A NOTE ON CONVERGENCE IN FUZZY METRIC SPACES

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Abstract. The sequential p-convergence in a fuzzy metric space, in the sense of George and Veeramani, was introduced by D. Mihet as a weaker concept than convergence. Here we introduce a stronger concept called s-convergence, and we characterize those fuzzy metric spaces in which convergent sequences are s-convergent. In such a case M is called an s-fuzzy metric. If $(N_M, *)$ is a fuzzy metric on X where $N_M(x, y) = \bigwedge\{M(x, y, t) : t > 0\}$ then it is proved that the topologies deduced from M and N_M coincide if and only if M is an s-fuzzy metric.

1. Introduction

George and Veeramani, [3, 5], introduced and studied a notion of fuzzy metric space with help of continuous t-norms. Several authors have contributed to the development of this theory, for instance [9, 13, 14, 22, 23, 24, 25]. In particular, it has been proved that the class of topological spaces which are fuzzy metrizable agrees with the class of metrizable topological spaces [4, 9] and then, some classical theorems on metric completeness and metric (pre) compactness have been adapted to the realm of fuzzy metric spaces [9]. Nevertheless, the theory of fuzzy metric completion is very different from the classical theories of metric completion. In fact, there are fuzzy metric spaces which are non-completable [10].

A significant difference between a classical metric and a fuzzy metric is that this last one includes in its definition a parameter t. This fact has been successfully used in engineering applications such as colour image filtering [1, 2, 17, 18, 19, 20, 21] and perceptual colour differences [7, 16].

From the mathematical point of view the parameter t allows to introduce novel (fuzzy) metric concepts that cannot be defined in the classical context. For instance, when working on contractivity, D. Mihet [15] defined for a sequence a concept weaker than convergence as follows: A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called p-convergent to x_0 if $\lim_n M(x_n, x_0, t_0) = 1$ for some $t_0 > 0$ (recall that $\{x_n\}$ converges to x_0 if and only if $\lim_n M(x_n, x_0, t) = 1$ for each $t > 0$). Those fuzzy metric spaces in which every p-convergent sequence is convergent were called principal [12], and they were characterized as ones in which $\{B(x, r, t) : r \in]0, 1[\}$ is a local base at x, for each $t > 0$ and each $x \in X$.

In this paper we continue the work started in [12, 15], but in the opposite way, that is, we strengthen the condition of convergence on t. So, we introduce the following concept: A sequence $\{x_n\}$ in $(X, M, *)$ is called s-convergent

Received: ; Revised: ; Accepted:
Key words and phrases: Fuzzy metric space; principal fuzzy metric; p-convergence.

if $\lim_n M(x_n, x_0, \frac{1}{n}) = 1$, for some $x_0 \in X$. This concept is close to convergence, and indeed, s-convergence implies convergence but the converse is not true, in general. A fuzzy metric space in which every convergent sequence is s-convergent will be called s-fuzzy metric space. Our first goal is to obtain a characterization of s-fuzzy metric spaces by means of local bases similar to the case of principal fuzzy metric spaces. Indeed, $(X, M, *)$ is an s-fuzzy metric space if and only if $\{\bigcap_{t>0} B(x, r, t) : r \in]0, 1[\}$ is a local base at x, for each $x \in X$ (Corollary 3.10).

The second goal is to characterize a certain class of fuzzy metrics by means of our concept. Indeed, for those fuzzy metrics M on X such that $N_M(x, y) = \bigwedge_{t>0} M(x, y, t)$ is a (stationary) fuzzy metric on X, we prove that the topologies on X deduced from M and N_M agree if and only if M is s-fuzzy metric (Theorem 4.2). Appropriate examples illustrate that the implications

$$s - \text{convergence} \Rightarrow \text{convergence} \Rightarrow p - \text{convergence},$$

have only one sense, in general.

Finally, to provide an overview, a classification of fuzzy metrics is drawn. This classification attends, specially, to the behaviour of fuzzy metrics with respect to the different types of convergence studied and it also involves some well-known families of fuzzy metrics used in this paper.

The structure of the paper is as follows. In Section 2 we give the preliminaries, in Section 3 we introduce and study the concept of s-convergence, in Section 4 we study a certain class of s-fuzzy metrics and in Section 5 we classify fuzzy metric spaces in accordance with the concepts of p and s-convergence.

2. **Preliminaries**

Definition 2.1. (George and Veeramani [3]). A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times]0, \infty[$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

(GV1) $M(x, y, t) > 0$;
(GV2) $M(x, y, t) = 1$ if and only if $x = y$;
(GV3) $M(x, y, t) = M(y, x, t)$;
(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(GV5) $M(x, y, \_) :]0, \infty[\to]0, 1]$ is continuous.

The continuous t-norms commonly used in fuzzy logic are the minimum ($\wedge$), the usual product ($\cdot$) and the Lukasiewicz t-norm ($\mathfrak{L}$).

If $(X, M, *)$ is a fuzzy metric space, we will say that $(M, *)$, or simply M, is a *fuzzy metric* on X. This terminology will be also extended in the paper to other related concepts, as usual, without explicit mention.

The following is a well-known result.

Lemma 2.2. *(Grabiec [6]). $M(x, y, \_)$ is non-decreasing for all $x, y \in X$.*

George and Veeramani proved in [3] that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form

$\{B_M(x, \epsilon, t) : x \in X, 0 < \epsilon < 1, t > 0\}$, where $B_M(x, \epsilon, t) = \{y \in X : M(x, y, t) > 1 - \epsilon\}$ for all $x \in X$, $\epsilon \in]0, 1[$ and $t > 0$. If confusion is not possible we write simply B instead of B_M.

Let (X, d) be a metric space and let M_d be a function on $X \times X \times]0, \infty[$ defined by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

Then $(X, M_d, \cdot)$ is a fuzzy metric space, [3], and M_d is called the *standard fuzzy metric* induced by d. The topology τ_{M_d} coincides with the topology $\tau(d)$ on X deduced from d.

Definition 2.3. (Gregori and Romaguera [11]). A fuzzy metric M on X is said to be *stationary* if M does not depend on t, i.e. if for each $x, y \in X$, the function $M_{x,y}(t) = M(x, y, t)$ is constant. In this case we write $M(x, y)$ instead of $M(x, y, t)$.

Proposition 2.4. *(George and Veeramani* [3]*). Let $(X, M, *)$ a fuzzy metric space. A sequence $\{x_n\}$ in X converges to x if and only if* $\lim_n M(x_n, x, t) = 1$*, for all $t > 0$.*

Definition 2.5. (Mihet [15]). Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be *p-convergent* to x_0 if $\lim_n M(x_n, x_0, t_0) = 1$ for some $t_0 > 0$. Each p-convergent sequence to x_0 is convergent (to x_0) if and only if $\{B(x_0, r, t) : r \in]0, 1[\}$ is a local base at x_0 for all $t > 0$, [12].

Definition 2.6. (Gregori et al. [12]). A fuzzy metric space $(X, M, *)$ is called *principal* (or simply, M is principal) if $\{B(x, r, t) : r \in]0, 1[\}$ is a local base at $x \in X$, for each $x \in X$ and each $t > 0$.

Theorem 2.7. *(Gregori et al.* [12]*). A fuzzy metric space is principal if and only if every p-convergent sequence is convergent.*

3. *s*-convergence

Definition 3.1. Let $(X, M, *)$ be a fuzzy metric space. We will say that a sequence $\{x_n\}$ in X is *s-convergent* to $x_0 \in X$ if $\lim_n M(x_n, x_0, \frac{1}{n}) = 1$.

Equivalently, $\{x_n\}$ is s-convergent to x_0 if for each $r \in]0, 1[$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_0, \frac{1}{n}) > 1 - r$ for all $n \geq n_0$, i.e. $x_n \in B(x_0, r, \frac{1}{n})$ for all $n \geq n_0$.

Under this terminology the following consequences are immediate:

Consequences 3.2.

(i) If M is stationary then convergent sequences are s-convergent.
(ii) Constant sequences are s-convergent.

In consequence:

(iii) If τ_M is the discrete topology then convergent sequences are s-convergent.

Proposition 3.3. *Let $(X, M, *)$ be a fuzzy metric space. Each s-convergent sequence in X is convergent.*

Proof. Suppose that $\{x_n\}$ is s-convergent to x_0. Let $t > 0$. We choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < t$. We have that $M(x_n, x_0, t) \geq M(x_n, x_0, \frac{1}{n})$ for all $n \geq n_0$, and so $\lim_n M(x_n, x_0, t) = 1$, for all $t > 0$ and so $\{x_n\}$ converges to x_0. □

Now we will see that the converse of the last proposition is not true, in general.

Example 3.4. ([12, 27]) On $[0, \infty[$ we consider the principal fuzzy metric $(M, \cdot)$ where M is defined by

$$M(x, y, t) = \frac{\min\{x, y\} + t}{\max\{x, y\} + t}, \quad x, y \in [0, \infty[, t > 0.$$

Since $\lim_n M(\frac{1}{n}, 0, t) = \lim_n \frac{0+t}{\frac{1}{n}+t} = 1$ for all $t > 0$, then $\left\{\frac{1}{n}\right\}$ converges to 0, but it is not s-convergent to 0, since $\lim_n M(\frac{1}{n}, 0, \frac{1}{n}) = \frac{0+\frac{1}{n}}{\frac{1}{n}+\frac{1}{n}} = \frac{1}{2}$.

Further, if $\{x_n\}$ is a sequence that converges to x_0 in a fuzzy metric space $(X, M, *)$ we cannot ensure, in general, that $\lim_n M(x_n, x_0, \frac{1}{n})$ exists. Indeed, in the current example, if we consider the sequence $\{x_n\}$ given by $x_n = \frac{1}{n}$ if n is odd and $x_n = \frac{1}{n^2}$ if n is even, then $\{x_n\}$ converges to 0 and it is easy to see $\lim_n M(x_n, 0, \frac{1}{n})$ does not exist.

Proposition 3.5. *Let $(X, M, *)$ be a fuzzy metric space.*

(i) *Each subsequence of an s-convergent sequence in X is s-convergent.*
(ii) *Each convergent sequence in X admits an s-convergent subsequence.*

Proof. (i) Suppose that $\{x_n\}$ is an s-convergent sequence to x_0 in X, and consider a subsequence $\{x_{n_k}\}$ of $\{x_n\}$. If we take a fix $r \in]0, 1[$, by our assumption there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_0, \frac{1}{n}) > 1 - r$ for each $n \geq n_0$. Now, for all $k \in \mathbb{N}$ we have that $M(x_{n_k}, x_0, \frac{1}{k}) \geq M(x_{n_k}, x_0, \frac{1}{n_k})$, since $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$. Thus if we take k_0 such that $n_{k_0} \geq n_0$, then $M(x_{n_k}, x_0, \frac{1}{k}) \geq M(x_{n_k}, x_0, \frac{1}{n_k}) > 1 - r$ for each $k \geq k_0$.

(ii) Let $\{x_n\}$ be a convergent sequence to x_0 in X. We will construct the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ as follows:

Since $\{B(x_0, \frac{1}{m}, \frac{1}{m}) : m \geq 2\}$ is a local base at x_0 and $\{x_n\}$ converges to x_0, then for $k = 2$ we can find $n_2 \in \mathbb{N}$ with $n_2 \geq 2$ such that $x_{n_2} \in B(x_0, \frac{1}{2}, \frac{1}{2})$. By induction on k ($k \geq 3$) we choose $x_{n_k} \in B(x_0, \frac{1}{k}, \frac{1}{k})$, with $n_k \geq \max\{n_{k-1}, k\}$ and so we construct the sequence $\{x_{n_k}\}$. By construction $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$. Finally, we will see that $\{x_{n_k}\}$ is s-convergent. Let $r \in]0, 1[$. We can find $k_0 \in \mathbb{N}$ such that $0 < \frac{1}{k_0} < r$ and then for all $k \geq k_0$ we have that $0 < \frac{1}{k} < \frac{1}{k_0} < r$. Thus $x_{n_k} \in B(x_0, \frac{1}{k}, \frac{1}{k}) \subset B(x_0, r, \frac{1}{k})$ for all $k \geq k_0$ and then $\{x_{n_k}\}$ is s-convergent.

□

Definition 3.6. We will say that $(X, M, *)$ is an *s-fuzzy metric space* or simply M is an s-fuzzy metric if every convergent sequence is s-convergent.

By Consequence 3.2 and the last definition we have the next corollary.

Corollary 3.7. *Let $(X, M, *)$ be a fuzzy metric space.*

(i) *If τ_M is the discrete topology then M is an s-fuzzy metric.*
(ii) *If M is stationary then M is an s-fuzzy metric.*

Theorem 3.8. *Let $(X, M, *)$ be a fuzzy metric space. Take $x_0 \in X$ and let $\{t_n\}$ be a sequence of positive real numbers that converges to 0 in the usual topology of $\mathbb{R}$ restricted to $[0, \infty[$. Then each convergent sequence $\{x_n\}$ to x_0 satisfies that $\lim_n M(x_n, x_0, t_n) = 1$ if and only if $\bigcap_{t>0} B(x_0, r, t)$ is a neighborhood of x_0 for each $r \in]0, 1[$.*

Proof. Suppose that $\bigcap_{t>0} B(x_0, r, t)$ is a neighborhood of x_0 for each $r \in]0, 1[$ and consider a convergent sequence $\{x_n\}$ to x_0 in X. Let $\epsilon \in]0, 1[$. Since $\bigcap_{t>0} B(x_0, \epsilon, t)$ is a neighborhood of x_0 there exists $n_\epsilon \in \mathbb{N}$ such that $x_n \in \bigcap_{t>0} B(x_0, \epsilon, t)$ for all $n \geq n_\epsilon$, i.e. $M(x_0, x_n, t) > 1 - \epsilon$ for all $t > 0$, and for all $n \geq n_\epsilon$. In particular $M(x_0, x_n, t_n) > 1 - \epsilon$ for all $n \geq n_\epsilon$. Thus $\lim_n M(x_0, x_n, t_n) = 1$.

Conversely, suppose that there exists $r_0 \in]0, 1[$ such that $\bigcap_{t>0} B(x_0, r_0, t)$ is not a neighborhood of x_0. Equivalently, $\bigcap_n B(x_0, r_0, t_n)$ is not a neighborhood of x_0. Recall that $\{B(x_0, \frac{1}{n}, \frac{1}{n}) : n \geq 2\}$ is a decreasing local base at x_0. So, for each $n \geq 2$ we have that $B(x_0, \frac{1}{n}, \frac{1}{n}) \nsubseteq \bigcap_n B(x_0, r_0, t_n)$. We construct a sequence $\{x_n\}$ taking $x_n \in B(x_0, \frac{1}{n}, \frac{1}{n}) \setminus (\bigcap_n B(x_0, r_0, t_n))$ for all $n \geq 2$. This sequence $\{x_n\}$ is convergent to x_0. (Indeed, let $\delta \in]0, 1[$ and $t > 0$, and consider $B(x_0, \delta, t)$, then there exists $n_0 \in \mathbb{N}$ such that $B(x_0, \frac{1}{n_0}, \frac{1}{n_0}) \subset B(x_0, \delta, t)$ and so for all $n \geq n_0$ we have that $B(x_0, \frac{1}{n}, \frac{1}{n}) \subset B(x_0, \delta, t)$ and then $x_n \in B(x_0, \delta, t)$ for all $n \geq n_0$). Now, we will see that $\lim_n M(x_n, x_0, t_n) \neq 1$, by reduction to absurd. Suppose that $\lim_n M(x_n, x_0, t_n) = 1$. Then for $r_0 \in]0, 1[$ there exists $n_{r_0} \in \mathbb{N}$ such that $M(x_n, x_0, t_n) > 1 - r_0$ for all $n \geq n_{r_0}$ and in consequence $x_n \in B(x_0, r_0, t_n)$ for all $n \geq n_{r_0}$, a contradiction. □

Using the sequence $\{\frac{1}{n}\}$ as $\{t_n\}$ in the above theorem and taking into account that for each $r \in]0, 1[$ and $t > 0$ we have that $\bigcap_{s>0} B(x_0, r, s) \subset B(x_0, r, t)$ for each $x_0 \in X$, we obtain the next corollary.

Corollary 3.9. *Let $(X, M, *)$ be a fuzzy metric space and let $x_0 \in X$. Then the following are equivalent:*

(i) *Each sequence converging to x_0 is s-convergent.*
(ii) *$\bigcap_{t>0} B(x_0, r, t)$ is a neighborhood of x_0 for each $r \in]0, 1[$.*
(iii) *$\{\bigcap_{t>0} B(x_0, r, t) : r \in]0, 1[\}$ is a local base at x_0.*

From this corollary it is immediate to obtain the following corollary.

Corollary 3.10. *Let $(X, M, *)$ be a fuzzy metric space. Then the following are equivalent:*

(i) *M is an s-fuzzy metric.*
(ii) *$\bigcap_{t>0} B(x, r, t)$ is a neighborhood of x for all $x \in X$, and for all $r \in]0, 1[$.*
(iii) *$\{\bigcap_{t>0} B(x, r, t) : r \in]0, 1[\}$ is a local base at x, for each $x \in X$.*

Taking into account Theorem 2.7 we have the next corollary.

Corollary 3.11. *Each p-convergent sequence $\{x_n\}$ in X is s-convergent if and only if X is a principal s-fuzzy metric space.*

Proposition 3.12. *Let (X,d) be a metric space. Then $(X,M_d,\cdot)$ is an s-fuzzy metric space if and only if $\tau(d)$ is the discrete topology.*

Proof. Fix $r \in]0,1[$ and $x_0 \in X$. We will see that $\bigcap_{t>0} B(x_0,r,t) = \{x_0\}$. Indeed, $B(x_0,r,t) = \{y \in X : d(x,y) < \frac{t\cdot r}{1-r}\}$ for all $t > 0$ and so $\bigcap_{t>0} B(x_0,r,t) = \bigcap_{t>0}\{y \in X : d(x,y) < \frac{t\cdot r}{1-r}\} = \{y \in X : d(x,y) \leq 0\} = \{x_0\}$. Then by Corollary 3.10 $(X,M_d,\cdot)$ is an s-fuzzy metric space if and only if x_0 is isolated, that is $\tau(d)$ is the discrete topology. □

4. On a class of s-fuzzy metrics

If $(X,M,*)$ is a fuzzy metric space we define the mapping N_M on X^2 given by $N_M(x,y) = \bigwedge_{t>0} M(x,y,t)$ for all $x,y \in X$. In this section we are interested in studying those non-stationary fuzzy metric spaces $(X,M,*)$ such that $(N_M,*)$ is a (stationary) fuzzy metric on X and we establish a relationship between those fuzzy metrics and s-fuzzy metrics. Notice that if X is a set with at least two elements and d is a metric on X it is obvious that $\bigwedge_{t>0} M_d(x,y,t) = 0$ for $x \neq y$, and so N_{M_d} is not a fuzzy metric on X.

We start with the following lemma (which proof we omit).

Lemma 4.1. *Let $(M,*)$ be a fuzzy metric on X. Then*

(i) *$(N_M,*)$ is a stationary fuzzy metric on X if and only if $N_M(x,y) > 0$ for all $x,y \in X$. In such a case:*

(ii) *$\tau_{N_M} \succ \tau_M$.*

Theorem 4.2. *Let $(M,*)$ be a fuzzy metric on X such that $N_M(x,y) > 0$ for each $x,y \in X$. Then*

$$\tau_{N_M} = \tau_M \text{ if and only if } M \text{ is an s-fuzzy metric.}$$

Proof. Suppose that $\tau_{N_M} = \tau_M$.

Fix $x_0 \in X$, $r \in]0,1[$. We will see that $\bigcap_{t>0} B_M(x_0,r,t)$ is a τ_M-neighborhood of x_0.

Consider the open ball $B_{N_M}(x_0,r)$ relative to N_M. Since $\tau_M = \tau_{N_M}$ we can find $r_1 \in]0,1[$, $t_1 > 0$ such that $B_M(x_0,r_1,t_1) \subset B_{N_M}(x_0,r)$. We will see that $B_{N_M}(x_0,r) \subset \bigcap_{t>0} B_M(x_0,r,t)$. Indeed, if $y \in B_{N_M}(x_0,r)$ then $N_M(x_0,y) > 1-r$, i.e. $\bigwedge_{t>0} M(x_0,y,t) > 1-r$ and so $M(x_0,y,t) > 1-r$ for all $t > 0$, i.e. $y \in B_M(x_0,r,t)$ for all $t > 0$. Then $y \in \bigcap_{t>0} B_M(x_0,r,t)$. Now, since $B_M(x_0,r_1,t_1) \subset B_{N_M}(x_0,r) \subset \bigcap_{t>0} B_M(x_0,r,t)$ then $\bigcap_{t>0} B_M(x_0,r,t)$ is a τ_M-neighborhood of x_0, and so by Corollary 3.10 M is an s-fuzzy metric.

Conversely, suppose that M is an s-fuzzy metric. By the last lemma we have that $\tau_{N_M} \succ \tau_M$. Now, we will see that $\tau_M \succ \tau_{N_M}$. Let $x_0 \in X$, $r \in]0,1[$ and consider $B_{N_M}(x_0,r)$. We will see that (the τ_M-neighborhood of x_0) $\bigcap_{t>0} B_M(x_0,\frac{r}{2},t)$ is contained in $B_{N_M}(x_0,r)$. Indeed, let $y \in \bigcap_{t>0} B_M(x_0,\frac{r}{2},t)$ then $y \in B_M(x_0,\frac{r}{2},t)$ for all $t > 0$, i.e $M(x_0,y,t) > 1-\frac{r}{2}$ for all $t > 0$, so $\bigwedge_{t>0} M(x_0,y,t) \geq 1-\frac{r}{2}$, thus $N_M(x_0,y) > 1-r$ and so $y \in B_{N_M}(x_0,r)$. □

An example of s-fuzzy metric fulfilling all conditions of Theorem 4.2 is given later in Example 5.1. On the other hand the next example shows that the class of fuzzy metrics M such that N_M is a fuzzy metric is not contained in the class of s-fuzzy metrics and *vice-versa*.

Example 4.3. (a) (N_M is a fuzzy metric and M is not an s-fuzzy metric)
Let $X =]0,1]$ be endowed with the usual metric d of $\mathbb{R}$. We define

$$M(x,y,t) = \begin{cases} 1 - \frac{1}{2}d(x,y)^t, & \text{if } 0 < t \leq 1 \\ 1 - \frac{1}{2}d(x,y), & \text{if } t > 1 \end{cases}$$

It is easy to verify that $\{(M_t, \mathfrak{L}) : t > 0\}$ is an increasing family of stationary fuzzy metrics on $]0,1]$, where $M_t(x,y) = M(x,y,t)$ for each $t > 0$. Also that τ_{M_t} is $\tau(d)$ (the usual topology of $\mathbb{R}$ restricted to $]0,1]$), for all $t > 0$. Then from [8, 26] one can conclude that $(X, M, \mathfrak{L})$ is a fuzzy metric space and τ_M is $\tau(d)$.

On the other hand, $N_M(x,y) = \bigwedge_{t>0} M(x,y,t) > 0$ for all $x, y \in]0,1]$, since

$$N_M(x,y) = \bigwedge_{t>0} M(x,y,t) = \begin{cases} 1, & \text{if } x = y \\ \frac{1}{2}, & \text{if } x \neq y \end{cases}$$

By the last lemma we have that $(N_M, \mathfrak{L})$ is a fuzzy metric on X, and it is obvious that τ_{N_M} is the discrete topology. Therefore $\tau_{N_M} \neq \tau_M$.

(b) (M is an s-fuzzy metric but N_M is not a fuzzy metric)
The fuzzy metric $(M, \cdot)$ of Example 5.3 is an s-fuzzy metric on X, but $N_M(\frac{1}{2}, \frac{1}{\pi}) = \bigwedge_{t>0} M(\frac{1}{2}, \frac{1}{\pi}, t) = \bigwedge_{t>0} \frac{2}{\pi} \cdot t = 0$ and so $(N_M, \cdot)$ is not a fuzzy metric on X.

5. **A classification of fuzzy metric spaces**

Let X be a non-empty set. Denote by $\mathcal{D}$ the family of fuzzy metrics that generate the discrete topology on X, and by $\mathcal{M}_s$ and $\mathcal{S}$ the families of s-fuzzy metrics and stationary fuzzy metrics on X, respectively. Attending to Consequences 3.2 we have that $\mathcal{D} \subset \mathcal{M}_s$ and $\mathcal{S} \subset \mathcal{M}_s$.

Also, denote the families of principal fuzzy metrics and standard fuzzy metrics on X by $\mathcal{P}$ and $\mathcal{M}_d$, respectively. From [12] we know that $\mathcal{S} \subset \mathcal{P}$ and $\mathcal{M}_d \subset \mathcal{P}$. Now, from our previous results and the implications

$$s - convergence \Rightarrow convergence \Rightarrow p - convergence$$

we can conclude the diagram of inclusions in Figure 1.

Next, we give examples which show that all (non-trivial) inclusions in the diagram are strict. In some cases, appropriate sequences with and without some type of convergence are also provided.

Notice that in Example 3.4 we have seen a principal non-s-fuzzy metric space.

FIGURE 1. Diagram of inclusions

Example 5.1. (A non-stationary principal s-fuzzy metric space). Let $(]0,\infty[, M, \cdot)$ the fuzzy metric space, where M is the fuzzy metric of Example 3.4. It is known that M is principal [12]. Now, $N_M(x,y) = \bigwedge_{t>0} \frac{\min\{x,y\}+t}{\max\{x,y\}+t} = \frac{\min\{x,y\}}{\max\{x,y\}} > 0$ for each $x, y \in]0,\infty[$. Then by Theorem 4.2 we have that M is an s-fuzzy metric, since $\tau_{N_M} = \tau_M$, [7].

Remark 5.2. Since the completion of the fuzzy metric space of Example 5.1 is the fuzzy metric space of Example 3.4, [7], then the completion of an s-fuzzy metric space is not necessarily an s-fuzzy metric space.

Example 5.3. (A non-stationary non-principal s-fuzzy metric space). Let $X =]0,1]$, $A = X \cap \mathbb{Q}$, $B = X \setminus A$. Define the function M on $X^2 \times \mathbb{R}^+$ by

$$M(x,y,t) = \begin{cases} \frac{\min\{x,y\}}{\max\{x,y\}} \cdot t, & (x \in A, y \in B) \text{ or } (x \in B, y \in A), t \in]0,1[, \\ \frac{\min\{x,y\}}{\max\{x,y\}}, & \text{elsewhere.} \end{cases}$$

In [12] it is proved that $(X, M, \cdot)$ is a fuzzy metric space which is not principal. (Notice that if we take $b \in B$ we have that the sequence $\{1 - \frac{b}{n}\}$ is p-convergent, since $\lim_n M(1-\frac{b}{n}, 1, 1) = \lim_n \frac{1-\frac{b}{n}}{1} = 1$, but it is not convergent, since $\lim_n M(1-\frac{b}{n}, 1, \frac{1}{2}) = \lim_n \frac{1-\frac{b}{n}}{1} \cdot \frac{1}{2} = \frac{1}{2}$.)

Now, we will see that M is an s-fuzzy metric on X. For it we will prove that $\bigcap_{t>0} B(x,r,t)$ is a neighborhood of x, for each $x \in X$ and each $r \in]0,1[$.

Fix $x \in X$ and $r \in]0,1[$. It is easy to verify that for $t \in]0, 1-r]$:

$$B(x,r,t) = \bigcap_{t>0} B(x,r,t) = \begin{cases} \left[x \cdot (1-r), \frac{x}{1-r}\right] \cap A, & x \in A, \\ \\ \left[x \cdot (1-r), \frac{x}{1-r}\right] \cap B, & x \in B. \end{cases}$$

On the other hand, if $n \geq 2$

$$B(x, \frac{1}{n}, \frac{1}{n}) = \left\{ \begin{array}{ll}]x \cdot (1-\frac{1}{n}), \frac{x}{1-\frac{1}{n}}[\cap A, & x \in A, \\ ]x \cdot (1-\frac{1}{n}), \frac{x}{1-\frac{1}{n}}[\cap B, & x \in B. \end{array} \right.$$

Therefore if we take $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < r$ we have that $B(x, \frac{1}{n}, \frac{1}{n}) \subset \bigcap_{t>0} B(x,r,t)$ and so $\bigcap_{t>0} B(x,r,t)$ is a neighborhood of x and then by Corollary 3.10 M is an s-fuzzy metric.

Example 5.4. (A non-principal non-s-fuzzy metric space). Let $A = \mathbb{R} \cap \mathbb{Q}$, $B = \mathbb{R} \setminus A$. Let d be the usual metric on $\mathbb{R}$. Define the function M on $\mathbb{R}^2 \times \mathbb{R}^+$ by

$$M(x,y,t) = \left\{ \begin{array}{ll} t \cdot M_d(x,y,t), & (x \in A, y \in B) \text{ or } (x \in B, y \in A), t \in]0,1[, \\ M_d(x,y,t), & \text{elsewhere.} \end{array} \right.$$

We will show that $(\mathbb{R}, M, \cdot)$ is a fuzzy metric space.

Obviously, M satisfies $(GV1)$, $(GV3)$ and $(GV5)$.

First, we will see that M satisfies $(GV2)$. Suppose that $M(x,y,t) = 1$ for $x \in A$, $y \in B$ and $t \in]0,1[$. Then $t \cdot M_d(x,y,t) = 1$, but since $t \in]0,1[$ we have that $M_d(x,y,t) > 1$, a contradiction. Therefore, $M(x,y,t) = M_d(x,y,t) = 1$ and so $x = y$. The converse is immediate.

Now, we will see that M satisfies $(GV4)$. Suppose that $x, y \in A$, $z \in B$ and let $t, s > 0$ such that $t + s \in]0,1[$. Then

$$M(x,z,t+s) = (t+s) \cdot M_d(x,z,t+s) > s \cdot M_d(x,y,t) \cdot M_d(y,z,s) =$$
$$M(x,y,t) \cdot M(y,z,s).$$

The other cases are proved in a similar way.

We will see that M is not principal and neither an s-fuzzy metric. For it, we will give a p-convergent sequence which is not convergent and a convergent sequence which is not s-convergent.

Consider the sequence $\{\frac{\pi}{n}\}$. Then $\lim_n M(\frac{\pi}{n}, 0, 1) = \lim_n \frac{1}{1+\frac{\pi}{n}} = 1$ and so $\{\frac{\pi}{n}\}$ is p-convergent, but $\lim_n M(\frac{\pi}{n}, 0, \frac{1}{2}) = \lim_n \frac{\left(\frac{1}{2}\right)^2}{\frac{1}{2}+\frac{\pi}{n}} = \frac{1}{2}$ and so $\{\frac{\pi}{n}\}$ is not convergent.

Now, consider the sequence $\{\frac{1}{n}\}$. For all $t > 0$, $\lim_n M(\frac{1}{n}, 0, t) = \lim_n \frac{t}{t+\frac{1}{n}} = 1$, then $\{\frac{1}{n}\}$ is convergent, but $\lim_n M(\frac{1}{n}, 0, \frac{1}{n}) = \frac{\frac{1}{n}}{\frac{1}{n}+\frac{1}{n}} = \frac{1}{2}$ and so $\{\frac{1}{n}\}$ is not s-convergent.

Example 5.5. (A non-stationary non-principal s-fuzzy metric which generates the discrete topology). Let $X =]0, \infty[$ and let $\varphi : \mathbb{R}^+ \to]0,1]$ be a function given by

$$\varphi(t) = \left\{ \begin{array}{ll} t, & \text{if } t \in]0,1[\\ 1, & \text{elsewhere} \end{array} \right.$$

Define the function M on $X^2 \times \mathbb{R}^+$ by

$$M(x,y,t) = \left\{ \begin{array}{ll} 1, & x = y \\ \frac{\min\{x,y\}}{\max\{x,y\}} \cdot \varphi(t), & x \neq y \end{array} \right.$$

In [12] it is proved that $(M, \cdot)$ is a non-principal fuzzy metric on X and that τ_M is the discrete topology, so M is an s-fuzzy metric. Clearly M is non-stationary.

Acknowledgements.
The authors are grateful to the referees for their valuable suggestions.

Valentín Gregori acknowledges the support of Spanish Ministry of Education and Science under Grant MTM 2012-37894-C02-01 and the support of Universitat Politècnica de València under Grant PAID-06-12 SP20120471.

Juan José Miñana acknowledges the support of Conselleria de Educación, Formación y Empleo (Programa Vali+d para investigadores en formación) of Generalitat Valenciana, Spain, ACIF/2012/040, and the support of Universitat Politècnica de València under Grant PAID-06-12 SP20120471.

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