Characterizing a class of completable fuzzy metric spaces

Valentín Gregori [a,*,1], Juan-José Miñana [a,2], Samuel Morillas [a], Almanzor Sapena [a,3]

[a] *Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, Camino de Vera s/n 46022 Valencia (SPAIN).*

Abstract

In this paper we give a characterization of the class of completable strong (non-Archimedean) fuzzy metric spaces, in the sense of George and Veeramani.

Key words: Fuzzy metric space; completable fuzzy metric space; strong (non-Archimedean) fuzzy metric space.
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1 Introduction

Probabilistic metric spaces were introduced by K. Menger [14] in 1942 who generalized the theory of metric spaces. In 1975 I. Kramosil and J. Michalek [12] gave a notion of fuzzy metric M on a set X, using a continuous t-norm $*$, in such a manner that the corresponding fuzzy metric space $(X, M, *)$ could be considered as a reformulation, in fuzzy setting, of the notion of Menger space. This fuzzy metric (space) will be denoted by KM-fuzzy metric (space). In a

* Corresponding author email: vgregori@mat.upv.es
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modern terminology, a fuzzy metric in the sense of Kramosil and Michalek is the original mentioned but removing the last axiom ($\lim_{t\to\infty} M(x,y,t) = 1$ for all $x, y \in X$). This fuzzy metric will be denoted here by G-fuzzy metric, since it was formulated in this way for the first time by M. Grabiec [5] (see also [2]), and it is also called generalized fuzzy Menger space [15,17].

Later, in 1994 V. George and P. Veeramani [2] modified the concept of G-fuzzy metric and introduced a new concept of fuzzy metric, which we denote GV-fuzzy metric. A GV-fuzzy metric can be regarded as a G-fuzzy metric if we define $M(x,y,0) = 0$ for each $x, y \in X$. Under this assumption the situation is described in the following diagram of implications

$$\begin{array}{ccc} KM\text{-fuzzy metric (Menger space)} & \rightarrow & G\text{-fuzzy metric} \\ & & \uparrow \\ & & GV\text{-fuzzy metric} \end{array}$$

By fuzzy metric (space) we mean one of the three mentioned fuzzy metrics (metric spaces) when distinction is not necessary.

A fuzzy metric space $(X, M, *)$ is called strong if it satisfies the additional condition $M(x,z,t) \geq M(x,y,t) * M(y,z,t)$ or all $x, y, z \in X$, $t > 0$. Strong fuzzy metrics are interesting because many results are stated in fuzzy setting under the assumption that M is strong (see for instance [13,16]). This class is so wide that, in fact, there are not many examples of non-strong fuzzy metrics in the literature [6,7,10].

Since Menger spaces are completable as proved H. Sherwood [22,21] (another proof was obtained by Sempi [20]) then KM-fuzzy metric spaces are also completable. With similar arguments to the ones found in [21] it can be proved that also G-fuzzy metrics are completable. (A different proof can be seen in [1].) The case of completion of GV-fuzzy metric spaces is very different from the classical case. Indeed, Gregori and Romaguera proved that there exist GV-fuzzy metric spaces which are not completable [8]. Later, the same authors gave a characterization of those GV-fuzzy metric spaces that are completable, which we reformulate, for our convenience, as follows:

Theorem 1 *[9] A GV-fuzzy metric space $(X, M, *)$ is completable if and only if for each pair of Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X the following three conditions are fulfilled:*

(c1) $\lim_n M(a_n, b_n, s) = 1$ *for some $s > 0$ implies* $\lim_n M(a_n, b_n, t) = 1$ *for all $t > 0$.*
(c2) $\lim_n M(a_n, b_n, t) > 0$ *for all $t > 0$.*
(c3) The assignment $t \rightarrow$ $\lim_n M(a_n, b_n, t)$ *for each $t > 0$ is a continuous*

function on $]0,\infty[$*, provided with the usual topology of* $\mathbb{R}$*.*

One can observe that these three conditions involve just the three differences between axioms $(KM1)$, $(KM2)$, $(KM5)$ in the definition of KM-fuzzy metric and the corresponding axioms $(GV1)$, $(GV2)$, $(GV5)$ in the concept of George and Veeramani (see Definition 2).

There were in the literature examples of non-completable strong GV-fuzzy metrics that do not satisfy $(c1)$ or $(c2)$ [8,9], and recently [6], the authors have constructed a non-completable fuzzy metric space which does not satisfy $(c3)$.

In this paper we first observe that $(c1)-(c3)$ constitute an independent axiomatic system and then we will proof, after several lemmas, that strong GV-fuzzy metrics satisfy $(c3)$, or in other words (Theorem 21): A strong GV-fuzzy metric space $(X, M, *)$ is completable if and only if M satisfies $(c1)$ and $(c2)$. Several corollaries can be obtained from this theorem, for instance a characterization of completable fuzzy ultrametrics (Corollary 23) and also we could obtain that metric spaces admit a unique completion, but we do not insist on it because it is well-known from the properties of the standard fuzzy metric. Several examples illustrate our results.

The structure of the paper is as follows. After the preliminaries section, in Section 3 we prove that $(c1)-(c3)$ constitute an independent axiomatic system. In Section 4 we give a characterization for the class of completable strong GV-fuzzy metrics.

2 Preliminaries

Definition 2 *(George and Veeramani [2].) A GV-fuzzy metric space is an ordered triple* $(X, M, *)$ *such that* X *is a (non-empty) set,* $*$ *is a continuous t-norm and* M *is a fuzzy set on* $X\times X\times]0,\infty[$ *satisfying the following conditions, for all* $x, y, z \in X,\ s, t > 0$*:*

(GV1) $M(x, y, t) > 0$
(GV2) $M(x, y, t) = 1$ *if and only if* $x = y$
(GV3) $M(x, y, t) = M(y, x, t)$
(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$
(GV5) $M(x, y, \_) :]0,\infty[\to]0, 1]$ *is continuous.*

If axioms (GV1), (GV2) and (GV5) are replaced by

(KM1) $M(x, y, 0) = 0$
(KM2) $M(x, y, t) = 1$ *for all* $t > 0$ *if and only if* $x = y$
(KM5) $M(x, y, \_) : [0,\infty[\to [0, 1]$ *is left continuous*

respectively, we obtain the concept of G-fuzzy metric space. If we add

(KM6) $\lim_{t\to\infty} M(x,y,t) = 1$ *for each* $x, y \in X$

then we obtain the (original) concept of fuzzy metric space due to Kramosil and Michalek that we denote KM-fuzzy metric.

If $(X, M, *)$ is a fuzzy metric space (in some sense), we will say that $(M, *)$ (or simply M) is a *fuzzy metric* on X.

The following is a well-known result.

Remark 3 $M(x, y, \_)$ *is non-decreasing for all* $x, y \in X$.

From now on, by a fuzzy metric we mean a GV-fuzzy metric.

George and Veeramani proved in [2] that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x,\epsilon,t) : x \in X, 0 < \epsilon < 1, t > 0\}$, where $B_M(x,\epsilon,t) = \{y \in X : M(x,y,t) > 1-\epsilon\}$ for all $x \in X$, $\epsilon \in]0,1[$ and $t > 0$.

Let (X, d) be a metric space and let M_d a fuzzy set on $X \times X \times]0,\infty[$ defined by

$$M_d(x,y,t) = \frac{t}{t + d(x,y)}$$

Then $(X, M_d, \cdot)$ is a fuzzy metric space, [2], and M_d is called the *standard fuzzy metric* induced by d. The topology on X deduced from d agrees with τ_{M_d}.

Definition 4 *(Gregori and Romaguera [9].) A fuzzy metric* M *on* X *is said to be* stationary *if* M *does not depend on* t*, i.e., if for each* $x, y \in X$*, the function* $M_{x,y}(t) = M(x,y,t)$ *is constant. In this case we write* $M(x,y)$ *instead of* $M(x,y,t)$.

Definition 5 *(Gregori et al. [7], Istrăţescu [11].) A fuzzy metric space* $(X, M, *)$ *is said to be* strong *(non-Archimedean) if for all* $x, y, z \in X$ *and all* $t > 0$ *satisfies*

$$M(x,z,t) \geq M(x,y,t) * M(y,z,t).$$

A strong fuzzy metric for the minimum t-norm is called a fuzzy ultrametric.

Proposition 6 *(George and Veeramani [2].) A sequence* $\{x_n\}$ *in* X *converges to* x *if and only if* $\lim_n M(x_n, x, t) = 1$*, for all* $t > 0$.

Definition 7 *(George and Veeramani [2].) A sequence* $\{x_n\}$ *in a fuzzy metric space* $(X, M, *)$ *is said to be* M-Cauchy*, or simply* Cauchy*, if for each* $\epsilon \in]0,1[$

and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1-\epsilon$ for all $n, m \geq n_0$. X is said to be complete if every Cauchy sequence in X is convergent with respect to τ_M. In such a case M is also said to be complete.

Definition 8 *(Gregori and Romaguera [8].) Let $(X, M, *)$ and $(Y, N, \diamond)$ be two fuzzy metric spaces. A mapping f from X to Y is said to be an* isometry *if for each $x, y \in X$ and $t > 0$, $M(x, y, t) = N(f(x), f(y), t)$ and, in this case, if f is a bijection, X and Y are called* isometric. *A* fuzzy metric completion *of (X, M) is a complete fuzzy metric space (X^*, M^*) such that (X, M) is isometric to a dense subspace of X^*. X is said to be* completable *if it admits a fuzzy metric completion.*

A t-norm $*$ is called integral (positive) if $x * y > 0$ whenever $x, y \in]0, 1]$.

Theorem 9 *(Gregori et al. [7]) Let $(X, M, *)$ be a strong fuzzy metric space and suppose that $*$ is integral. If $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and $t > 0$ then $\{M(x_n, y_n, t)\}_n$ converges in $]0, 1]$.*

Proposition 10 *(Gregori and Romaguera [9].) A stationary fuzzy metric space $(X, M, *)$ is completable if and only if $\lim_n M(a_n, b_n) > 0$ for each pair of Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X.*

Remark 11 *Obviously, if $(X, M, *)$ is a stationary fuzzy metric space, then M satisfies conditions $(c1)$ and $(c3)$ of Theorem 1.*

3 Non-completable fuzzy metric spaces

In this section we will show that the axioms $(c1) - (c3)$ constitute an independent axiomatic system. To that end, we show three examples of non-completable fuzzy metric space, which do not satisfy anyone of these three axioms but they satisfy the other two.

Example 12 *(Gregori and Romaguera [9].) Let $\{x_n\}$ and $\{y_n\}$ be two strictly increasing sequences of positive real numbers, which converge to 1 with respect to the usual topology of $\mathbb{R}$, with $A \cap B = \emptyset$, where $A = \{x_n : n \in \mathbb{N}\}$ and $B = \{y_n : n \in \mathbb{N}\}$. Put $X = A \cup B$ and define a fuzzy set M on $X \times X \times]0, \infty[$ by:*

$M(x_n, x_n, t) = M(y_n, y_n, t) = 1$ *for all* $n \in \mathbb{N}, t > 0$,

$M(x_n, x_m, t) = x_n \wedge x_m$ *for all* $n, m \in \mathbb{N}$ *with* $n \neq m, t > 0$,

$M(y_n, y_m, t) = y_n \wedge y_m$ *for all* $n, m \in \mathbb{N}$ *with* $n \neq m, t > 0$,

$M(x_n, y_m, t) = M(y_m, x_n, t) = x_n \wedge y_m$ *for all* $n, m \in \mathbb{N}, t \geq 1$,

$M(x_n, y_m, t) = M(y_m, x_n, t) = x_n \wedge y_m \wedge t$ *for all* $n, m \in \mathbb{N}, t \in]0, 1[$.

As pointed out in [9], an easy computation shows that $(X, M, *)$ *is a fuzzy metric space, where* $*$ *is the minimum t-norm, and it satisfies conditions* $(c2)$ *and* $(c3)$ *of Theorem 1. But* M *does not satisfy condition* $(c1)$ *of Theorem 1. Indeed, in [9] it was observed that* $\{x_n\}$ *and* $\{y_n\}$ *are Cauchy sequences in* X *such that* $\lim_n M(x_n, y_n, t) = 1$ *for all* $t \geq 1$*, but* $\lim_n M(x_n, y_n, t) = t$ *for all* $t \in]0, 1[$.

Example 13 *(Gregori and Romaguera [8].) Let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences of distinct points such that* $A \cap B = \emptyset$*, where* $A = \{x_n : n \geq 3\}$ *and* $B = \{y_n : n \geq 3\}$*. Put* $X = A \cup B$ *and define a fuzzy set* M *on* $X \times X \times]0, \infty[$ *by:*

$M(x_n, x_m, t) = M(y_n, y_m, t) = 1 - \left[\frac{1}{n \wedge m} - \frac{1}{n \vee m}\right]$,

$M(x_n, y_m, t) = M(y_m, x_n, t) = \frac{1}{n} + \frac{1}{m}$,

for all $n, m \geq 3$*. In [8], it was proved that* $(X, M, *)$ *is a fuzzy metric space, where* $*$ *is the Luckasievicz t-norm* ($a * b = \max\{0, a + b - 1\}$)*, for which both* $\{x_n\}_{n \geq 3}$ *and* $\{y_n\}_{n \geq 3}$ *are Cauchy sequences. Clearly,*

$$\lim_n M(x_n, y_n, t) = \lim_n \left(\frac{1}{n} + \frac{1}{n}\right) = 0.$$

Therefore, M *does not satisfy condition* $(c2)$.

On the other hand, M *is a stationary fuzzy metric on* X*, and so by Remark 11 we have that this fuzzy metric space satisfies conditions* $(c1)$ *and* $(c3)$.

Example 14 *(Gregori et al. [6].) Let* d *be the usual metric on* $\mathbb{R}$ *restricted to* $]0, 1]$ *and consider the standard fuzzy metric* M_d *induced by* d*. Put* $X =]0, 1]$ *and define a fuzzy set* M *on* $X \times X] \times]0, \infty[$ *by*

$$M(x,y,t)=\begin{cases} M_d(x,y,t), & 0<t\leq d(x,y)\\ M_d(x,y,2t)\cdot\frac{t-d(x,y)}{1-d(x,y)}+M_d(x,y,t)\cdot\frac{1-t}{1-d(x,y)}, & d(x,y)<t\leq 1\\ M_d(x,y,2t), & t>1\end{cases}$$

In [6] it is proved that $(X,M,*)$ is a fuzzy metric space, where $*$ is the usual product. Also, it is obtained that for the Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X, given by $a_n=\frac{1}{n}$ and $b_n=1$ for all $n\in\mathbb{N}$, the assignment

$$\lim_n M(a_n,b_n,t)=\begin{cases}\frac{t}{t+1}, & 0<t<1\\ \\ \frac{2t}{2t+1}, & t\geq 1\end{cases}$$

is a well-defined function on $]0,\infty[$ which is not continuous at $t=1$. Therefore, M does not satisfy condition $(c3)$.

Next, we will see that M satisfies conditions $(c1)$ and $(c2)$.

For proving that M satisfies $(c1)$, we suppose that $\{a_n\}$ and be $\{b_n\}$ are two Cauchy sequences in $]0,1]$ such that $\lim_n M(a_n,b_n,s)=1$ for some $s>0$. By Lemma 3, we can find $t_0>1$, with $t_0>s$, such that $\lim_n M(a_n,b_n,t_0)=1$. Then,

$$\lim_n M(a_n,b_n,t_0)=\lim_n M_d(a_n,b_n,2t_0)=\lim_n\frac{2t_0}{2t_0+|a_n-b_n|}=1$$

and thus $\lim_n |a_n-b_n|=0$.

Let $t>0$. We distinguish two cases:

(1) If $t\in]0,1]$, then there exists $n_0\in\mathbb{N}$ such that $|a_n-b_n|<t$ for all $n\geq n_0$, since $\lim_n|a_n-b_n|=0$. Then

$$\lim_n M(a_n,b_n,t)=\lim_n\left(\frac{2t}{2t+|a_n-b_n|}\cdot\frac{t-|a_n-b_n|}{1-|a_n-b_n|}+\frac{t}{t+|a_n-b_n|}\cdot\frac{1-t}{1-|a_n-b_n|}\right)=$$

$$=t+1-t=1$$

(2) If $t>1$, then

$$\lim_n M(a_n,b_n,t)=\lim_n\frac{2t}{2t+|a_n-b_n|}=1$$

Therefore, $\lim_n M(a_n, b_n, t) = 1$ *for all* $t > 0$*, and so* M *satisfies* $(c1)$*.*

Now, we will prove that M *satisfies* $(c2)$*. Suppose the contrary, i.e., there exist two Cauchy sequences* $\{a_n\}$ *and* $\{b_n\}$ *such that* $\lim_n M(a_n, b_n, s) = 0$ *for some* $s > 0$*. First, we claim that* M*-Cauchy sequences are Cauchy for the usual metric* d *of* $\mathbb{R}$ *restricted to* $]0, 1]$*. Indeed, if* $\{a_n\}$ *is a Cauchy sequence in* $(X, M, *)$*, then* $\lim_{n,m} M(a_n, a_m, t) = 1$ *for all* $t > 0$*. In particular, for* $t > 1$ *we have that* $\lim_{n,m} M(a_n, a_m, t) = \lim_{n,m} \frac{2t}{2t+|a_n-a_m|} = 1$*, and so* $\lim_{n,m} |a_n - a_m| = 0$*, i.e.,* $\{a_n\}$ *is Cauchy in* $(\mathbb{R}, d)$*.*

Then, there exist $a, b \in [0, 1]$ *such that* $\{a_n\}$ *and* $\{b_n\}$ *converge to* a *and* b*, respectively, for the usual topology of* $\mathbb{R}$ *restricted to* $[0, 1]$*. Therefore,* $\lim_n |a_n - b_n| = |a - b|$*.*

We distinguish two cases:

(1) Suppose that $|a - b| = 0$*. Then for* $t_0 > 1$ *we have that*

$$\lim_n M(a_n, b_n, t_0) = \lim_n \frac{2t_0}{2t_0 + |a_n - b_n|} = \frac{2t_0}{2t_0 + |a - b|} = 1.$$

So $M(a_n, b_n, t) = 1$ *for all* $t > 0$*, since* M *satisfies condition* $(c1)$*, a contradiction.*

(2) Suppose that $|a - b| \in]0, 1]$*. Taking into account our assumption and Lemma 3, we can find* $0 < t_0 < |a-b|$*, with* $t_0 < s$*, such that* $\lim_n M(a_n, b_n, t_0) = 0$*. Then there exists* $n_0 \in \mathbb{N}$ *such that* $|a_n - b_n| > t_0$ *for all* $n \geq n_0$*, and so*

$$\lim_n M(a_n, b_n, t_0) = \lim_n \frac{t_0}{t_0 + |a_n - b_n|} = \frac{t_0}{t_0 + |a - b|} > 0,$$

a contradiction.

Therefore, M *satisfies* $(c2)$*.*

Consequently, $(c1) - (c3)$ constitute an independent axiomatic system.

4 Completable strong fuzzy metrics

In this section we will show that condition $(c3)$ in Theorem 1 can be omitted when $(X, M, *)$ is a strong fuzzy metric space.

We begin this section giving five lemmas.

Lemma 15 *Let* $(X, M, *)$ *be a strong fuzzy metric space and let* $\{a_n\}$*,* $\{b_n\}$ *be two Cauchy sequences in* X*. For each* $t > 0$*, the sequence* $\{M(a_n, b_n, t)\}_n$

converges in $[0,1]$ *with the usual topology of* $\mathbb{R}$ *restricted to* $[0,1]$.

Proof. Fix $t > 0$. Let $\{a_n\}$ and $\{b_n\}$ be two Cauchy sequences in X. Since $[0,1]$ is compact the sequence $M(a_n, b_n, t) \in [0,1]$ has a subsequence $\{M(a_{n_k}, b_{n_k}, t)\}_k$ that converges to some $c \in [0,1]$. We will see that $\{M(a_n, b_n, t)\}_n$ converges to c.

Contrary, suppose that $\{M(a_n, b_n, t)\}_n$ does not converge to c. Then, we can find a subsequence $\{M(a_{m_i}, b_{m_i}, t)\}_i$ of $\{M(a_n, b_n, t)\}_n$ converging to $a \in [0,1]$, with $a \neq c$.

Now, since M is strong, for each $i, k \in \mathbb{N}$ we have that

$$M(a_{n_k}, b_{n_k}, t) \geq M(a_{n_k}, a_{m_i}, t) * M(a_{m_i}, b_{m_i}, t) * M(b_{m_i}, b_{n_k}, t)$$

and taking limit as $i, k \to \infty$, we have that

$$\lim_k M(a_{n_k}, b_{n_k}, t) \geq \lim_i M(a_{m_i}, b_{m_i}, t).$$

With a similar argument, we can also obtain

$$\lim_i M(a_{m_i}, b_{m_i}, t) \geq \lim_k M(a_{n_k}, b_{n_k}, t).$$

So, $c = \lim_k M(a_{n_k}, b_{n_k}, t) = \lim_i M(a_{m_i}, b_{m_i}, t) = a$, a contradiction.

Therefore, $\lim_n M(a_n, b_n, t) = c$.

Lemma 16 *Let* $(X, M, *)$ *be a fuzzy metric space, let* $\{a_n\}$ *be a Cauchy sequence in* X *and let* $\{t_n\}$ *be a strictly increasing (decreasing) sequence of positive real numbers converging to* $t_0 > 0$ *(for the usual topology of* $\mathbb{R}$*). Then* $\lim_{n,m} M(a_n, a_m, t_n) = 1$.

Proof. It is immediate.

Lemma 17 *Let* $(X, M, *)$ *be a strong fuzzy metric space. Let* $\{a_n\}$, $\{b_n\}$ *be two Cauchy sequences in* X *and let* $\{t_n\}$ *be a strictly increasing (decreasing) sequence of positive real numbers converging to* $t_0 > 0$ *(for the usual topology of* $\mathbb{R}$*). Then, the sequence* $\{M(a_n, b_n, t_n)\}_n$ *converges in* $[0,1]$, *with the usual topology of* $\mathbb{R}$ *restricted to* $[0,1]$.

Proof. Let $\{a_n\}$, $\{b_n\}$ be two Cauchy sequences in X and let $\{t_n\}$ be a strictly increasing sequence of positive real numbers converging to $t_0 > 0$. Consider the sequence $\{M(a_n, b_n, t_n)\}_n \subset [0,1]$. Since $[0,1]$ is compact then, there exists a subsequence $\{M(a_{n_k}, b_{n_k}, t_{n_k})\}_k$ of $\{M(a_n, b_n, t_n)\}_n$ converging to $c \in [0,1]$.

Suppose that $\{M(a_n, b_n, t_n)\}_n$ does not converge to c. Then, we can find a subsequence $\{M(a_{m_i}, b_{m_i}, t_{m_i})\}_i$ of $\{M(a_n, b_n, t_n)\}_n$ converging to $a \in [0,1]$, with $a \neq c$.

Suppose, without loss of generality, that $a > c$. We will construct, by induction, two subsequences $\{M(a_{n_{k_l}}, b_{n_{k_l}}, t_{n_{k_l}})\}_l$ and $\{M(a_{m_{i_j}}, b_{m_{i_j}}, t_{m_{i_j}})\}_j$ of $\{M(a_{n_k}, b_{n_k}, t_{n_k})\}_k$ and $\{M(a_{m_i}, b_{m_i}, t_{m_i})\}_i$, respectively, as follows.

Take $m_{i_1} = m_1 \in \mathbb{N}$. We can choose $n_{k_1} \in \mathbb{N}$ such that $n_{k_1} > m_{i_1}$ and $t_{n_{k_1}} > t_{m_{i_1}}$ (since $\{t_{n_k}\}$ is strictly increasing). By Lemma 3 and using that M is strong, we have that

$$M(a_{n_{k_1}}, b_{n_{k_1}}, t_{n_{k_1}}) \geq M(a_{n_{k_1}}, b_{n_{k_1}}, t_{m_{i_1}}) \geq$$

$$M(a_{n_{k_1}}, a_{m_{i_1}}, t_{m_{i_1}}) * M(a_{m_{i_1}}, b_{m_{i_1}}, t_{m_{i_1}}) * M(b_{m_{i_1}}, b_{n_{k_1}}, t_{m_{i_1}}).$$

Now, we choose $m_{i_2} \in \mathbb{N}$ such that $m_{i_2} > n_{k_1}$. Given m_{i_2}, we can choose $n_{k_2} \in \mathbb{N}$ such that $n_{k_2} > m_{i_2}$ and $t_{n_{k_2}} > t_{m_{i_2}}$. By Lemma 3 and using that M is strong, we have that

$$M(a_{n_{k_2}}, b_{n_{k_2}}, t_{n_{k_2}}) \geq M(a_{n_{k_2}}, b_{n_{k_2}}, t_{m_{i_2}}) \geq$$

$$M(a_{n_{k_2}}, a_{m_{i_2}}, t_{m_{i_2}}) * M(a_{m_{i_2}}, b_{m_{i_2}}, t_{m_{i_2}}) * M(b_{m_{i_2}}, b_{n_{k_2}}, t_{m_{i_2}}).$$

Therefore, by induction on j we have that

$$M(a_{n_{k_j}}, b_{n_{k_j}}, t_{n_{k_j}}) \geq$$

$$M(a_{n_{k_j}}, a_{m_{i_j}}, t_{m_{i_j}}) * M(a_{m_{i_j}}, b_{m_{i_j}}, t_{m_{i_j}}) * M(b_{m_{i_j}}, b_{n_{k_j}}, t_{m_{i_j}}).$$

Taking limit as $j \to \infty$, by Lemma 16 we have that.

$$c = \lim_j M(a_{n_{k_j}}, b_{n_{k_j}}, t_{n_{k_j}}) \geq \lim_j M(a_{m_{i_j}}, b_{m_{i_j}}, t_{m_{i_j}}) = a,$$

a contradiction.

Therefore, $\lim_n M(a_n, b_n, t_n) = c$.

If $\{t_n\}$ is strictly decreasing, it is proved in a similar way.

Lemma 18 *Let $(X, M, *)$ be a strong fuzzy metric space. Let $\{a_n\}$, $\{b_n\}$ be two Cauchy sequences in X and let $\{t_n\}$, $\{s_n\}$ be two strictly increasing (decreasing) sequences of positive real numbers converging to $t_0 > 0$ (for the usual topology of $\mathbb{R}$). Then,* $\lim_n M(a_n, b_n, t_n) = \lim_n M(a_n, b_n, s_n)$.

Proof. Let $\{a_n\}$, $\{b_n\}$ be two Cauchy sequences in X and let $\{t_n\}$, $\{s_n\}$ be two strictly increasing sequence of positive real numbers converging to $t_0 > 0$. By Lemma 17, there exist $a, c \in [0, 1]$ such that $\lim_n M(a_n, b_n, t_n) = a$ and $\lim_n M(a_n, b_n, s_n) = c$. Contrary, suppose that $\lim_n M(a_n, b_n, t_n) \neq \lim_n M(a_n, b_n, s_n)$. Suppose, without loss of generality, that $a < c$.

In a similar way that in the proof of the above lemma, we will construct two subsequences $\{M(a_{n_k}, b_{n_k}, t_{n_k})\}_k$ and $\{M(a_{m_i}, b_{m_i}, s_{m_i})\}_i$ of $\{M(a_n, b_n, t_n)\}_n$ and $\{M(a_n, b_n, s_n)\}_n$, respectively, such that $t_{n_k} > s_{m_k}$ for all $k \in \mathbb{N}$ and we have that

$$M(a_{n_k}, b_{n_k}, t_{n_k}) \geq$$
$$M(a_{n_k}, a_{m_k}, s_{m_k}) * M(a_{m_k}, b_{m_k}, s_{m_k}) * M(b_{m_k}, b_{n_k}, s_{m_k})$$

for each $k \in \mathbb{N}$.

Taking limit as $k \to \infty$, by Lemma 16 we have that

$$a = \lim_k M(a_{n_k}, b_{n_k}, t_{n_k}) \geq \lim_k M(a_{m_k}, b_{m_k}, s_{m_k}) = c,$$

a contradiction.

Therefore, $\lim_n M(a_n, b_n, t_n) = \lim_n M(a_n, b_n, s_n)$.

The case in which $\{t_n\}$ and $\{s_n\}$ are strictly decreasing is proved in a similar way.

Lemma 19 *Let $(X, M, *)$ be a strong fuzzy metric space. Let $\{a_n\}$, $\{b_n\}$ be two Cauchy sequences in X and let $\{t_n\}$ be a strictly increasing (decreasing) sequence of positive real numbers converging to $t_0 > 0$ (for the usual topology of $\mathbb{R}$). Then, $\lim_n M(a_n, b_n, t_n) = \lim_n M(a_n, b_n, t_0)$.*

Proof. Let $\{a_n\}$, $\{b_n\}$ be two Cauchy sequences in X and let $\{t_n\}$ be a strictly increasing sequence of positive real numbers converging to $t_0 > 0$.

By Lemma 17, there exists $a \in [0, 1]$ such that $\lim_n M(a_n, b_n, t_n) = a$ and by Lemma 15, there exists $c \in [0, 1]$ such that $\lim_n M(a_n, b_n, t_0) = c$. Note that, by Lemma 3, since $\{t_n\}$ is strictly increasing converging to t_0, we have that for each $n \in \mathbb{N}$ we have that $M(a_n, b_n, t_n) \leq M(a_n, b_n, t_0)$ and so $a \leq c$.

Since $\lim_n M(a_n, b_n, t_0) = c$, for each $\epsilon \in]0, 1[$, with $\epsilon < c$, we can find $n_\epsilon \in \mathbb{N}$ such that $M(a_{n_\epsilon}, b_{n_\epsilon}, t_0) \in]c - \epsilon/2, c + \epsilon/2[$. By axiom $(GV5)$ we can find $\delta_{n_\epsilon} > 0$ such that $M(a_{n_\epsilon}, b_{n_\epsilon}, t) \in]c - \epsilon, c + \epsilon[$ for each $t \in]t_0 - \delta_{n_\epsilon}, t_0[$.

Suppose that $c > a$. Taking into account the last paragraph, we will construct a sequence $\{M(a_{n_k}, b_{n_k}, s_k)\}_k$, where $\{a_{n_k}\}$ and $\{b_{n_k}\}$ are subsequences of $\{a_n\}$ and $\{b_n\}$, respectively, converging to c, as follows.

Let $i_1 \in \mathbb{N}$, with $\frac{1}{i_1} < \min\{c, t_0\}$, then there exist $n_1 \in \mathbb{N}$ and $s_1 \in]t_0 - \frac{1}{i_1}, t_0[$ such that $M(a_{n_1}, b_{n_1}, s_1) > c - \frac{1}{i_1}$. Choose $i_2 \in \mathbb{N}$, with $\frac{1}{i_2} < t_0 - s_1$, then we can find $n_2 \in \mathbb{N}$, with $n_2 > n_1$ and $s_2 \in]t_0 - \frac{1}{i_2}, t_0[$, such that $M(a_{n_2}, b_{n_2}, s_2) > c - \frac{1}{i_2}$. Thus, in this way by induction on k, we construct the sequence $\{M(a_{n_k}, b_{n_k}, s_k)\}_k$, which obviously satisfies $\lim_k M(a_{n_k}, b_{n_k}, s_k) = c$. On the other hand, $\{s_k\}$ is a strictly increasing sequence of positive real numbers converging to t_0. Therefore, by Lemma 18 $\lim_k M(a_{n_k}, b_{n_k}, r_k) = c$ for each strictly increasing sequence $\{r_k\}$ of positive real numbers converging to t_0. In particular, if we consider the subsequence $\{t_{n_k}\}$ of $\{t_n\}$, then $\lim_k M(a_{n_k}, b_{n_k}, t_{n_k}) = c$, a contradiction, since $\lim_n M(a_n, b_n, t_n) = a < c$.

Therefore, $\lim_n M(a_n, b_n, t_n) = c$.

The case of $\{t_n\}$ strictly decreasing is proved in a similar way.

Theorem 20 *Let $(X, M, *)$ be a strong fuzzy metric space, and let $\{a_n\}$, $\{b_n\}$ be two Cauchy sequences in X. Then the assignment*

$$t \to \lim_n M(a_n, b_n, t), \text{ for each } t > 0$$

is a continuous function on $]0, \infty[$ provided with the usual topology of $\mathbb{R}$.

Proof. Let $\{a_n\}$ and $\{b_n\}$ be two Cauchy sequences in X. By Lemma 15, the assignment $t \to \lim_n M(a_n, b_n, t)$ for each $t > 0$, is a well-defined function on $]0, \infty[$ to $[0, 1]$.

Next, we will see that this function is continuous. First we see that for each $t > 0$ the mentioned function is left-continuous.

Fix $t_0 > 0$. By Lemma 15, we have that there exists $c \in [0, 1]$ such that $\lim_n M(a_n, b_n, t_0) = c$. We distinguish two cases:

(1) Suppose that $c = 0$. By Lemma 3 and Lemma 15 we have that $\lim_n M(a_n, b_n, s) = 0$ for all $s \in]0, t_0[$. So, the function $t \to \lim_n M(a_n, b_n, t)$ is left-continuous at t_0.

(2) Suppose that $c \in]0, 1]$. Contrary, suppose the function $t \to \lim_n M(a_n, b_n, t)$ is not left-continuous at t_0.

Then, there exists $\epsilon_0 \in]0, 1[$ such that for each $\delta \in]0, t_0[$ we can find $t_\delta \in]t_0 - \delta, t_0[$ such that $b_\delta = \lim_n M(a_n, b_n, t_\delta) \notin]c - \epsilon_0, c + \epsilon_0[$. Note that, by Lemma 3, $b_\delta \leq c$ and so $b_\delta < c - \epsilon_0$.

On the other hand, given $t_\delta \in]t_0 - \delta, t_0[$, since $\lim_n M(a_n, b_n, t_\delta) = b_\delta < c - \epsilon_0$, for $\epsilon_0/2$ we can find $n(\delta) \in \mathbb{N}$ such that $M(a_n, b_n, t_\delta) \in]b_\delta - \epsilon_0/2, b_\delta + \epsilon_0/2[$ for each $n \geq n(\delta)$. Therefore, $M(a_n, b_n, t_\delta) < c - \epsilon_0/2$ for each $n \geq n(\delta)$.

Now, we will construct a sequence $\{M(a_{n_k}, b_{n_k}, t_k)\}_k$, where $\{a_{n_k}\}$ and $\{b_{n_k}\}$ are subsequences of $\{a_n\}$ and $\{b_n\}$, respectively, as follows.

Consider $i_1 \in \mathbb{N}$, with $\frac{1}{i_1} < t_0$. We can find $t_1 \in]t_0 - \frac{1}{i_1}, t_0[$ such that $\lim_n M(a_n, b_n, t_1) < c - \epsilon_0$. Then, we can find $n(i_1) \in \mathbb{N}$ such that $M(a_n, b_n, t_1) < c - \epsilon_0/2$ for each $n \geq n(i_1)$. We choose $n_1 = n(i_1)$.

Consider now, $i_2 \in \mathbb{N}$, with $\frac{1}{i_2} \in]t_1, t_0[$. We can find $t_2 \in]t_0 - \frac{1}{i_2}, t_0[$ such that $\lim_n M(a_n, b_n, t_2) < c - \epsilon_0$. Then, we can find $n(i_2) \in \mathbb{N}$ such that $M(a_n, b_n, t_2) < c - \epsilon_0/2$ for each $n \geq n(i_2)$. We choose $n_2 \geq n(i_2)$, with $n_2 > n_1$.

So, by induction on k we construct the sequence $\{M(a_{n_k}, b_{n_k}, t_k)\}_k$, where $\{a_{n_k}\}$ and $\{b_{n_k}\}$ are subsequences of $\{a_n\}$ and $\{b_n\}$, respectively, such that $M(a_{n_k}, b_{n_k}, t_k) < c - \epsilon_0/2$ for each $k \in \mathbb{N}$. Also, $\{t_k\}$ is a strictly increasing sequence of positive real numbers converging to t_0. Therefore, by Lemma 19, we have that $\lim_k M(a_{n_k}, b_{n_k}, t_k) = \lim_k M(a_{n_k}, b_{n_k}, t_0) = \lim_n M(a_n, b_n, t_0) = c$, a contradiction.

So, the above assignment is a left-continuous function at t_0.

In a similar way it is proved that $t \to \lim_n M(a_n, b_n, t)$ is right-continuous at t_0 using a strictly decreasing sequence $\{t_n\}$ converging to t_0 and thus it is continuous at t_0.

Hence, the assignment $t \to \lim_n M(a_n, b_n, t)$ is a continuous function on $]0, \infty[$.

Theorem 21 *A strong fuzzy metric space $(X, M, *)$ is completable if and only if for each pair of Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X the following conditions are fulfilled:*

(c1) $\lim_n M(a_n, b_n, s) = 1$ *for some* $s > 0$ *implies* $\lim_n M(a_n, b_n, t) = 1$ *for all* $t > 0$.

(c2) $\lim_n M(a_n, b_n, t) > 0$ *for all* $t > 0$.

Proof. The proof is immediate using Theorem 20 and Theorem 1.

By Theorem 9 and the fact that the minimum t-norm is integral, the following corollaries are immediate.

Corollary 22 *Let $(X, M, *)$ be a strong fuzzy metric space and suppose that $*$ is integral. Then $(X, M, *)$ is completable if and only if for each pair of Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X the condition (c1) is satisfied.*

Corollary 23 *Let $(X, M, *)$ be a fuzzy ultrametric space. Then $(X, M, *)$ is completable if and only if for each pair of Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X the condition (c1) is satisfied.*

Remark 24 *The fuzzy metric of Example 12 is a fuzzy ultrametric, and it is not completable because it does not satisfies* $(c1)$*. The fuzzy metric of Example 13 is strong and satisfy* $(c1)$ *but it is not completable because* $*$ *is not integral.*

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