]0, 1]$ *is continuous.*

The continuous t-norms used in this paper are the minimum, denoted by $\wedge$, the usual product, denoted by $\cdot$, and the Lukasievicz t-norm, denoted by $\mathfrak{L}$ ($x \mathfrak{L} y = max\{0, x + y - 1\}$).

If $(X, M, *)$ is a fuzzy metric space, we will say that $(M, *)$ is a *fuzzy metric* on X. Also, if confusion is not possible, we will say that (X, M) is a fuzzy metric space or M is a fuzzy metric on X. This terminology will be also extended along the paper in other concepts, as usual, without explicit mention.

The following is a well-known result.

Lemma 2 *(Grabiec [10]) The real function* $M(x, y, \_)$ *of Axiom (GV5) is increasing for all* $x, y \in X$.

In the definition of Kramosil and Michalek, [25], M is a fuzzy set on $X^2 \times [0, \infty[$ that satisfies (GV3) and (GV4), and (GV1), (GV2), (GV5) are replaced by (KM1), (KM2), (KM5), respectively, below:

(KM1) $M(x, y, 0) = 0$;
(KM2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;
(KM5) $M(x, y, \_) : [0, \infty[\to [0, 1]$ is left continuous.

We will refer to these fuzzy metric spaces as KM fuzzy metric spaces. It is worth nothing that, by defining the probabilistic metric $F_{xy}(t) = M(x, y, t)$, every KM fuzzy metric space $(X, M, *)$ becomes a generalized Menger space, [41], under the continuous t-norm $*$. On the other hand a fuzzy metric space can be considered a KM fuzzy metric space if we extend M defining $M(x, y, 0) = 0$ for all $x, y \in X$.

George and Veeramani proved in [7] that every fuzzy metric M on X generates a topology τ_M on X which has a base the family of open sets of the form $\{B_M(x, \epsilon, t) : x \in X, 0 < \epsilon < 1, t > 0\}$, where $B_M(x, \epsilon, t) = \{y \in X : M(x, y, t) > 1 - \epsilon\}$ for all $x \in X$, $\epsilon \in]0, 1[$ and $t > 0$.

Let (X, d) be a metric space and let M_d a function on $X \times X \times]0, \infty[$ defined by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

Then $(X, M_d, \cdot)$ is a fuzzy metric space, [7], and M_d is called the *standard fuzzy metric* induced by d. The topology τ_{M_d} coincides with the topology on X deduced from d.

Definition 3 *A fuzzy metric M on X is said to be stationary, [17], if M does not depend on t, i.e. if for each $x, y \in X$, the function $M_{x,y}(t) = M(x, y, t)$ is constant. In this case we write $M(x, y)$ instead of $M(x, y, t)$.*

Proposition 4 *(George and Veeramani [7]). A sequence $(x_n)_n$ in X converges to x if and only if $\lim_n M(x_n, x, t) = 1$, for all $t > 0$.*

Definition 5 *(George and Veeramani [7]), Schweizer and Sklar [48]). A sequence $(x_n)_n$ in a fuzzy metric space (X, M) is said to be M-Cauchy if for each $\epsilon \in]0, 1[$ and each $t > 0$ there is $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$. Equivalently, $(x_n)_n$ is M-Cauchy if $\lim_{n,m} M(x_n, x_m, t) = 1$, where $\lim_{n,m}$ denotes the double limit as $n \to \infty$, and $m \to \infty$. X is called M-complete if every Cauchy sequence in X is convergent with respect to τ_M. In such a case M is also said to be complete.*

If confusion is not possible we will say, simply, that $(x_n)_n$ is Cauchy.

Definition 6 *(Gregori et al. [12]). We say that the fuzzy metric space $(X, M, *)$ is principal (or simply, M is principal) if $\{B_M(x, r, t) : r \in]0, 1[\}$ is a local base at $x \in X$, for each $x \in X$ and each $t > 0$.*

Definition 7 *(Gregori and Romaguera [16]). Let (X, M) and (Y, N) be two fuzzy metric spaces. A mapping f from X to Y is called an* isometry *if for each $x, y \in X$ and $t > 0$, $M(x, y, t) = N(f(x), f(y), t)$ and, in this case, if f is a bijection, X and Y are called* isometric. *A* fuzzy metric completion *of (X, M) is a complete fuzzy metric space (X^*, M^*) such that (X, M) is isometric to a dense subspace of X^*. X is called* completable *if it admits a fuzzy metric completion.*

Proposition 8 *(Gregori and Romaguera [16]). If a fuzzy metric space has a fuzzy metric completion then it is unique up to isometry.*

Remark 9 *Suppose $(X^*, M^*, \diamond)$ is a fuzzy metric completion of $(X, M, *)$. Attending to the last proposition and the construction of the completion, [17], we can consider that $X \subset X^*$, $\diamond$ is $*$, and that M^* is defined on X^* by*

$$M^*(x, y, t) = \lim_n M(x_n, y_n, t)$$

for all $x, y \in X^*, t > 0$, where $(x_n)_n$ and $(y_n)_n$ are sequences in X that converges to x and y, respectively.

In [17] is given the following characterization about completion of a fuzzy metric space.

Theorem 10 *Let $(X, M, *)$ be a fuzzy metric space, and let $(a_n)_n$ and $(b_n)_n$ be two Cauchy sequences in X. Then $(X, M, *)$ is completable if and only if it satisfies the following conditions:*

(C1) The function $t \to \lim_n M(a_n, b_n, t)$ is a continuous function on $]0, \infty[$ with values in $]0, 1]$.

(C2) If $\lim_n M(a_n, b_n, s) = 1$ for some $s > 0$ then $\lim_n M(a_n, b_n, t) = 1$ for all $t > 0$.

Remark 11 *Cauchy sequences are defined in the same way in fuzzy metric spaces and KM fuzzy metric spaces. Then it is easy to verify, [43], that a fuzzy metric space (X, M) is complete if and only if the corresponding KM fuzzy metric space is also complete. Further if (X, M) admits completion this completion agrees with the completion of the corresponding KM fuzzy metric space. Recall that every KM fuzzy metric space has a completion which is unique up to an isometry, [43,51].*

It will be left to the reader to point out the analogies or differences between the results obtained for fuzzy metric spaces and the corresponding ones for KM fuzzy metric spaces, in the next sections.

Definition 12 *Let $(X, M, *)$ be a fuzzy metric space. The fuzzy metric M (or the fuzzy metric space $(X, M, *)$) is said to be* strong *if it satisfies for each $x, y, z \in X$ and each $t > 0$*

$$M(x, z, t) \geq M(x, y, t) * M(y, z, t) \ (GV4')$$

Let $(X, M, *)$ be a non-stationary fuzzy metric. Define the family of functions $\{M_t : t > 0\}$ where, for each $t > 0$, $M_t : X^2 \to]0, 1]$ is given by $M_t(x, y) = M(x, y, t)$. Then $(X, M, *)$ is strong if and only if $(X, M_t, *)$ is a stationary fuzzy metric for each $t > 0$. In this case we will say that $\{M_t : t > 0\}$ is the *family of stationary fuzzy metrics associated* to M. Clearly, this family characterizes M in the sense that $M(x, y, t) = M_t(x, y)$ for all $x, y \in X, t > 0$. If $(X, M, *)$ is strong then $\tau_M = \bigvee\{\tau_{M_t} : t > 0\}$.

Moreover, it is easy to verify that the sequence $(x_n)_n$ in X is M-Cauchy if and only if $(x_n)_n$ is M_t-Cauchy for each $t > 0$.

Remark 13 *(About terminology) If* $(X, M, \wedge)$ *is strong then (GV4') becomes*

$$M(x,z,t) \geq min\{M(x,y,t), M(y,z,t)\} \ (GV4'')$$

and in this case we say that M *is a* fuzzy ultrametric *[13].*

Let d *be a metric on* X*. Now, we can consider the standard fuzzy metric* M_d *on* X*. Further, if* $d(x,y) < 1$ *for all* $x, y \in X$ *then we can also consider the stationary fuzzy metric* $(N, \mathfrak{L})$ *on* X*, where* $N(x,y) = 1 - d(x,y)$*. Then* d *is an ultrametric (a non-Archimedean metric) if and only if* M_d *is a fuzzy ultrametric, [44], if and only if* N *is a fuzzy ultrametric [13]. Further, condition (GV4") is stronger than (GV4) in the same way that* $d(x,z) \leq max\{d(x,y), d(y,z)\}$ *is stronger than the usual triangular inequality.*

Following terminology of probabilistic metric spaces, [11,22], some authors call non-Archimedean fuzzy metrics those that also satisfy equation (GV4'). Notice that in this case there is not any correspondence, in the above sense, between non-Archimedean metrics and non-Archimedean fuzzy metrics since M_d always satisfies $M_d(x,z,t) \geq M_d(x,y,t) \cdot M_d(y,z,t)$ and also because all stationary fuzzy metrics would be non-Archimedean. Further (GV4') is not stronger than (GV4) and it means that if we replace (GV4) by (GV4') then M could not be a fuzzy metric on X (Indeed, $M(x,y,t) = \frac{1/t}{1/t + d(x,y)}$ satisfies (GV1)-(GV3), (GV4') and (GV5) and it does not satisfies (GV4)).

From now on $\mathbb{R}$ and $\mathbb{N}$ will denote the sets of real numbers and positive integers, respectively.

3 Introducing the examples. On completeness and completion.

Throughout the paper $(]0, \infty[, M_0, \cdot)$ will be the stationary fuzzy metric space where M_0 is defined by $M_0(x,y) = \frac{min\{x,y\}}{max\{x,y\}}$, [7]. It is easy to verify that τ_{M_0} is the usual topology of $\mathbb{R}$ restricted to $]0, \infty[$.

Also, $([0, \infty[, M^*, \cdot)$ will be the fuzzy metric space where M^* is defined by $M^*(x,y,t) = \frac{min\{x,y\}+t}{max\{x,y\}+t}$, [56]. Its subspace $(]0, \infty[, M^*, \cdot)$ will take an interesting role in this section.

We omit the proof of the next proposition.

Proposition 14 *Consider the fuzzy metric* M^* *on* $[0, \infty[$ *(respectively, on* $]0, \infty[$*).*

(i) τ_{M^*} *is the usual topology of* $\mathbb{R}$ *restricted to* $[0, \infty[$ *(respectively, to* $]0, \infty[$*).*

(ii) M^* *is principal.*

(iii) M^* *is strong.*

Since M^* *is strong so we can consider its associated family of stationary fuzzy metrics* $\{M_t^* : t > 0\}$ *defined on* $[0, \infty[$ *(respectively, on* $]0, \infty[$*), i.e.* $M_t^*(x, y) = M^*(x, y, t)$*, for each* $t > 0$*, and by (ii) we have:*

(iv) $\tau_{M_t^*}$ *is the usual topology of* $\mathbb{R}$ *restricted to* $[0, \infty[$ *(respectively, to* $]0, \infty[$*), for each* $t > 0$.

The infimum (denoted by $\wedge$) of a family of stationary fuzzy metrics associated to a strong fuzzy metric was studied in [13]. In the case of M^* we have the next proposition.

Proposition 15

(i) Consider M^* *on* $[0, \infty[$*. Then* $\bigwedge_{t>0} M_t^*$ *is not a fuzzy metric on* $[0, \infty[$.

(ii) Consider M^* *on* $]0, \infty[$*. Then* $\bigwedge_{t>0} M_t^*$ *is the fuzzy metric* M_0.

Proof. (i) If we take $y \neq 0$ then $\bigwedge_{t>0} M_t^*(0, y) = \inf\left\{\frac{t}{y+t} : t > 0\right\} = 0$ and then $\bigwedge_{t>0} M_t^*$ is not a fuzzy metric on $[0, \infty[$.
(ii) For each $x, y, t \in]0, \infty[$ we have that $\bigwedge_{t>0} M_t^*(x, y) = \inf\left\{\frac{\min\{x,y\}+t}{\max\{x,y\}+t} : t > 0\right\} = \frac{\min\{x,y\}}{\max\{x,y\}} > 0$ and so, $\bigwedge_{t>0} M_t^*$ is the fuzzy metric M_0.

From now on, for simplicity, by a convergent sequence (in reference to τ_{M^*} or τ_{M_0}) we mean that it is convergent with respect to the usual topology of $\mathbb{R}$ restricted to the corresponding domain.

Taking into account Remark 11 we could obtain the next theorem using results of KM fuzzy metric spaces, [41], but we choose to prove it, since it is illustrative within the context of the paper (see Remark 24).

Theorem 16 $(]0, \infty[, M_0, \cdot)$ *is complete.*

Proof. Recall that τ_{M_0} is the usual topology of $\mathbb{R}$ restricted to $]0, \infty[$. We will characterize the M_0-Cauchy sequences.
Firstly, we will see that M_0-Cauchy sequences in $]0, \infty[$ are bounded for the usual metric of $\mathbb{R}$. Indeed, if $(a_n)_n$ is a non-bounded sequence in $]0, \infty[$, then for a given $\epsilon \in]0, \infty[$ and for any $n \in \mathbb{N}$ we can find $m \in \mathbb{N}$ with $m > n$ such that $\epsilon \cdot a_m > a_n$ and so $M_0(a_n, a_m) = \frac{a_n}{a_m} < \epsilon$ and thus $(a_n)_n$ is not M_0-Cauchy.

Now we will see that if $(a_n)_n$ is a sequence in $]0,\infty[$ that converges to 0 then $(a_n)_n$ is not M_0-Cauchy. Indeed, if $(a_n)_n$ converges to 0 then for a fixed $\epsilon \in]0,1[$ and for any $n \in \mathbb{N}$ we can find $m \in \mathbb{N}$ with $m > n$ such that $a_m < \epsilon \cdot a_n$ and so $M_0(a_n, a_m) = \frac{a_m}{a_n} < \epsilon$ and then $(a_n)_n$ is not M_0-Cauchy.

Finally, we will see that if $(a_n)_n$ is an M_0-Cauchy sequence in $]0,\infty[$ then $(a_n)_n$ converges in $]0,\infty[$. Let $(a_n)_n$ an M_0-Cauchy sequence in $]0,\infty[$ and hence, as we have seen above, $(a_n)_n$ is bounded. Then there exist $a \in [0,\infty[$ and a subsequence $(a_{n_i})_i$ of $(a_n)_n$ such that $\lim_i a_{n_i} = a$. Now, $(a_{n_i})_i$ is also an M_0-Cauchy sequence and hence, for the last paragraph, $a > 0$. We will show that $(a_n)_n$ converges to a.

If $(a_n)_n$ does not converges to a then there exist $\delta' > 0$ such that infinite terms of $(a_n)_n$ are in (the compact of $\mathbb{R}$) $I = [0, a-\delta'] \cup [a+\delta', K]$, where K is an upper bound of $(a_n)_n$. Then there exist a subsequence $(a_{n'_j})_j$ of $(a_n)_n$ in I and $b \in I$ such that $\lim_j a_{n'_j} = b$, and, as above, $b > 0$. Suppose that $b < a$. Let $\delta > 0$ with $\delta < \min\left\{b, \frac{a-b}{3}\right\}$ and let $\epsilon = \frac{b+\delta}{a-\delta} > 1$. Since $\lim_i a_{n_i} = a$ and $\lim_j a_{n'_j} = b$ then there exists $p \in \mathbb{N}$ such that $a_{n_i} \in]a-\delta, a+\delta[$ for each $i \geq p$ and $a_{n'_j} \in]b-\delta, b+\delta[$ for each $j \geq p$.

Given $n \in \mathbb{N}$ we choose $q_n = \max\{n, p\}$ and then for $i, j \geq q_n$ we have $M_0(a_{n_i}, a_{n'_j}) < \frac{b+\delta}{a-\delta} = \epsilon$ and so $(a_n)_n$ is not M_0-Cauchy, a contradiction.

A similar argument can be made if $b > a$.

In consequence $(a_n)_n$ is M_0-Cauchy iff $(a_n)_n$ converges in $]0,\infty[$.

Since a compact fuzzy metric space is precompact and complete, [15], then we have the next corollary.

Corollary 17 $(]0,\infty[, M_0, \cdot)$ *is not precompact.*

Proposition 18 $(]0,\infty[, M_t^*, \cdot)$ *is not complete for each* $t > 0$.

Proof. Recall that $\tau_{M_t^*}$ is the usual topology of $\mathbb{R}$ restricted to $]0,\infty[$, for each $t > 0$.

Now, the sequence $(\frac{1}{n})_n$ is not convergent in $]0,\infty[$ because $0 \notin]0,\infty[$, but it is M_t^*-Cauchy for each $t > 0$. Indeed,

$$\lim_{m,n} M_t^*\left(\frac{1}{n}, \frac{1}{m}\right) = \lim_{m,n} \frac{\min\{\frac{1}{n}, \frac{1}{m}\} + t}{\max\{\frac{1}{n}, \frac{1}{m}\} + t} = 1, \text{ for each } t > 0.$$

In the proof of the last proposition we have just obtained that $(\frac{1}{n})_n$ is Cauchy in $(]0,\infty[, M^*, \cdot)$ and so the next corollary is immediate.

Corollary 19 $(]0,\infty[, M^*, \cdot)$ *is not complete.*

Lemma 20 *Take $t > 0$ and consider the fuzzy metric space $(]0,\infty[, M_t^*, \cdot)$. Let $(x_n)_n$ be a sequence in $]0,\infty[$. Then $(x_n)_n$ is M_t^*-Cauchy if and only if $(x_n)_n$ converges in $[0,\infty[$.*

Proof. Fix $t > 0$, and let $(x_n)_n$ be an M_t^*-Cauchy sequence in $]0,\infty[$. Then $\lim_{m,n} M_t^*(x_n, x_m) = \lim_{m,n} \frac{\min\{x_n,x_m\}+t}{\max\{x_n,x_m\}+t} = 1$, but this expression is equivalent to $\lim_{m,n} \frac{\min\{x_n+t,x_m+t\}}{\max\{x_n+t,x_m+t\}} = 1$ and so $(x_n + t)_n$ is an M_0-Cauchy sequence in $]0,\infty[$, so by Theorem 16 $(x_n + t)_n$ converges in $]0,\infty[$, then $(x_n)_n$ is convergent and clearly $(x_n)_n$ converges in $[0,\infty[$.

Conversely, if $(x_n)_n$ converges in $]0,\infty[$, then clearly it is M_t^*-Cauchy for each $t > 0$. Now, suppose $(x_n)_n$ is a sequence in $]0,\infty[$ that converges to 0. Then, $\lim_{m,n} \min\{x_n, x_m\} = \lim_{m,n} \max\{x_n, x_m\} = 0$ and therefore, for a fixed $t > 0$ we have that $\lim_{m,n} M_t^*(x_n, x_m) = \lim_{m,n} \frac{\min\{x_n,x_m\}+t}{\max\{x_n,x_m\}+t} = 1$, and so $(x_n)_n$ is M_t^*-Cauchy.

Since M^* is strong by the above lemma we have the next corollary.

Corollary 21 *Consider the fuzzy metric space $(]0,\infty[, M^*, \cdot)$. Then a sequence $(x_n)_n$ in $]0,\infty[$ is M^*-Cauchy if and only if $(x_n)_n$ converges in $[0,\infty[$.*

Theorem 22 $(]0,\infty[, M^*, \cdot)$ *is completable.*

Proof. Let $(a_n)_n$ and $(b_n)_n$ be two M^*-Cauchy sequences in $(]0,\infty[, M^*, \cdot)$. First we will prove that $(C1)$ of Theorem 10 is satisfied.

From [45,13] $(a_n)_n$ and $(b_n)_n$ are M_t^*-Cauchy sequences in $]0,\infty[$ for all $t > 0$ and so, by the previous lemma, $(a_n)_n$ and $(b_n)_n$ converge to a and b, respectively, in $[0,\infty[$.

Suppose, without lost of generality, that $a \leq b$. Then, it is an easy exercise to prove that $\lim_n(min\{a_n, b_n\}) = a$ and $\lim_n(max\{a_n, b_n\}) = b$.

Thus, for $t > 0$ we have that

$$\lim_n M^*(a_n, b_n, t) = \lim_n \frac{\min\{a_n, b_n\} + t}{\max\{a_n, b_n\} + t} = \frac{a + t}{b + t} > 0.$$

We have just obtained that the function $t \longrightarrow \lim_n M^*(a_n, b_n, t)$ is a continuous function on $]0, \infty[$ with values in $]0, 1]$, and $(C1)$ of Theorem 10 is satisfied.

Next we will prove that $(C2)$ of Theorem 10 is also satisfied.

Suppose that for some $t_0 > 0$ $\lim_n M^*(a_n, b_n, t_0) = \lim_n \frac{\min\{a_n,b_n\}+t_0}{\max\{a_n,b_n\}+t_0} = 1$. Then, as we have seen in the first part of the proof, we can assert that there exist $\lim_n(\min\{a_n, b_n\})$ and $\lim_n(\max\{a_n, b_n\})$ and obviously, in this case, $\lim_n(\min\{a_n, b_n\}) = \lim_n(\max\{a_n, b_n\})$. Consequently

$$\lim_n M^*(a_n, b_n, t) = \lim_n \frac{\min\{a_n, b_n\} + t}{\max\{a_n, b_n\} + t} = 1, \text{ for all } t > 0$$

and $(C2)$ of Theorem 10 is satisfied. So $(]0, \infty[, M^*, \cdot)$ is completable.

The completion of $(]0, \infty[, M^*, \cdot)$**.**

Denote by $(\tilde{X}, \tilde{M}, \cdot)$ the completion of $(]0, \infty[, M^*, \cdot)$. By Corollary 21 M^*-Cauchy sequences in $]0, \infty[$ are the convergent sequences in $[0, \infty[$, then attending to [17] we can identify the equivalent class of M^*-Cauchy sequences in $]0, \infty[$ that converge to $p \in [0, \infty[$ with p and so $\tilde{X}$ is identified with $[0, \infty[$.

Now, attending to Remark 9 the fuzzy completion $\tilde{M}$ of M^* is defined in a such manner that if $(a_n)_n$ is a convergent sequence to 0 and $b \in]0, \infty[$ then for $t > 0$, $\tilde{M}(0, b, t) = \tilde{M}(b, 0, t) = \lim_n \frac{\min\{a_n,b\}+t}{\max\{a_n,b\}+t} = \frac{t}{b+t}$. On the other hand $\tilde{M}(0, 0, t) = 1$ for all $t > 0$ and then $\tilde{M}$ is given by $\tilde{M}(a, b, t) = \frac{\min\{a,b\}+t}{\max\{a,b\}+t}$ for each $a, b \in [0, \infty[$, $t > 0$ and therefore $\tilde{M}$ is the fuzzy metric M^* on $[0, \infty[$ defined at the beginning of this section.

From [13] Theorem 40, the following corollary is immediate.

Corollary 23 $([0, \infty[, M_t^*, \cdot)$ *is the completion of* $(]0, \infty[, M_t^*, \cdot)$ *for each* $t > 0$.

Remark 24 *Using similar arguments to the above ones in Theorem 16 one can shows that* $([0, \infty[, M^*, \cdot)$ *is complete. Now, the mapping* $i : (]0, \infty[, M^*, \cdot) \to ([0, \infty[, M^*, \cdot)$ *given by* $i(x) = x$ *for each* $x \in]0, \infty[$, *is an isometry and by* (i) *of Proposition 14* $]0, \infty[$ *is dense in* $([0, \infty[, \tau_{M^*})$, *and since the completion of a fuzzy metric space is unique, up to isometry [16], then* $([0, \infty[, M^*, \cdot)$ *is the completion of* $(]0, \infty[, M^*, \cdot)$.

For obtaining the completion of $(]0, \infty[, M^*, \cdot)$ *we have preferred the above* *constructive method because it allows us to introduce in its appropriate context the following open question.*

Problem 25 *To find a fuzzy metric space $(X, M, *)$ where for two M-Cauchy sequences $(a_n)_n$ and $(b_n)_n$ in X the assignment $f(t) = \lim_n M(a_n, b_n, t)$ for all $t > 0$, does not define a continuous function on t.*

It is known that the completion of a strong fuzzy metric is strong, [13] Lemma 39. On the other hand we have just obtained above that the completion of the principal fuzzy metric space $(]0, \infty[, M^*, \cdot)$ is $([0, \infty[, M^*, \cdot)$, which is also principal. Now, the next is an open question.

Problem 26 *If the principal fuzzy metric space $(X, M, *)$ admits completion $(\tilde{X}, \tilde{M}, *)$, is it also principal?*

4 On continuity and uniform continuity

We have just seen above that the fuzzy metric M^* on $]0, \infty[$ can be extended to $[0, \infty[$ by means of the fuzzy metric $\tilde{M}$ in such a manner that $]0, \infty[$ is dense in $([0, \infty[, \tau_{\tilde{M}})$. Now, this situation is not possible for $(]0, \infty[, M_0)$ as shows the next proposition.

Proposition 27 *Consider the fuzzy metric space $(]0, \infty[, M_0, \cdot)$ and let $\tilde{M_0}$ an extension of M_0 to $[0, \infty[$. Then $\{0\}$ is $\tau_{\tilde{M_0}}$-open.*

Proof. $]0, \infty[$ is $\tilde{M_0}$-complete and then it is $\tau_{\tilde{M_0}}$-closed.

Consequently, we cannot find an extension $\tilde{M_0}$ of M_0 such that $\tau_{\tilde{M_0}}$ coincides with the usual topology of $\mathbb{R}$ restricted to $[0, \infty[$.

Example 28 *The fuzzy metric $\tilde{M_0}$ on $[0, \infty[$ given by*

$$\tilde{M_0}(y, x) = \tilde{M_0}(x, y) = \begin{cases} M_0(x, y), & x, y \in]0, \infty[\\ \frac{1}{2y}, & x = 0, y \geq 1 \\ \frac{y}{2}, & x = 0, y < 1 \\ 1, & x = y = 0 \end{cases}$$

is an extension of M_0 to $[0, \infty[$ and $\{0\}$ is clearly open of $\tau_{\tilde{M_0}}$.

From [42] we know that $M_0(x,y)$ *is continuous on* $]0,\infty[^2$ *(endowed with the producto topology). Now, the continuous function* M_0 *does not admit any continuous extension* N *to* $[0,\infty[^2$ *endowed with the usual topology of* $\mathbb{R}$*. Indeed, if* N *were so, then since* $(\frac{1}{n})_n$ *and* $(\frac{1}{n^2})_n$ *converge to* 0 *it should be* $N(0,0) = \lim_n M_0(1/n, 1/n) = 1$ *and also* $N(0,0) = \lim_n M_0(1/n, 1/n^2) = \lim_n \frac{1/n^2}{1/n} = 0$*, a contradiction.*

Definition 29 *We will say that the fuzzy metrics* M_1 *and* M_2 *on* X *are uniformly equivalent if the identity mappings* $i : (X, M_1) \to (X, M_2)$ *and* $i : (X, M_2) \to (X, M_1)$ *are uniformly continuous [8]. In that case, obviously* $(x_n)_n$ *is an* M_1*-Cauchy sequence if and only if* $(x_n)_n$ *is an* M_2*-Cauchy sequence.*

Now the fuzzy metrics M^* and M_0 on $]0,\infty[$ are topologically equivalent on $]0,\infty[$, i.e. $\tau_{M^*} = \tau_{M_0}$ on $]0,\infty[$, but they are not uniformly equivalent on $]0,\infty[$ because $(]0,\infty[, M_0)$ is complete but $(]0,\infty[, M^*)$ is not complete (Notice that the identity mapping $i : (]0,\infty[, M_0) \to (]0,\infty[, M^*)$ is uniformly continuous since $M_0(x,y) \leq M^*(x,y,t)$ for each $x, y \in]0,\infty[$, $t > 0$, but $i : (]0,\infty[, M^*) \to (]0,\infty[, M_0)$ is not uniformly continuous since $(\frac{1}{n})_n$ is a Cauchy sequence in $(]0,\infty[, M^*)$ but it is not M_0-Cauchy).

Definition 30 *(Gregori, Romaguera and Sapena [19]) Let* $(X, M, *)$ *be a fuzzy metric space. A mapping* $f : X \to \mathbb{R}$ *is called* $\mathbb{R}$-uniformly continu*ous if given* $\epsilon > 0$ *we can find* $s > 0$, $\delta \in]0,1[$ *such that* $M(x,y,s) > 1 - \delta$ *implies* $|f(x) - f(y)| < \epsilon$*.*

Proposition 31 *Consider the fuzzy metric space* $(]0,\infty[, M_0)$*. For a fixed* $y > 0$ *the mapping* $M_0^y :]0,\infty[\to]0,\infty[$ *given by* $M_0^y(x) = \frac{\min\{x,y\}}{\max\{x,y\}}$ *for all* $x \in]0,\infty[$ *is* $\mathbb{R}$*-uniformly continuous.*

Proof. Let $\epsilon > 0$. We distinguish three cases: (a) $x, x' \leq y$, (b) $x, x' \geq y$, (c) $x \leq y, x' > y$ (or $x' \leq y, x > y$).

(a) Choose $\delta \in]0,1[$ with $\delta < \epsilon$. Suppose that $x, x' \in]0,\infty[$ satisfy $M_0(x,x') > 1 - \delta$. Without lost of generality we can suppose $x \leq x'$. Then we have that $\frac{x}{x'} > 1 - \delta$ and hence

$$|M_0^y(x') - M_0^y(x)| = \frac{x'}{y} - \frac{x}{y} = \frac{1}{y}(x' - x) < \frac{1}{y}(x' - x'(1-\delta)) = \frac{x'}{y}\delta \leq \delta < \epsilon$$

With similar arguments the other cases can be proved, and then M_0^y is $\mathbb{R}$-uniformly continuous.

The next is an open question.

Problem 32 *Let $(X, N, *)$ be a stationary fuzzy metric space. Is the real function $N_y(x) = N(x, y)$ for each $x \in X$, $\mathbb{R}$-uniformly continuous for all $y \in X$?*

5 Extending fuzzy metrics

5.1 A related fuzzy pseudo-metric

Consider the fuzzy set N on $\mathbb{R}^2 \times]0, \infty[$ given by

$$N(x, y, t) = \frac{\min\{|x|, |y|\} + t}{\max\{|x|, |y|\} + t} \tag{1}$$

It is easy to verify that N satisfies axioms (GV1),(GV3) and (GV5). Also, N satisfies the triangular inequality. Indeed, for $x, y, z \in \mathbb{R}$, $t > 0$ we have

$$N(x, z, t+s) = \frac{\min\{|x|, |z|\} + t + s}{\max\{|x|, |z|\} + t + s} = M^*(|x|, |z|, t+s) \geq$$

$$\geq M^*(|x|, |y|, t) \cdot M^*(|y|, |z|, s) = \frac{\min\{|x|, |y|\} + t}{\max\{|x|, |y|\} + t} \cdot \frac{\min\{|y|, |z|\} + s}{\max\{|y|, |z|\} + s} =$$
$$= N(x, y, t) \cdot N(y, z, s)$$

Also, for $x = y$ we have that $N(x, y, t) = 1$ for all $t > 0$ but the converse is, in general, false since for $x \neq 0$ we have that $N(x, -x, t) = 1$ but $x \neq -x$. Consequently $(\mathbb{R}, N, \cdot)$ is a fuzzy pseudo-metric space, [18], but it is not a fuzzy metric space.

The mapping $j :]-\infty, 0] \to [0, \infty[$ defined by $j(x) = -x$ is a bijection and then $(]-\infty, 0], M', \cdot)$ and $([0, \infty[, M^*, \cdot)$ are two fuzzy isometric spaces [16], where M' is given by $M'(x, y, t) = M^*(j(x), j(y), t) = M^*(-x, -y, t) = M^*(|x|, |y|, t)$ for all $x, y \in]-\infty, 0]$, $t > 0$. So M' is, obviously, strong and principal.

Notice that M^* and M' can be defined both two in their corresponding domains by the expresion (1), i.e. $N|_{[0,\infty[} = M^*$ and $N|_{]-\infty,0]} = M'$.

Remark 33 *Section 5.1 admits the following easy generalization. Let $(M, *)$ be a fuzzy metric on a set of non-negative real numbers A. Put $-A = \{x \in \mathbb{R} : -x \in A\}$. Define $N(x, y, t) = M(|x|, |y|, t)$ for all $x, y \in -A \cup A$, $t > 0$. Then, $(N, *)$ is a fuzzy pseudo-metric on $-A \cup A$.*

5.2 A fuzzy metric extension of M^*

We have just seen that the fuzzy pseudometric N on $\mathbb{R}$ satisfies

$$N|_{[0,\infty[} = M^* \text{ and } N|_{]-\infty,0]} = M' \tag{2}$$

Now we will construct a fuzzy metric $\bar{M}$ on $\mathbb{R}$ such that $\bar{M}|_{[0,\infty[} = M^*$ and $\bar{M}|_{]-\infty,0]} = M'$. For it we consider the family $\{M_t^* : t > 0\}$ of stationary fuzzy metrics on $[0,\infty[$ associated to M^*, and the family $\{M_t' : t > 0\}$ of stationary fuzzy metrics on $]-\infty,0]$ associated to M'.

Then, since $]-\infty,0] \cap [0,\infty[= \{0\}$, from [14] Proposition 19 we have for each fixed $t > 0$ that the function

$$\bar{M}_t(x,y) = \begin{cases} M_t^*(x,y) & \text{if } x,y \in [0,\infty[\\ M_t'(x,y) & \text{if } x,y \in]-\infty,0] \\ M_t^*(x,0) \cdot M_t'(0,y) & \text{if } x \in]0,\infty[, y \in]-\infty,0[\\ M_t'(x,0) \cdot M_t^*(0,y) & \text{if } x \in]-\infty,0[, y \in]0,\infty[\end{cases}$$

is a stationary fuzzy metric on $\mathbb{R}$, such that $\bar{M}_t|_{]-\infty,0]} = M_t'$ and $\bar{M}_t|_{[0,\infty[} = M_t^*$.

Attending (2), we can be written

$$\bar{M}_t(x,y) = \begin{cases} \frac{\min\{|x|,|y|\}+t}{\max\{|x|,|y|\}+t} & x,y \in [0,\infty[\text{ or } x,y \in]-\infty,0] \\ \frac{t}{|x|+t} \cdot \frac{t}{|y|+t} & \text{elsewhere} \end{cases}$$

Obviously $\{\bar{M}_t : t > 0\}$ is an increasing family, i.e. $t < t'$ implies $\bar{M}_t \leq \bar{M}_{t'}$.

Now we define $\bar{M}(x,y,t) = \bar{M}_t(x,y)$ for all $x,y \in \mathbb{R}$, $t > 0$. Then, obviously $\bar{M}$ satisfies (GV1)-(GV3) and (GV5).

We prove that $\bar{M}$ satisfies the triangular inequality. Let $x,y,z \in \mathbb{R}$, $t,s > 0$. Then, since $\{\bar{M}_t : t > 0\}$ is an increasing family we have $\bar{M}(x,z,t+s) = \bar{M}_{t+s}(x,z) \geq \bar{M}_{t+s}(x,y) \cdot \bar{M}_{t+s}(y,z) \geq \bar{M}_t(x,y) \cdot \bar{M}_s(y,z) = \bar{M}(x,y,t) \cdot \bar{M}(y,z,s)$ and so $(\bar{M},\cdot)$ is a fuzzy metric on $\mathbb{R}$ which obviously satisfy $\bar{M}|_{[0,\infty[} = M^*$ and $\bar{M}|_{]-\infty,0]} = M'$.

The following is an open question.

Problem 34 *Let H and K be two distinct sets with $H \cap K \neq \emptyset$. Let $(M_H,*)$ and $(M_K,*)$ be two non-stationary fuzzy metrics on H and K, respectively, that agree in $H \cap K$. Does it exist a fuzzy metric M on $H \cup K$ such that $M|_H = M_H$ and $M|_K = M_K$?*

6 Contractivity in $(]0,\infty[, M_0, \cdot)$

6.1 On contractivity

Let (X, M) be a fuzzy metric space.

In order to obtain satisfactory results in the fuzzy setting, related to the classical Banach contraction theorem, several concepts of M-contractivity on a mapping $f : (X, M) \to (X, M)$ have been given, for instance [10,20,21,30–33,41,49,50,52,54,55] among others.

The weaker contractivity condition on f which makes sense when M is stationary is given by the formula

$$M(f(x), f(y)) \geq M(x, y) \text{ for } x, y \in X$$

and in fact, it is obtained from the concept of B-contraction, [10,49], given by the expression $M(f(x), f(y), kt) \geq M(x, y, t)$ for all $x, y \in X$, $t > 0$ and some fixed $k \in]0, 1[$. Now, for stationary fuzzy metrics this concept is not really appropriate (in the same way that the contractivity condition $d(f(x), f(y)) \leq d(x, y)$ is not appropriate for a metric space (X, d)). Indeed, the identity mapping $i : X \to X$ satisfies $M(f(x), f(y)) = M(x, y)$ for all $x, y \in X$ and all points of X are fixed of i. Further, in the case of the fuzzy metric space $(]0, \infty[, M_0, \cdot)$ the mapping $f :]0, \infty[\to]0, \infty[$ given by $f(x) = ax$, where $a \in \mathbb{R}^+ \sim \{1\}$, also satisfies $M(f(x), f(y)) = M(x, y)$ for all $x, y \in]0, \infty[$ but f has not any fixed point. Then, a stronger contractivity condition than the above one is needed. So, we adopt the next definition.

Definition 35 *Let M be a stationary fuzzy metric on X. A mapping $f : X \to X$ is fuzzy M-contractive (a fuzzy contraction) if*

$$M(f(x), f(y)) > M(x, y) \text{ for } x, y \in X, x \neq y \tag{3}$$

This concept comes from the fuzzy Edelstein contractives notion stated by Grabiec [10] as $M(f(x), f(y), t) > M(x, y, t)$ for $x, y \in X$, $x \neq y$, $t > 0$, where M is a fuzzy metric on X. The author proved that a fuzzy Edelstein contractive mapping on a compact KM fuzzy metric space has a unique fixed point.

Notice that (3) is satisfied by almost all fuzzy M-contractive concepts in the literature when M is stationary.

We can get a class of fuzzy M_0-contractive mappings with a unique fixed point in $]0, \infty[$ as follows. Consider the continuous increasing functions $f : [0, \infty[\to$

$[0,\infty[$ with $f(0)=0$ such that $f''(x)<0$ for all $x\in]0,\infty[$ (f'' denotes the second derivative of f). Using arguments from Analysis one can verify that for $0<x<y$ it is satisfied that $\frac{f(x)}{x}>\frac{f(y)}{y}$, i.e. $\frac{f(x)}{f(y)}>\frac{x}{y}$ and hence f is fuzzy M_0-contractive. It is easy to verify that such functions have at most a unique fixed point in $]0,\infty[$. Further, f has a (unique) fixed point if and only if $f'(x)=1$ for some $x\in]0,\infty[$. Notice that $\ln(1+x)$ satisfies $f''(x)<0$ for $x\in]0,\infty[$ but $f'(x)\neq 1$ for $x\in]0,\infty[$, and clearly $\ln(1+x)$ has not any fixed point in $]0,\infty[$. The mappings $f_\lambda(x)=\sqrt{x+\lambda}$ for $x\in]0,\infty[$, with a fixed $\lambda>0$, fulfill all conditions of this paragraph and they play an interesting role in the following.

Mihet [30] pointed out that the mapping $f(x)=x+a$ for $x\in]0,\infty[$, with a fixed $a>0$, is fuzzy M_0-contractive but it has not any fixed point in $]0,\infty[$. Then, in order to guarantee the existence of fixed points for such a mappings Mihet introduced and studied the next concept for KM fuzzy metric spaces that we rewrite in our context.

Definition 36 *Let $(X,M,*)$ be a fuzzy metric space and let φ be a decreasing continuous mapping $\varphi:[0,1]\to[0,1]$ such that $\varphi(t)>t$ for all $t\in]0,1[$. A mapping $f:X\to X$ is called φ-contractive if $M(f(x),f(y),t)\geq\varphi(M(x,y,t))$ for all $x,y\in X$, $t>0$. Obviously in this case f satisfies (3).*

The author proved, [32], that a fuzzy φ-contractive mapping in a strong complete fuzzy metric space has a unique fixed point.

As a consequence, since the above commented mappings $f(x)=x+a$ and $\ln(1+x)$ satisfy (3) and they have not any fixed point in $]0,\infty[$, these mappings are fuzzy M_0-contractive but they are not φ-contractive in $(]0,\infty[,M_0)$.

We see that the mappings $f_\lambda:]0,\infty[\to]0,\infty[$, with $\lambda>0$, defined by $f_\lambda(x)=\sqrt{x+\lambda}$ are φ-contractive. Indeed, if $x<y$ we have $M(f_\lambda(x),f_\lambda(y))=\frac{\sqrt{x+\lambda}}{\sqrt{y+\lambda}}\geq\sqrt{\frac{x}{y}}=\varphi(M(x,y))$ where $\varphi(t)=\sqrt{t}$, independently of $\lambda>0$.

Then each mapping f_λ has a unique fixed point $a_\lambda\in]0,\infty[$.

Now we can define the mapping $g:]0,\infty[\to]0,\infty[$ by $g(\lambda)=a_\lambda$. So, $g(\lambda)=\frac{1+\sqrt{1+4\lambda}}{2}$ and thus g is a continuous function on $]0,\infty[$. Then it arises the following question.

Problem 37 *Let $(X,M,*)$ be a strong complete fuzzy metric space and let $f_\lambda:X\to X$ be a family of φ-contractive mappings for the same function φ, for all $\lambda>0$. Write a_λ the unique fixed point of f_λ for each $\lambda>0$. Is the mapping $g:]0,\infty[\to X$ defined by $g(\lambda)=a_\lambda$ continuous?*

Remark 38 *This problem has been formulated according to the previous re-*

sults but obviously it admits other versions. We notice that the analogous problem formulated in metric spaces has positive answered [46].

7 Application of the fuzzy metric M_0 to measure perceptual colour differences

Apart from the interesting theoretical properties of the fuzzy metrics studied in previous sections, it is interesting as well to note that they have application in a variety of practical problems. Indeed, they have been previously used to filter colour images and to measure the degree of consistency of elements in a dataset [3,34,35,40].

Here we focus on a different application of the fuzzy metric M_0 that takes advantage of the homotetique invariant property that this fuzzy metric satisfies. Indeed, M_0 fulfills that, for any $\lambda \in \mathbb{R}$:

$$(I)\ M_0(\lambda x, \lambda y) = M_0(x, y)$$

Also, if $z > 0$,

$$(II)\ M_0(x + z, y + z) > M_0(x, y) \text{ if } x \neq y$$

As we will see later on, there exist practical problems where these properties are pretty interesting. However, in practical applications it is more appropriate to use the M^* fuzzy metric (which also satisfies (II)), instead of the M_0, because the presence of the t parameter makes this fuzzy metric more adaptive to the particular problem. On the other hand, M_0 is in fact M^* when $t = 0$. Notice that both M_0 and M^* are suitable only for scalar values and that for vector values the combination of several fuzzy metrics needs to be considered.

In particular, one application that matches the behaviour of these two fuzzy metrics regards the modeling of the perception of physical magnitudes such as colours, sounds or weights. It is known that the perception threshold of changes in these magnitudes increases as the magnitudes themselves increase [4,5,53]. That is to say, the perceived difference between two magnitude values x, y is different that for the values $x + k, y + k$, whenever $k > 0$. In particular, the perceived difference will be larger in the former case than in the latter, which agrees with (II). This situation can be observed in the case of perceptual colour differences and, since the M^* fuzzy metric behaves accordingly to this situation, M^* can be used to appropriately devise colour difference formulas as explained in the following.

A colour sample is usually represented as a tern in a particular colour space. Among the different colour spaces, a well-known one, specially in computer graphics, is the Hue-Chroma-Lightness (HCL) colour space [23], where a colour sample $\mathbf{s}$ is represented as a tern $\mathbf{s} = (H_s, C_s, L_s)$. In such a tern: Hue, H_s, is usually represented as an angle in $[0°, 360°]$ where 0°, 90°, 180°, and 270° correspond to approximately pure red, yellow, green and blue, respectively. $C_s \in [0, 100]$ represents the Chroma of the colour, where 0 is associated with neutral gray, black or white; and $L_s \in [0, 100]$ represents the Lightness of the sample, where 0 represents no lightness (absolute black colour) and 100 represents the maximum lightness (absolute white colour).

A series of experimental datasets: BFD-P, Leeds, RIT-Dupont, and Witt, which are combined to form the COM dataset, have been obtained in order to characterize the perceptual difference between pairs of colour samples [1,24,26,27,57,58]. In these datasets each pair of colour samples is associated with a value ΔV which represents the experimental perceptual difference between them. On the other hand, colour difference formulas are used to obtain, from two terns representing a pair of colour samples, the computed perceptual difference between them, usually denoted by ΔE. Since the objective of colour difference formulas is to model human perception, all formulas try to obtain ΔE values as close (or correlated) as possible to the ΔV values. One well-known colour difference formula is the CIELAB formula [59], that corresponds with the Euclidean distance in the CIELAB colour space.

The performance of a colour difference formula is assessed by measuring how close the ΔE values computed for the experimental datasets are to the ΔV values. A well established figure of merit for this closeness is the STRESS coefficient [28], which provides values in the interval $[0, 1]$, where lower values indicate a higher closeness. In Table 1, we can see that the value of STRESS for the CIELAB formula over the COM dataset is 0.428.

By analysing the experimental datasets, it has been observed that the sensitivity to differences in Chroma decreases as the value of Chroma increases. Notice that this fact is related to the Weber-Frechner and Stevens observations [4,5,53]. According to this, we propose to use the M^* fuzzy metric to model the similarity between two Chroma values C_s, C_r as

$$M^*(C_s, C_r) = \frac{min\{(C_s, C_r)\} + k_C}{max\{(C_s, C_r)\} + k_C},$$

where k_C is a parameter to adjust the behaviour as desired.

An analogous observation can be made with respect to Lightness. So we pro-

Fig. 1. Values of STRESS obtained by different colour difference formulas for the COM dataset.

Color difference formula	STRESS
CIELAB	0.428
CIE94	0.335
CIEDE2000	0.292
$\Delta E_{M_1^*}$	0.347
$\Delta E_{M_1^*}$	0.348

pose to measure the similarity between two Lightness values L_s, L_r as

$$M^*(L_s, L_r) = \frac{min\{(L_s, L_r)\} + k_L}{max\{(L_s, L_r)\} + k_L},$$

where k_L is another adjusting parameter.

Using these two expressions we build a more complex expression to obtain a new colour difference formula. We want also to take into account the CIELAB colour difference, ΔE^*_{ab}, so, we employ the standard fuzzy metric deduced from ΔE^*_{ab}. Given that the product of these fuzzy metrics is as well a fuzzy metric, [44], we can use a productory to join these three criteria. Finally, to obtain a difference formula we use the involutive negation as follows:

$$\Delta E_{M_1^*}(\mathbf{s}, \mathbf{r}) = 1 - \left(M^*(L_s, L_r) M^*(C_s, C_r) \frac{t}{t + \Delta E^*_{ab}} \right), \tag{4}$$

where k_L, k_C and t are parameters able to tune the importance of each criterion. However, since ΔE^*_{ab} also includes Lightness and Chroma differences, alternatively we propose to replace ΔE^*_{ab} in Eq. (4) with ΔH, which represents only Hue differences in ΔE^*_{ab} and is given by $\Delta H = \sqrt{\Delta E^{*2}_{ab} - |L_s - L_r|^2 - |C_s - C_r|^2}$, and so obtaining

$$\Delta E_{M_2^*}(\mathbf{s}, \mathbf{r}) = 1 - \left(M^*(L_s, L_r) M^*(C_s, C_r) \frac{t}{t + \Delta H} \right), \tag{5}$$

where we have three adjusting parameters, as above.

It is interesting to point out that $\Delta E_{M_1^*}$ can be seen as a modification of the ΔE^*_{ab} using a correction term inspired in the Weber-Fechner and Stevens laws which are represented by an appropriate fuzzy metric. On the other hand, $\Delta E_{M_2^*}$ is a color difference formula that corresponds with the representation of the Weber-Frechner and Stevens laws by means of fuzzy metrics.

We have performed extensive experimental assessments varying the values of the adjusting parameters k_L, k_C and t in the range $[0, 100]$ to obtain the optimal parameter setting for the formulas proposed in Eq. (4)-(5). With optimal parameter setting, $\Delta E_{M_1^*}$ is able to obtain a STRESS value for the COM dataset of 0.347 (with $k_L = 2, k_C = 4, t = 11$), whereas $\Delta E_{M_2^*}$ obtained STRESS of 0.348 (with $k_L = 4, k_C = 12, t = 40$). Notice that, in both cases, a significative improvement with respect to ΔE^*_{ab} is obtained. This means that M^* has been successfully used to take into account the facts related to the Weber-Fechner and Stevens laws. It should be also noted that whereas ΔE^*_{ab} does not incorporate these laws, they are considered in more recent colour difference formulas such as the CIE94 [60] and CIEDE2000 [61] formulas. We also compare the performance of the proposed formulas with these recent ones in Table 1, where we can see that the performance of our formulas are pretty close to the one of the CIE94.

8 Conclusions

In this paper we have studied from the mathematical point of view the fuzzy metric defined by $M^*(x, y, t) = \frac{min\{x,y\}+t}{max\{x,y\}+t}$ on $[0, \infty[$ (the set of non-negative real numbers) and other fuzzy metrics related to it. As a consequence of this study, we have introduced five questions in fuzzy metric spaces (relative to completion, uniform continuity, extension and contractivity) that we think provide the basis of much future research. Finally, from the practical application point of view, we have shown that this fuzzy metric can be used to approach the problem of measuring perceptual colour differences between colour samples.

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