Some remarks on fuzzy contractive mappings

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Abstract

Recently it has been introduced a new concept of fuzzy contraction in fuzzy metric spaces in the sense of George and Veeramani [D. Wardowski, *Fuzzy contractive mappings and fixed points in fuzzy metric spaces*, Fuzzy Sets and Systems **222** (2013) 108-114]. Here we provide some comments to this paper.

Key words: Fuzzy metric space; contractive mapping; contractive sequence.

1 Introduction

In [5] the author has introduced a concept of fuzzy contractive mapping that generalizes the one due to Gregori and Sapena [2], and then a fixed point theorem is given. The main Wardowski's theorem is correct and it is different from the ones known in the literature. However, in this note we make some remarks about some assertions made in [5], which are not true.

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2 Preliminaries

In the following $(X, M, *)$ is a fuzzy metric space in the sense of George and Veeramani. For concepts and basic results the reader is referred to [1]. Next, we briefly recall some other definitions and results.

Definition 1 *(Gregori and Sapena [2]). A mapping $f : X \to X$ is said to be fuzzy contractive if there exists $k \in]0, 1[$ such that*

$$\frac{1}{M(f(x), f(y), t)} - 1 \leq k \left(\frac{1}{M(x, y, t)} - 1 \right) \tag{1}$$

for all $x, y \in X$ and $t > 0$.

A sequence $\{x_n\}$ in X is said to be fuzzy contractive if there exists $k \in]0, 1[$ such that

$$\frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1 \leq k \left(\frac{1}{M(x_n, x_{n+1}, t)} - 1 \right) \tag{2}$$

for all $t > 0$, $n \in \mathbb{N}$.

Theorem 2 *(Gregori and Sapena [2]). Let $(X, M, *)$ be a complete fuzzy metric space in which fuzzy contractive sequences are Cauchy. If $f : X \to X$ is a fuzzy contractive mapping then f has a unique fixed point.*

In our context the concept of fuzzy ψ-contractive mapping in [4] becomes as follows:

Definition 3 *(Mihet [4]). Let Ψ be the class of all mappings $\psi :]0, 1] \to]0, 1]$ such that ψ is continuous, nondecreasing and $\psi(t) > t$ for all $t \in]0, 1[$. Let $\psi \in \Psi$. A mapping $f : X \to X$ is said to be fuzzy ψ-contractive mapping if:*

$$M(f(x), f(y), t) \geq \psi(M(x, y, t)). \tag{3}$$

for all $x, y \in X$ and $t > 0$.

Definition 4 *(Wardowski [5]). Denote by $\mathcal{H}$ the family of mappings $\eta :]0, 1] \to [0, \infty[$ satisfying the following two conditions:*

(H1) η transforms $]0, 1]$ onto $[0, \infty[$;
(H2) η is strictly decreasing.

A mapping $f : X \to X$ is said to be fuzzy $\mathcal{H}$-contractive with respect to $\eta \in \mathcal{H}$ if there exists $k \in]0, 1[$ satisfying the following condition:

$$\eta(M(f(x), f(y), t)) \leq k\eta(M(x, y, t)) \tag{4}$$

for all $x, y \in X$ and $t > 0$.

Theorem 5 *(Wardowski [5]). Let $(X, M, *)$ be a complete fuzzy metric space and let $f : X \to X$ be a fuzzy $\mathcal{H}$-contractive mapping with respect to $\eta \in \mathcal{H}$ such that:*

(a) $\Pi_{i=1}^{k} M(x, f(x), t_i) \neq 0$, for all $x \in X$, $k \in \mathbb{N}$ and any sequence $\{t_i\} \subset]0, \infty[$, $t_i \searrow 0$;
(b) $r * s > 0 \Rightarrow \eta(r * s) \leq \eta(r) + \eta(s)$, for all $r, s \in \{M(x, f(x), t) : x \in X, t > 0\}$;
(c) $\{\eta(M(x, f(x), t_i)) : i \in \mathbb{N}\}$ is bounded for all $x \in X$ and any sequence $\{t_i\} \subset]0, \infty[$, $t_i \searrow 0$.

Then f has a unique fixed point $x^* \in X$ and for each $x_0 \in X$ the sequence $\{f^n(x_0)\}$ converges to x^*.

3 Main results

Consider the fuzzy metric space $(X, M, *)$, where $X = \mathbb{N}$, $M(x, y, t) = \mathrm{e}^{-\frac{|x-y|}{t}}$ and $*$ is the usual product on $[0, 1]$. Consider the self-mapping f on $\mathbb{N}$ given by $f(x) = 2x$ for each $x \in \mathbb{N}$, and let $\eta(s) = \frac{1}{s} - 1$ for all $s \in]0, 1]$.We have

$$\eta(M(f(x), f(y), t)) = \mathrm{e}^{\frac{2|x-y|}{t}} - 1 > \mathrm{e}^{\frac{|x-y|}{t}} - 1 = \eta(M(x, y, t))$$

if $x \neq y$, and so $\eta(M(f(x), f(y), t)) > k\eta(M(x, y, t))$ for all $k \in]0, 1[$, $x \neq y$ and for all $t > 0$, that is f is not fuzzy $\mathcal{H}$-contractive with respect to η.

This means that Example 3.4 in [5] is not correct. As such, the question "is every fuzzy contractive sequence a Cauchy sequence?" (see [2]) still remains open. Note that in fuzzy metric spaces in the sense of Kramosil and Michalek there exist contractive sequences which are not Cauchy, see [3].

In [5] it is proved that the class of fuzzy contractive mappings is included in the class of fuzzy $\mathcal{H}$-contractive mappings. We can add the next proposition.

Proposition 6 *The class of fuzzy $\mathcal{H}$-contractive mappings are included in the class of fuzzy ψ-contractive mappings.*

Proof. Suppose that $f : X \to X$ is $\mathcal{H}$-contractive with respect to $\eta \in \mathcal{H}$. Then there exists $k \in]0, 1[$ such that for all $x, y \in X$ and for all $t > 0$ we have that

$$\eta(M(f(x), f(y), t)) \leq k\eta(M(x, y, t)). \tag{5}$$

By conditions $(H1)$ and $(H2)$ in Definition 4 we have that η is continuous and bijective. Further, the mappings $k \cdot \eta :]0,1] \to [0,\infty[$ and $\eta^{-1} : [0,\infty[\to]0,1]$, defined in its obvious sense, are two bijective continuous mappings which are strictly decreasing. So, the mapping $\psi :]0,1] \to]0,1]$ defined by $\psi(t) = \eta^{-1}(k \cdot \eta(t))$ is continuous, bijective and strictly increasing. Now, since $k \cdot \eta(t) < \eta(t)$ for all $t \in]0,1[$ then $\psi(t) = \eta^{-1}(k \cdot \eta(t)) > \eta^{-1}(\eta(t)) = t$ for all $t \in]0,1[$ and therefore $\psi \in \Psi$.

Finally, from (5) we have

$$M(f(x), f(y), t) = \eta^{-1}(\eta(M(f(x), f(y), t))) \geq$$

$$\eta^{-1}(k \cdot \eta(M(f(x), f(y), t)) = \psi(M(x, y, t)).$$

Remark 7 *(Drawback of the condition (c) of Theorem 5).*

Let (X, d) be a complete metric space and suppose that f is a contractive non-constant self-mapping on X. Then f is fuzzy contractive (see [2]) in the complete standard fuzzy metric space $(X, M_d, \cdot)$ (see [1]), and applying Theorem 2 (or Corollary 4.5 of [2]) one can assert that f has a unique fixed point.

Although f is fuzzy $\mathcal{H}$-contractive (Example 3.1 of [5]) Theorem 5 cannot be applied on $(X, M_d, \cdot)$ since for any $\eta \in \mathcal{H}$ condition (c) of this theorem is never satisfied. (Indeed, if we take $x \in X$ such that $f(x) \neq x$ then $\lim_n(M_d(x, f(x), \frac{1}{n})) = \lim_n \frac{\frac{1}{n}}{\frac{1}{n} + d(x, f(x))} = 0$ and then $\{\eta(M_d(x, f(x), \frac{1}{n})) : n \in \mathbb{N}\}$ is not bounded).

The last remark points out that the interesting result of Wardowski, Theorem 5, is not a generalization of Theorem 2.

Remark 8 *Theorem 5 can be written in a more elegant way, although less general, as follows:*

Let $(X, M, *)$ be a complete fuzzy metric space where $*$ is positive (i.e. $x * y \neq 0$, for $x, y \neq 0$) and M satisfies that $\bigwedge_{t>0} M(x, y, t) > 0$ for each $x, y \in X$. Let $f : X \to X$ be a fuzzy $\mathcal{H}$-contractive mapping with respect to $\eta \in \mathcal{H}$ such that $\eta(r * s) \leq \eta(r) + \eta(s)$ for all $r, s \in]0,1]$. Then f has a unique fixed point.

This result suggests the following question: Can the condition “$*$ is positive” be omitted in the last theorem? (Compare with the question at the end of [5]).

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