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A duality relationship between fuzzy metrics and metrics

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ABSTRACT
Based on the duality relationship between indistinguishability operators and (pseudo-)metrics, we address the problem of establishing whether there is a relationship between the last ones and fuzzy (pseudo-)metrics. We give a positive answer to the posed question. Concretely, we yield a method for generating fuzzy (pseudo-)metrics from (pseudo)-metrics and vice-versa. The aforementioned methods involve the use of the pseudo-inverse of the additive generator of a continuous Archimedean t-norm. As a consequence we get a method to generate non-strong fuzzy (pseudo-)metrics from (pseudo-)metrics. Examples that illustrate the exposed methods are also given. Finally, we show that the classical duality relationship between indistinguishability operators and (pseudo)-metrics can be retrieved as a particular case of our results when continuous Archimedean t-norms are under consideration.

KEYWORDS
Fuzzy (pseudo-)metric; indistinguishability operator; continuous Archimedean t-norm; additive generator; pseudo-inverse; (pseudo-)metric.

1. Introduction

In 1975, it was intorduced a concept of fuzzy (pseudo-)metric space in Kramosil and Michalek (1975) (see Section 2) as an adaptation to the fuzzy context of the notion of probabilistic metric space given in Menger (1942). Later, in 1982, E. Trillas introduced the concept of indistinguishability operator (see Section 2) as a way to measure the degree of equivalence, in fuzzy logic, between the elements of a set (see Trillas (1982)). In both cases, the fuzzy measures satisfy an inequality (triangle inequality for fuzzy pseudo-metrics and transitivity for indistinguishability operators) which involves a t-norm (a continuous t-norm in the case of fuzzy pseudo-metrics). Although both notions allow to measure the dissimilarity or similarity between objects at some level or degree, there is an essential difference between a fuzzy pseudo-metric and an indistinguishability operator. Concretely the first one provides measurements with respect to a parameter and the second one does not.

Since they were defined, these concepts have led to independent fields of research and many authors have contributed to their study. An interesting topic in the study of indistinguishability operators, which has been addressed for different authors in the literature (see De Baets and Mesiar (2002); Recasens (2000) and references therein),

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is the study of the relation between classical pseudo-metrics (or distinguishability operators) and indistinguishability operators. The aforementioned relationship can be stated in the following way. On the one hand, we can construct a (pseudo-)metric from an indistinguishability operator through the additive generator of the involved t-norm (see assertion (ii) in the statement of Theorem 2.3). On the other hand, we can construct an indistinguishability operator from a (pseudo-)metric by means of the pseudo-inverse of an additive generator of the involved t-norm when this one is continuous (see assertion (i) in the statement of Theorem 2.3).

In the light of the existence of the exposed duality relationship and the fact that an indistinguishability operator defined for a continuous t-norm is, really, a stationary fuzzy (pseudo-)metric, i.e., a fuzzy (pseudo-)metric whose values are independent from the parameter, it seems natural to wonder whether there exists the possibility to state a kind of duality relationship between fuzzy (pseudo-)metrics and (pseudo-)metrics in such a way that the classical one between indistinguishability operators and (pseudo-)metrics can be retrieved as a particular case.

In this paper, we focus our investigation on the posed question and get a positive answer. Thus, we provide a concrete method for generating fuzzy (pseudo-)metrics from (pseudo)-metrics using the pseudo-inverse of the additive generator of a continuous t-norm (see Theorem 3.1) and, in addition, we show that such a method allows to construct non-strong fuzzy (pseudo-)metrics (see Section 2). Moreover, we yield a method to generate (pseudo-)metrics from fuzzy (pseudo-)metrics by means of the additive generator of the continuous t-norm under consideration (see Theorem 4.1). Besides, we show that the classical results stated for indistinguishability operators can be obtained as a consequence of our main results (see Corollary 3.5 and Corollary 4.4).

The interest on our study is twofold. On the one hand, fuzzy (pseudo-)metrics have been used successfully for engineering problems, in particular in colour image filtering (see for instance Camarena et al. (2008); Morillas et al. (2009, 2005, 2007)). In these papers, the classical metrics, used in the algorithms, were replaced by fuzzy ones and it was shown that then the algorithms yielded improved results. In addition, in Celebi (2009) a comparative study of several dissimilarities (classical and fuzzy) used in the "Reduced Ordering Based Vectors Filters" problem and the conclusions of such a study were that the algorithms using fuzzy (pseudo-)metrics significantly outperform the results obtained by means of those algorithms that implement the classical ones. Nevertheless, an important drawback in the applicability of fuzzy (pseudo-)metrics is the lack of examples in the literature, which makes it difficult for engineers to use them in solving their problems. In particular, there are not many examples of fuzzy (pseudo-)metrics that, on the one hand, involve a t-norm distinct from the product or Lukasievicz and, on the other hand, they are non-strong. Therefore, for each continuous Archimedean t-norm, the method provided by Theorem 3.1 "fuzzifies" a classical (pseudo-)metric and, thus, we can construct a collection of examples of fuzzy (pseudo-)metrics to experiment them in engineering problems in which the solution is obtained implementing algorithms that use classical (pseudo-)metrics.

On the other hand, in Castro-Company et al. (2015) (see also references therein) the problem of constructing a "compatible" metric from a given fuzzy one have been studied. In this direction the method given by Theorem 4.1 allows to construct a classical (pseudo-)metrics from a fuzzy (pseudo)-metrics in such a way that a (large) class of continuous t-norms, the Archimedean ones, can be taken under consideration. Moreover, the technique introduced in Castro-Company et al. (2015) needs to use an additional function which is not related to the continuous t-norm under consideration. However, our method presents the advantage of needing only to consider the additive generator of the t-norm. Despite the aforesaid benefit, our method should be restricted to take under consideration only Archimedean continuous t-norm which excludes, for instance, the minimum t-norm.

The structure of the remainder of the paper is as follows: in Section 2 we recall the essential notions of indistinguishability operators and fuzzy pseudo-metrics which will be crucial in our subsequent work. In Section 3, we introduce the method for inducing fuzzy (pseudo-)metrics from (pseudo)-metrics using the pseudo-inverse of the additive generator of a continuous t-norm. Moreover, in the same section, we discuss conditions under which our method gives as a result non-strong fuzzy pseudo-metrics. Furthermore, we give examples in order to illustrate the introduced method. Finally, in Section 4, we study the converse of the aforesaid method. Thus we generate (pseudo-)metrics from fuzzy (pseudo-)metrics by means of an additive generator of the continuous t-norm under consideration.

2. Preliminaries on indistinguishability operators and fuzzy metrics

In this section we will recall a few pertinent notions about t-norms, indistinguishability operators and fuzzy pseudo-metrics that will play a central role in the remainder of the paper.

Our main reference for t-norms is Klement et al. (2000).

Let us recall that a t-norm is a function $* : [0,1]^2 \to [0,1]$ such that for all $x, y, z \in [0,1]$ the following four axioms are satisfied:

(T1) $x * y = y * x$; (Commutativity)
(T2) $x * (y * z) = (x * y) * z$; (Associativity)
(T3) $x * y \leq x * z$ whenever $y \leq z$; (Monotonicity)
(T4) $x * 1 = x$. (Boundary Condition)

If in addition, the t-norm $*$ is a continuous function on $[0,1]^2$, then we will say that $*$ is a continuous t-norm. Moreover, a t-norm is called Archimedean if for each $x, y \in]0,1[$ there exists $n \in \mathbb{N}$ such that $x^{(n)} < y$, where $\mathbb{N}$ stands for the set of positive integer numbers and $x^{(n)}$ is defined as follows: $x^{(1)} = x$ and $x^{(n+1)} = x^{(n)} * x$ for all $n \in \mathbb{N}$. The following proposition characterizes the class of continuous Archimedean t-norms.

Proposition 2.1. *A continuous t-norms $*$ is Archimedean if and only if it satisfies $x * x < x$ for each $x \in]0,1[$.*

Two well-known examples of continuous Archimedean t-norms are the usual product $*_P$ and the Luckasievicz t-norm $*_L$, where $x *_P y = x \cdot y$ and $x *_L y = \max\{x + y - 1, 0\}$ for all $x, y \in [0,1]$. An example of continuous t-norm which is non-Archimedean is the minimum t-norm $\wedge$, i.e., $x \wedge y = \min\{x, y\}$ for all $x, y \in [0,1]$.

Essential concepts, in our work, which allow us to represent a class of t-norms are the notions of pseudo-inverse and additive generator. Let us recall that, given a strictly decreasing continuous function $f : [0,1] \to [0,\infty]$ such that $f(1) = 0$, the pseudo-inverse $f^{(-1)}$ of f is the function $f^{(-1)} : [0,\infty] \to [0,1]$ defined as follows:

$$f^{(-1)}(y) = \begin{cases} f^{-1}(y), & \text{if } y < f(0) \\ 0, & \text{elsewhere} \end{cases}.$$

Moreover, given a t-norm $*$, a strictly decreasing continuous function $f_* : [0,1] \to [0,\infty]$ is said to be an additive generator of $*$ provided that $f_*(1) = 0$ and

$$x * y = f_*^{(-1)}(f_*(x) + f_*(y))$$

for all $x, y \in [0,1]$, where $f_*^{(-1)}$ is the pseudo-inverse of the additive generator f_*.

Note that in this case, the t-norm $*$ induced by means of the pseudo-inverse of the additive generator is continuous, since the continuity of the t-norm is equivalent to the continuity of the additive generator.

It is known that each t-norm with an additive generator is Archimedean. Nevertheless, the converse of this assertion is not true in general, but the next result states that continuous Archimedean t-norms always admit an additive generator.

Theorem 2.2. *A function $* : [0,1]^2 \to [0,1]$ is a continuous Archimedean t-norm if and only if there exists an additive generator f_* of $*$.*

2.1. *Indistinguishability operators*

Based on the notion of similarity relation introduced in Zadeh (1971), it was introduced the notion of indistinguishability operator in Trillas (1982). According to Trillas (see also Recasens (2000)), an indistinguishability operator E on a non-empty set X for $*$ is a fuzzy set $E : X \times X \to [0,1]$ which satisfies for each $x, y, z \in X$ the following axioms:

(E1) $E(x,x) = 1$; (Reflexivity)
(E2) $E(x,y) = E(y,x)$; (Symmetry)
(E3) $E(x,z) \geq E(x,y) * E(y,z)$. (Transitivity)

If in addition, E satisfies for all $x, y \in X$ the following condition:

(E1') $E(x,y) = 1 \Rightarrow x = y$,

then E is said to separate points.

The notion of indistinguishability operator is essentially interpreted as a measure of similarity (in contrast to dissimilarity modeled by pseudo-metrics). Thus, $E(x,y)$ matches up with the degree of indistinguishability between the objects x and y. In fact, the greater $E(x,y)$ the most similar are x and y. In such a way that when $x = y$, then the measure of similarity is exactly $E(x,x) = 1$.

Several authors have studied the relationship between indistinguishability operators and (pseudo-)metrics (see De Baets and Mesiar (1997, 2002); Gottwald (1992); Höle (1993); Klement et al. (2000); Ovchinnikov (1984); Recasens (2000); Valverde (1985)). It must be pointed out that, from now on, we will consider (pseudo-)metrics which can take the value ∞ (which are also known as extended (pseudo-)metrics in Deza M.M. and Deza E. (2016)). The following well-known result provides a method of constructing an indistinguishability operator from a pseudo-metric and vice-versa (see for instance Klement et al. (2000)).

Theorem 2.3. *Let X be a non-empty-set and let $*$ be a continuous Archimedean t-norm with additive generator $f_* : [0,1] \to [0,\infty]$. Then we have:*

(i) If d is a pseudo-metric on X, then the fuzzy set $E_d : X \times X \to [0,1]$ is an indistinguishability operator for $*$ on X, where $E_d(x,y) = f_*^{(-1)}(d(x,y))$ for each $x,y \in X$. Furthermore, E_d separates points if and only if d is a metric.

(ii) If E is an indistinguishability operator for $*$ on X, then the function $d_E : X \times X \to [0,\infty]$ is a pseudo-metric on X, where $d_E(x,y) = f_*(E(x,y))$ for each $x,y \in X$. In addition, d_E is a metric if and only if E separates points.

It must be stressed that the method, provided by assertion (ii) in the preceding result, to construct pseudo-metrics from indistinguishability operators is presented for continuous Archimedean t-norms in Klement et al. (2000). However, such a method can be stated for t-norms admitting an additive generator (so Archimedean t-norms) according to De Baets and Mesiar (1997, 2002); Gottwald (1992); Höle (1993); Klement et al. (2000); Ovchinnikov (1984); Recasens (2000); Valverde (1985)). Anyway, we have decided to use the approach given in Klement et al. (2000) because it will be sufficient for our further discussion.

2.2. *Fuzzy pseudo-metric spaces*

In our subsequent work we will focus our study on the notion of fuzzy pseudo-metric introduced in Kramosil and Michalek (1975). Let us recall, following the reformulation of the original definition presented in Grabiec (1988) and in George and Veeramani (1994), that a fuzzy pseudo-metric space is an ordered triple $(X, M, *)$ such that X is a non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times [0,\infty[$ satisfying the following conditions, for all $x,y,z \in X$ and $s,t > 0$:

(KM1) $M(x,y,0) = 0$;
(KM2) $M(x,x,t) = 1$ for all $t > 0$;
(KM3) $M(x,y,t) = M(y,x,t)$;
(KM4) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$;
(KM5) The function $M_{x,y} : [0,\infty[\to [0,1]$ is left-continuous, where $M_{x,y}(t) = M(x,y,t)$ for each $t \in [0,\infty[$.

A fuzzy pseudo-metric space $(X, M, *)$ is said to be a *fuzzy metric space* when the axiom $(KM2)$ is replaced by the following one:

(KM2') $M(x,y,t) = 1$ for all $t > 0$ if and only if $x = y$.

According to George and Veeramani (1994), given a fuzzy pseudo-metric space $(X, M, *)$, the value $M(x,y,t)$ can be interpreted as the degree of nearness or similarity between x and y with respect to a positive real parameter t. Nevertheless, one can observe that the value $M(x,y,t)$ is not used in the classical metric sense, i.e. 0 for close and 1 for distant, but as a similarity, i.e. 0 for distant and 1 close.

In the light of the preceding interpretation, it is clear that axiom $(KM1)$ does not play any interesting role from a fuzzy measurement viewpoint. In fact, the properties of fuzzy measurement that can be attributed to a fuzzy metric involves only the remainder of axioms, $(KM2) - (KM5)$, for all $t \in]0,\infty[$. So, in the following, we will refer as a fuzzy pseudo-metric a fuzzy set M on $X \times X \times]0,\infty[$ satisfying axioms $(KM2) - (KM5)$ for all $t \in]0,\infty[$.

From now on, if $(X, M, *)$ is a fuzzy (pseudo-)metric space we will say that $(M, *)$ is a *fuzzy (pseudo-)metric* on X.

A paradigmatic example of a fuzzy metric space is given by the so-called *standard* *fuzzy metric* space. If (X,d) is a metric space, then the standard fuzzy metric induced by d is the fuzzy set M_d defined on $X \times X \times]0,\infty[$ by

$$M_d(x,y,t) = \frac{t}{t+d(x,y)},$$

for all $x,y \in X$ and for all $t \in]0,\infty[$. According to George and Veeramani (1994), $(X, M_d, \wedge)$ is a fuzzy metric space, where $\wedge$ denotes the minimum t-norm. It must be stressed that the original proof given in George and Veeramani (1994) remains valid in the case in which the pseudo-metric can take the value ∞.

One can observe that the preceding example provides a way to construct a fuzzy (pseudo-)metric from a given classical one for each continuous t-norm, since the minimum t-norm is the greatest one.

Finally, we recall a well-known and distinguished class of fuzzy pseudo-metric spaces and its relation with indistinguishability operators.

Following Gregori and Romaguera (2004), a fuzzy pseudo-metric space $(X,M,*)$ is said to be *stationary* if M does not depend on $t \in]0,\infty[$, i.e., if the function $M_{x,y} :]0,\infty[\rightarrow [0,1]$ given by $M_{x,y}(t) = M(x,y,t)$ is constant for each $x,y \in X$. From now on, we will write $M(x,y)$ instead of $M(x,y,t)$ provided that the fuzzy pseudo-metric M is stationary.

We end this subsection noting that if E is an indistinguishability operator, for a continuous t-norm $*$, on a non-empty set X then it is not hard to check that the triple $(X, M_E, *)$ is a stationary fuzzy pseudo-metric space, where the stationary fuzzy pseudo-metric M_E is given by

$$M_E(x,y,t) = E(x,y),$$

for all $x,y \in X$ and $t \in]0,\infty[$. If in addition, E separates points, then $(M_E, *)$ is a fuzzy metric. So, indistinguishability operators induced by means of the pseudo-inverse of the additive generator of a continuous Archimedean t-norm (see assertion (i) in Theorem 2.3) can be considered as a stationary fuzzy pseudo-metrics.

3. A method for generating fuzzy pseudo-metrics from pseudo-metrics

In this section we will construct a fuzzy metric space from a given metric space (X,d). Our method will be based on the pseudo-inverse of a continuous Archimedean t-norm preserving the spirit of the construction given in assertion (i) in Theorem 3.1.

First, we will try to motivate the way of constructing our fuzzy metric from the classical one.

As we have pointed out in Section 2, given a fuzzy (pseudo-)metric space $(X,M,*)$, the value $M(x,y,t)$ can be interpreted as the degree of nearness or similarity between x and y, with respect to a positive real parameter t. Under this interpretation if we consider a (pseudo-)metric space (X,d) we can consider the parameter t as a threshold from which x and y would be indistinguishable, and so the degree of nearness between them will be 1. Before that threshold, the degree of nearness is fuzzified, taking values in $[0,1)$ smaller as t decreases, since $M_{x,y}$ is a decreasing function on $]0,\infty[$ (the fact that $M_{x,y}$ is a increasing function was proved in George and Veeramani (1994)).

The following fuzzy metric illustrates the aforementioned idea. Notice that it is a generalization of a well-known example of probabilistic metric space introduced in Schweizer and Sklar (1983).

It is not hard to see that, given a (pseudo-)metric space (X, d), $(X, M^d, \wedge)$ is a fuzzy (pseudo-)metric space, where M^d is given by

$$M^d(x, y, t) = \begin{cases} 0, & \text{if } 0 < t \leq d(x, y) \\ 1, & \text{if } t > d(x, y) \end{cases} .$$

Observe that the preceding example constitutes a drastic fuzzification of the classical (pseudo-)metric from which is defined. Indeed, the degree of nearness between two points $M(x, y, t)$ is 0, x and y are totally distinguishable before a threshold parameter value t ($t = d(x, y)$) and, in addition, x and y are indistinguishable from the threshold, i.e., $M(x, y, t) = 1$ when $t > d(x, y)$.

Taking into account the method of construction of indistinguishability operators given in assertion (i) in Theorem 2.3 and the fact that every indistinguishability operator, for a continuous and Archimedean t-norm, can be considered as a stationary fuzzy pseudo-metrics, we introduce a method to fuzzify a classical pseudo-metric in such a way that the spirit of the construction of the fuzzy metric M^d is preserved.

Theorem 3.1. *Let (X, d) be a pseudo-metric space and let $*$ be a continuous t-norm with additive generator f_*. Then, $(M_{d,f_*}, *)$ is a fuzzy pseudo-metric on X, where M_{d,f_*} is the fuzzy set defined on $X \times X \times]0, \infty[$ as follows:*

$$M_{d,f_*}(x, y, t) = f_*^{(-1)}\left(\max\{d(x, y) - t, 0\}\right),$$

for all $x, y \in X$ and for all $t \in]0, \infty[$. Furthermore, $(M_{d,f_*}, *)$ is a fuzzy metric on X if and only if d is a metric on X.

Proof. Let (X, d) be a pseudo-metric space and let $*$ be a continuous Archimedean t-norm with additive generator f_*. We define, for each $x, y \in X$ and each $t \in]0, \infty[$, the mapping

$$M_{d,f_*}(x, y, t) = f_*^{(-1)}\left(\max\{d(x, y) - t, 0\}\right).$$

We will see that $(M_{d,f_*}, *)$ is a fuzzy pseudo-metric on X.

First, note that the axiom $(KM3)$ is obviously fulfilled by definition of M_{d,f_*} and by the fact that $d(x, y) = d(y, x)$ for all $x, y \in X$. So, we only need to see that, for each $x, y \in X$ and each $t \in]0, \infty[$, the fuzzy set M_{d,f_*} also satisfies $(KM2), (KM4)$ and $(KM5)$.

Next we show that M_{d,f_*} satisfies $(KM2)$. To this end, let $x \in X$ and $t \in]0, \infty[$. Since (X, d) is a pseudo-metric space we have that $d(x, x) = 0$ and so

$$M_{d,f_*}(x, x, t) = f_*^{(-1)}\left(\max\{0 - t, 0\}\right) = f_*^{(-1)}(0) \text{ for each } t \in]0, \infty[.$$

Since $f_*^{(-1)}(0) = 1$ we deduce that $M(x, x, t) = 1$ for each $t \in]0, \infty[$ and $(KM2)$ is hold. In order to show that M_{d,f_*} satisfies $(KM4)$, let $x, y, z \in X$ and $t, s \in]0, \infty[$.

First, note that $d(x,z) - t - s \leq d(x,y) - t + d(y,z) - s$, since d is a pseudo-metric on X. Then,

$$\max\{d(x,y) - t, 0\} + \max\{d(y,z) - s, 0\} \geq \max\{d(x,z) - t - s, 0\}.$$

Hence, we obtain that

$$M_{d,f_*}(x,z,t+s) = f_*^{(-1)}(\max\{d(x,z) - t - s, 0\}) \geq$$
$$f_*^{(-1)}(\max\{d(x,y) - t, 0\} + \max\{d(x,y) - t, 0\}) \qquad ,$$

since $f_*^{(-1)}$ is decreasing.

Moreover, we have that

$$M_{d,f_*}(x,y,t) * M_{d,f_*}(y,z,s) =$$
$$f_*^{(-1)}(\max\{d(x,y) - t, 0\}) * f_*^{(-1)}(\max\{d(x,y) - s, 0\}) =$$
$$f_*^{(-1)}\left(f\left(f_*^{(-1)}(\max\{d(x,y) - t, 0\})\right) + f\left(f_*^{(-1)}(\max\{d(x,y) - s, 0\})\right)\right),$$

since f_* is an additive generator of $*$.

Since

$$f_*^{(-1)}(\max\{d(x,y) - t, 0\} + \max\{d(x,y) - s, 0\}) \geq$$
$$f_*^{(-1)}\left(f\left(f_*^{(-1)}(\max\{d(x,y) - t, 0\})\right) + f\left(f_*^{(-1)}(\max\{d(x,y) - s, 0\})\right)\right)$$

we deduce that

$$M_{d,f_*}(x,z,t+s) \geq M_{d,f_*}(x,y,t) * M_{d,f_*}(y,z,s).$$

Thus, $(KM4)$ is satisfied.

Next we show that $(KM5)$ is hold. Fix $x, y \in X$ and consider the function $M_{x,y} :]0, \infty[\to [0,1]$ given by $M_{x,y}(t) = M_{d,f_*}(x,y,t)$ for all $t \in]0, \infty[$. Then,

$$M_{x,y}(t) = \begin{cases} f_*^{(-1)}(d(x,y) - t), & \text{if } 0 < t \leq d(x,y) \\ 1, & \text{if } t > d(x,y) \end{cases}.$$

An straightforward computation, and taking into account that $f_*^{(-1)}$ is decreasing and continuous, gives that $M_{x,y}$ is continuous on $]0, \infty[$ and so left-continuous on $]0, \infty[$.

Therefore, $(M_{d,f_*}, *)$ is a fuzzy pseudo-metric on X.

It remains to prove that M_{d,f_*} is a fuzzy metric on X if and only if, d is a metric on X. To this end, note that M_{d,f_*} satisfies $(KM2')$ if and only if $M_{d,f}(x,y,t) = 1$ for all $t \in]0,\infty[$ implies $x = y$. Moreover, $M_{d,f_*}(x,y,t) = 1$ for all $t \in]0,\infty[$ is equivalent to $f_*^{(-1)}(\max\{d(x,y) - t, 0\}) = 1$ for all $t \in]0,\infty[$. Since $f_*^{(-1)}$ is the pseudo-inverse of an additive generator of $*$, then $f_*^{(-1)}(a) = 1$ if and only if $a = 0$. Therefore, $M(x,y,t) = 1$ for each $t \in]0,\infty[$ if and only if $\max\{d(x,y) - t, 0\} = 0$ for each $t \in]0,\infty[$, or equivalently, if and only if $d(x,y) = 0$. Thus, the fuzzy pseudo-metric $(M_{d,f_*}, *)$ is a fuzzy metric on X if and only if d is a metric space on X.

□

In the following two corollaries, we specify the method given in Theorem 3.1 for the case of the usual product $*_P$ and the Luckasievicz t-norm $*_L$.

Corollary 3.2. *Let (X,d) be a pseudo-metric space. Then, $(M_{d,f_{*_P}}, *_P)$ is a fuzzy pseudo-metric on X, where $M_{d,f_{*_P}}$ is the fuzzy set defined on $X \times X \times]0,\infty[$ as follows:*

$$M_{d,f_{*_P}}(x,y,t) = \begin{cases} \mathsf{e}^{t-d(x,y)}, & \text{if } t \leq d(x,y) \\ 1, & \text{if } t > d(x,y) \end{cases},$$

for all $x,y \in X$ and for all $t \in]0,\infty[$. Furthermore, $(M_{d,f_{*_P}}, *_P)$ is a fuzzy metric on X if and only if d is a metric on X.

Corollary 3.3. *Let (X,d) be a pseudo-metric space. Then, $(M_{d,f_{*_L}}, *_L)$ is a fuzzy pseudo-metric on X, where $M_{d,f_{*_L}}$ is the fuzzy set defined on $X \times X \times]0,\infty[$ as follows:*

$$M_{d,f_{*_L}}(x,y,t) = \begin{cases} 0, & \text{if } t \leq d(x,y) - 1 \\ 1 + t - d(x,y) & \text{if } d(x,y) - 1 < t \leq d(x,y) \\ 1, & \text{if } t \geq d(x,y) \end{cases},$$

for all $x,y \in X$ and for all $t \in]0,\infty[$. Furthermore, $(M_{d,f_{*_L}}, *_L)$ is a fuzzy metric on X if and only if d is a metric on X.

3.1. *Stationary fuzzy metric spaces and indistinguishability operators*

In this subsection, we will show that the method yielded in assertion (i) in Theorem 2.3 can be retrieved as a particular case of our construction provided in Theorem 3.1. The following result, that gives a method to construct an indistinguishability operator from a given fuzzy pseudo-metric, will be crucial to this end.

Lemma 3.4. *Let $(M,*)$ be a fuzzy pseudo-metric on X and let E_M be the fuzzy set defined on $X \times X$ as follows:*

$$E_M(x,y) = \bigwedge_{s>0} M(x,y,s) \text{ for each } x,y \in X.$$

Then, E_M is an indistinguishability operator for $*$ on X. Moreover, E_M separates points if and only if $(M,*)$ is a fuzzy metric on X.

Proof. First of all, we note that the numerical value $\bigwedge_{s>0} M(x,y,s)$ exists. Indeed, the set $\{M(x,y,s)\}_{s\in]0,\infty[}$ is bounded below by 0.

Next we show that the fuzzy set E_M satisfies axioms $(E1)$, $(E2)$ and $(E3)$. Clearly, $(E2)$ is fulfilled by definition of E_M and the fact that $M(x,y,s) = M(y,x,s)$ for all $x, y \in X$ and for all $s \in]0, \infty[$.

In order to show that E_M satisfies $(E1)$, let $x \in X$. Since $(M, *)$ is a fuzzy pseudo-metric on X we have that $M(x,x,s) = 1$ for each $s \in]0, \infty[$ and so $E_M(x,x) = \bigwedge_{s>0} M(x,x,s) = 1$. Thus, $(E1)$ is hold.

With the aim of showing that E_M satisfies $(E3)$, let $x, y, z \in X$. Since $(M, *)$ is a fuzzy pseudo-metric on X it is hold that

$$M(x,z,s) \geq M(x,y,s/2) * M(y,z,s/2), \text{for each } s \in]0, \infty[.$$

Then,

$$E_M(x,z) = \bigwedge_{s>0} M(x,z,s) \geq \bigwedge_{s>0} \left(M(x,y,s/2) * M(y,z,s/2)\right).$$

Since $\bigwedge_{s>0} M(u,v,s/2) \leq M(u,v,t/2)$ for all $u, v \in X$ and $t \in]0, \infty[$ we have that

$$\left(\bigwedge_{s>0} M(x,y,s/2)\right) * \left(\bigwedge_{s>0} M(y,z,s/2)\right) \leq M(x,y,s/2) * M(y,z,s/2)$$

for all $s \in]0, \infty[$ and, hence, that

$$\left(\bigwedge_{s>0} M(x,y,s/2)\right) * \left(\bigwedge_{s>0} M(y,z,s/2)\right) \leq \bigwedge_{s>0} \left(M(x,y,s/2) * M(y,z,s/2)\right).$$

Whence we deduce that

$$E_M(x,z) \geq \left(\bigwedge_{s>0} M(x,y,s/2)\right) * \left(\bigwedge_{s>0} M(y,z,s/2)\right) = E_M(x,y) * E_M(y,z).$$

Therefore, E_M satisfies axiom $(E3)$ too and so E_M is an indistinguishability operator for $*$ on X.

Finally, it remains to prove that E_M separates points if and only $(M, *)$ is a fuzzy metric space on X. It is easy to check that, given $x, y \in X$, $E_M(x,y) = 1 \Leftrightarrow M(x,y,s) = 1$ for each $s \in]0, \infty[$. Whence we immediately obtain that $E_M(x,y) = 1 \Leftrightarrow x = y$ if and only if $M(x,y,s) = 1$ for each $s \in]0, \infty[\Leftrightarrow x = y$.

□

In the light of Lemma 3.4 and Theorem 3.1 we are bale to achieve our promised target, i.e., that the method given in assertion (i) in Theorem 2.3 can be retrieved from the method provided in Theorem 3.1.

Corollary 3.5. *Let (X, d) be a pseudo-metric space and let $*$ be a continuous Archimedean t-norm with additive generator $f_* : [0,1] \to [0, \infty]$. Then, the fuzzy set $E_{d,f_*} : X \times X \to [0,1]$ is an indistinguishability operator for $*$ on X, where*

$E_{d,f_*}(x,y) = f^{(-1)}(d(x,y))$ *for each* $x, y \in X$. *Furthermore,* E_{d,f_*} *separates points if and only if* d *is a metric on* X.

Proof. Let (X, d) be a pseudo-metric space and let $*$ be a continuous Archimedean t-norm with additive generator f_*.

On the one hand, Theorem 3.1 ensures that $(M_{d,f_*}, *)$ is a fuzzy pseudo-metric, where M_{d,f_*} is given by

$$M_{d,f_*}(x,y,t) = f_*^{(-1)}\left(\max\{d(x,y) - t, 0\}\right), \text{ for each } x, y \in X, t \in]0, \infty[.$$

On the other hand, we define the fuzzy set $E_{M_{d,f_*}}$ on $X \times X\times$ given by

$$E_{M_{d,f_*}}(x,y) = \bigwedge_{s>0} M_{d,f_*}(x,y,s), \text{ for each } x, y \in X.$$

Then, by Lemma 3.4, we have that $E_{M_{d,f_*}}$ is an indistinguishability operator for $*$ on X. In addition, $E_{M_{d,f_*}}$ separates points if and only if $(M_{d,f_*}, *)$ is a fuzzy metric on X.

Now, observe that for each $x, y \in X$ we have that

$$E_{M_{d,f_*}}(x,y) = \bigwedge_{s>0} M_{d,f_*}(x,y,s) = \bigwedge_{s>0} \left(f_*^{(-1)}(d(x,y) - s)\right) = f_*^{(-1)}(d(x,y)),$$

since $f_*^{(-1)}$ is a decreasing function. Thus, the fuzzy set E_{d,f_*}, given by $E_d(x,y) = f_*^{(-1)}(d(x,y))$ for each $x, y \in X$, matches up with $E_{M_{d,f_*}}$ on $X \times X$ and, therefore, it is an indistinguishability operator for $*$ on X. Furthermore, $E_{M_{d,f_*}}$ separates points if and only if $(M_{d,f_*}, *)$ is a fuzzy metric on X. By Theorem 3.1 we have that $(M_{d,f_*}, *)$ is a fuzzy metric on X if and only if d is a metric on X. Therefore we conclude that E_{d,f_*} separates points if and only if d is a metric on X.

$\square$

3.2. *Non-strong fuzzy (pseudo-)metric spaces*

Many examples of fuzzy pseudo-metrics that can be found in the literature are strong. Let us recall that, according to Gregori et al. (2010), a fuzzy (pseudo-)metric $(M, *)$ on a non-empty set X is said to be *strong* (or *non-Archimedean*) if, in addition to $(KM2) - (KM5)$, M satisfies the following stronger version of the triangle inequality

(KM4') $M(x,z,t) \geq M(x,y,t) * M(y,z,t)$, for each $x, y, z \in X$ and $t \in]0, \infty[$.

So, taking into account that each stationary fuzzy pseudo-metric on a non-empty set X is an indistinguishability operator, each strong fuzzy pseudo-metrics can be seen as a parametric family of indistinguishability operators.

Fuzzy metric spaces satisfying the property of being strong constitute a large class. In fact, in the literature it is very difficult to find examples of non-strong fuzzy (pseudo-)metrics. Inspired by this handicap a few authors have focused their efforts on finding examples of non-strong fuzzy (pseudo-)metrics (see, for instance, Gregori et al. (2015); Gutierrez and Romaguera (2011)). As an instance of this kind of fuzzy pseudo-metrics we have the fuzzy pseudo-metric $(M^d, \wedge)$ introduced in Section 3. Indeed, it is easily seen that $(M^d, \wedge)$ is a non-strong fuzzy metric on $\mathbb{R}$ when d is taken as the Euclidean metric.

Based on the preceding observation and motivated by the fact lack of examples of non-strong fuzzy pseudo-metrics, our purpose in this subsection is, on the on the hand, to show that the method given in Theorem 3.1 does not yield in general strong fuzzy pseudo-metrics and, on the other hand, to provide conditions that guarantee when our construction gives a strong fuzzy pseudo-metric.

The next example gives an instance of non-strong fuzzy pseudo-metric which is obtained by means of Theorem 3.1.

Example 3.6. Consider the Euclidean metric $|\cdot|$on $\mathbb{R}$. Attending to Corollary 3.2 we have that $M_{d,f_{*P}}$ is a fuzzy metric on $\mathbb{R}$, where recall that $M_{|\cdot|,f_{*P}}$ is given by

$$M_{|\cdot|,f_{*P}}(x,y,t) = \begin{cases} \mathsf{e}^{t-|y-x|}, & \text{if } 0 < t \leq |y-x| \\ 1, & \text{if } t > |y-x| \end{cases}.$$

Next we show that $M_{|\cdot|,f_{*P}}(0,2,1) < M_{|\cdot|,f_{*P}}(0,1,1) *_P M_{|\cdot|,f_{*P}}(1,2,1)$. Indeed, it is clear that $|1-0| = 1$, $|2-1| = 1$ and $|2-0| = 2$. Thus

$$M_{|\cdot|,f_{*P}}(0,2,1) = \mathsf{e}^{1-2} = \mathsf{e}^{-1} < 1 = \mathsf{e}^{1-1} *_P \mathsf{e}^{1-1} = M_{|\cdot|,f_{*P}}(0,1,1) *_P M_{|\cdot|,f_{*P}}(1,2,1).$$

Thus, the fuzzy metric $(M_{|\cdot|,f_{*P}}, *_P)$ is not strong.

The next result ensures that our method, given by Theorem 3.1, allows always to construct non-strong fuzzy pseudo-metrics when we consider that the metric fulfills an extra condition.

Theorem 3.7. *Let $*$ be a continuous Archimedean t-norm with additive generator f_* and let (X,d) be a pseudo-metric space such that there exist $a,b,c \in X$ and $t_0 \in]0,\infty[$ satisfying $d(a,c) \in]t_0, f_*(0)[$ and $d(a,b), d(b,c) \in [0,t_0]$. Then $(M_{d,f_*}, *)$ is a non-strong fuzzy pseudo-metric on X, where M_{d,f_*} is the fuzzy set defined on $X \times X \times]0,\infty[$ as follows:*

$$M_{d,f_*}(x,y,t) = f_*^{(-1)}\left(\max\{d(x,y)-t, 0\}\right),$$

for all $x,y \in X$ and for all $t \in]0,\infty[$. Furthermore, $(M_{d,f_*}, *)$ is a non-strong fuzzy metric on X if and only if d is a metric on X.

Proof. By Theorem 3.1 we have that $(M_{d,f_*}, *)$ is a fuzzy pseudo-metric space on X. Furthermore, $(M_{d,f_*}, *)$ is a fuzzy metric on X if and only if d is a metric on X.

Now, we will see that in both cases the fuzzy (pseudo-)metric is not strong. With this aim, we will show that $(KM4')$ is not fulfilled. Indeed, let $a,b,c \in X$ and $t_0 \in]0,\infty[$ such that $d(a,c) \in]t_0, f_*(0)[$ and $d(a,b), d(b,c) \in [0,t_0]$. Since $f_*^{(-1)}$ is strictly monotone on $[0, f_*(0)]$ then

$$M_{d,f_*}(a,c,t_0) = f_*^{(-1)}\left(\max\{d(a,c)-t_0, 0\}\right) = f_*^{(-1)}(d(a,c)-t_0) < f_*^{(-1)}(0) = 1$$

and

$$M_{d,f_*}(a,b,t_0) * M_{d,f_*}(b,c,t_0) =$$

$$f_*^{(-1)}(\max\{d(a,b)-t_0,0\}) * f_*^{(-1)}(\max\{d(b,c)-t_0,0\}) =$$

$$= f_*^{(-1)}(0) * f_*^{(-1)}(0) = 1.$$

Whence we conclude that

$$M_{d,f_*}(a,c,t_0) < M_{d,f_*}(a,b,t_0) * M_{d,f_*}(b,c,t_0)$$

and, thus, that M_{d,f_*} is not strong, as we claimed. □

In Gutierrez and Romaguera (2011) it was provided two examples of non-strong fuzzy metrics, one for the product t-norm and another one for the Lukasievicz t-norm. Besides they posed the question of finding examples of non-strong fuzzy metrics when the continuous t-norms that are under consideration are greater than the product but different from minimum. It must be stressed that they posed the aforesaid question in the framework of GV-fuzzy metrics (fuzzy metrics in the sense of George and Veeramani, see George and Veeramani (1994)). Motivated by the fact that GV-fuzzy metrics are closely related to the fuzzy metrics studied in the present paper, we introduce in the following example an instance of non-strong fuzzy pseudo-metric when the considered continuous t-norm is greater than the product t-norm and different form the minimum.

Example 3.8. Consider the Hamacher t-norm $*_H$ defined by $a *_H b = \frac{ab}{a+b-ab}$ for each $a, b \in (0,1]$ and $0 *_H 0 = 0$ (see Klement et al. (2000) for more information about the family of Hamacher t-norms). It is clear that $*_H$ is a continuous Archimedean t-norm.

Moreover, it is easy to check that for each $a, b \in [0,1]$ we have that $a *_H b \geq a *_P b$, so $*_H$ is greater than $*_P$.

An additive generator of $*_H$ is given by the function $f_{*_H} : [0,1] \to [0,\infty]$ given by $_{*_H}(x) = \frac{1-x}{x}$ for all $x \in [0,1]$. Hence an easy computation shows that the pseudo-inverse $f_{*_H}^{(-1)}$ of f_{*_H} is given as follows:

$$f_{*_H}^{(-1)}(y) = \frac{1}{1+y}, \text{ for each } y \in [0,\infty].$$

Furthermore, note that $f_{*_H}(0) = \infty$.

Now, consider the Euclidean metric $|\cdot|$ on $\mathbb{R}$. Taking $x = 0$, $y = 1$, $z = 2$ and $t_0 = 1$ we have that the condition in the statement of Theorem 3.7 is hold, since $|2-0| \in]1,\infty[$, $|1-0| \in [0,1]$ and $|2-1| \in [0,1]$.

By Theorem 3.1, $(M_{|\cdot|,f_{*_H}}, *_H)$ is a fuzzy metric on $\mathbb{R}$, where the fuzzy set $M_{|\cdot|,f_{*_H}}$ on $\mathbb{R} \times \mathbb{R} \times]0,\infty[$ is given by

$$M_{|\cdot|,f_{*_H}}(x,y,t) = \begin{cases} \frac{1}{1+d(x,y)-t}, & \text{if } t \leq d(x,y); \\ 1, & \text{if } t > d(x,y). \end{cases} .$$

Similar arguments to those given in Example 3.6 remain valid to show that $(M_{|\cdot|,f_{*_H}}, *_H)$ is not strong.

4. A method for generating pseudo-metrics from fuzzy pseudo-metrics

The problem of obtaining metrics from fuzzy metrics has been treated in the literature by several authors. Some approaches to the aforementioned problem have been obtained in Radu (1978, 2004) and in Hicks (1983). Recently, the aforementioned results of Radu and Hicks have been generalized in Castro-Company et al. (2015). In all results given in the aforesaid references, a metric is constructed from a fuzzy metric using an additional function which is not related to the continuous t-norm under consideration and must satisfy many constraints.

Taking into account the mentioned results, we continue the study on the duality relationship between pseudo-metrics and fuzzy pseudo-metrics in this section. Thus our goal is twofold. On the one hand, we have interested in generating pseudo-metrics from fuzzy pseudo-metrics by means of the pseudo-inverse of an additive generator of the continuous t-norm under consideration. On the other hand, we aspire to retrieve the method provided by assertion (ii) in Theorem 2.3 as a particular case when stationary fuzzy pseudo-metrics are under consideration.

Our method, in contrast to those mentioned before, presents the advantage of needing only to consider the pseudo-inverse of an additive generator of the t-norm under consideration. Despite the aforesaid benefit, it must be pointed out that our method should be restricted to take under consideration only Archimedean continuous t-norm which excludes, for instance, the minimum t-norm.

The next result provides a method to obtain a pseudo-metric from a fuzzy pseudo-metric. The theorem is proved following similar arguments to those used in the proof of Theorem 3 in Castro-Company et al. (2015). First we point out a comment on the notation used in it inspired in Klement et al. (2000).

Remark 1. In a lattice $(L, \preceq)$, the meet (i.e. , the greatest lower bound) and the join (i.e. , the least upper bound) of a subset A of L are denoted by $\inf A$ and $\sup A$, respectively. Observe that each element of L is both an upper and a lower bound of the empty set $\emptyset$, so $\inf \emptyset$ and $\sup \emptyset$ depend on the underlying set L. In particular, in the lattice $((a, b], \leq)$ (with $[a, b] \subseteq [-\infty, \infty]$ and where $\leq$ is the usual order on the (extended) real line) we obtain $\inf \emptyset = b$ and $\sup \emptyset = a$.

Now, we are able to show the annunciated theorem.

Theorem 4.1. *Let $(M, *)$ be a fuzzy pseudo-metric on X, where $*$ is a continuous Archimedean t-norm. Then the function $d_{M,f_*} : X \times X \to [0, \infty]$ defined as*

$$d_{M,f_*}(x, y) = \sup\{t \in]0, f_*(0)[: M(x, y, t) \leq f_*^{(-1)}(t)\},$$

is a pseudo-metric on X, where f_* is an additive generator of $*$. Moreover, d is a metric on X if and only if $(M, *)$ is a fuzzy metric on X.

Proof. First we show that $d_{M,f_*}(x, x) = 0$. To this end, let $x \in X$. Since $(M, *)$ is a fuzzy pseudo-metric on X we have that $M(x, x, t) = 1$ for all $t \in]0, f_*(0)[$. It follows that $M(x,x,t) > f_*^{(-1)}(t)$ for all $t \in]0, f_*(0)[$, since $f_*^{(-1)}(t) \in [0,1[$ for each $t \in]0, f_*(0)[$. Thus, $d_{M,f_*}(x,x) = 0$.

The symmetry of d_{M,f_*} is obvious attending to its definition and the fact that $M(x,y,t) = M(y,x,t)$ for all $x, y \in X$ and $t \in]0,\infty[$.

Next we show that $d_{M,f_*}(x,z) \leq d_{M,f_*}(x,y) + d_{M,f_*}(y,z)$ for all $x, y, z \in X$. To this end, we will assume that $d_{M,f_*}(x,y) < f_*(0)$ and $d_{M,f_*}(y,z) < f_*(0)$ because otherwise the triangle inequality is hold trivially.

By definition of d_{M,f_*} we have

$$M(x,y,d_{M,f_*}(x,y)+\epsilon) > f_*^{(-1)}(d_{M,f_*}(x,y)+\epsilon)$$

and

$$M(y,z,d_{M,f_*}(y,z)+\epsilon) > f_*^{(-1)}(d_{M,f_*}(y,z)+\epsilon)$$

for each $x, y, z \in X$ and for each $\epsilon \in]0, K[$, where $K = \min\{f_*(0) - d_{M,f_*}(x,y), f_*(0) - d_{M,f_*}(y,z)\}$.

Then, for each $\epsilon \in]0,\infty[$, we have that

$$f_*(M(x,y,d_{M,f_*}(x,y)+\epsilon)) < d_{M,f_*}(x,y)+\epsilon \tag{1}$$

and

$$f_*(M(y,z,d_{M,f_*}(y,z)+\epsilon)) < d_{M,f_*}(y,z)+\epsilon, \tag{2}$$

since f_* is a strictly decreasing function and $(f_* \circ f_*^{(-1)})(a) \leq a$ for each $a \in [0,\infty]$.

Attending to axiom $(KM4)$ and taking into account that f_* is an additive generator of $*$, we have

$$\begin{aligned}
&M(x,z,d_{M,f_*}(x,y)+d_{M,f_*}(y,z)+2\epsilon) \geq \\
&M(x,y,d_{M,f_*}(x,y)+\epsilon) * M(y,z,d_{M,f_*}(y,z)+\epsilon) = \\
&f_*^{(-1)}\left(f_*(M(x,y,d_{M,f_*}(x,y)+\epsilon)) + f_*(M(y,z,d_{M,f_*}(y,z)+\epsilon))\right) \geq \\
&f_*^{(-1)}(d_{M,f_*}(x,y)+\epsilon+d_{M,f_*}(y,z)+\epsilon) \qquad .
\end{aligned}$$

Thus, by definition of $d_{M,f_*}(x,z)$ it is hold

$$d_{M,f_*}(x,z) \leq d_{M,f_*}(x,y) + d_{M,f_*}(y,z) + 2\epsilon.$$

Taking into account that $\epsilon \in]0, \infty[$ is arbitrary, we obtain

$$d_{M,f_*}(x, z) \leq d_{M,f_*}(x, y) + d_{M,f_*}(y, z).$$

$\square$

In the following two corollaries, we specify the method given in Theorem 4.1 for the case of the usual product $*_P$ and the Luckasievicz t-norm $*_L$.

Corollary 4.2. *Let $(M, *_P)$ be a fuzzy pseudo-metric on X. Then the function $d_{M,f_{*_P}} : X \times X \to [0, \infty]$ defined as*

$$d_{M,f_{*_P}}(x, y) = \sup\{t \in]0, \infty[: M(x, y, t) \leq \mathsf{e}^{-t}\},$$

is a pseudo-metric on X. Moreover, $d_{M,f_{*_P}}$ is a metric on X if and only if $(M, *_P)$ is a fuzzy metric on X.

Corollary 4.3. *Let $(M, *_L)$ be a fuzzy pseudo-metric on X. Then the function $d_{M,f_{*_L}} : X \times X \to [0, \infty]$ defined as*

$$d_{M,f_{*_L}}(x, y) = \sup\{t \in]0, 1[: M(x, y, t) \leq 1 - t\},$$

is a pseudo-metric on X. Moreover, $d_{M,f_{*_L}}$ is a metric on X if and only if $(M, *_L)$ is a fuzzy metric on X.

Next we are able to show that the method given in assertion (ii) in Theorem 2.3 can be retrieved from the method provided in Theorem 4.1.

Corollary 4.4. *Let X be a non-empty-set and $*$ a continuous Archimedean t-norm with additive generator $f_* : [0, 1] \to [0, \infty]$. If E is an indistinguishability operator for $*$ on X, then the function $d_E : X \times X \to [0, \infty]$ is a pseudo-metric on X, where $d_E(x, y) = f_*(E(x, y))$ for each $x, y \in X$. In addition, d_E is a metric on X if and only if E separates points.*

Proof. Define the mapping $M_E(x, y, t) = E(x, y)$ for each $x, y \in X$ and each $t \in]0, \infty[$. Then $(M_E, *)$ is a fuzzy pseudo-metric on X (see remark at the end of Section 2).

We will show that $\sup\{t \in]0, f_*(0)[: M_E(x, y, t) \leq f_*^{(-1)}(t)\} = f_*(E(x, y))$ for each $x, y \in X$.

Fix $x, y \in X$. Since $(f_*^{(-1)} \circ f)(a) = a$ for all $a \in [0, 1]$ then

$$f_*^{(-1)}(f_*(E(x, y)) = E(x, y) = M_E(x, y, t), \text{ for all } t \in]0, \infty[.$$

Thus we deduce that $f_*^{(-1)}(f_*(E(x, y)) = E(x, y) = M_E(x, y, f_*(E(x, y))$ and $f_*(E(x, y)) \in \{t \in]0, f_*(0)[: M_E(x, y, t) \leq f_*^{(-1)}(t)\}$. The fact that $f_*^{(-1)}$ is an strictly decreasing function on $]0, f_*(0)[$ guarantees $M_E(x, y, t) > f_*^{(-1)}(t)$ for all $t > f_*(E(x, y))$. Therefore

$$\sup\{t \in]0, f_*(0)[: M_E(x, y, t) \leq f_*^{(-1)}(t)\} = f_*(E(x, y)),$$

as we claimed.

Since $d_{M_E,f_*}(x,y) = d_E(x,y)$ for all $x,y \in X$, by Theorem 4.1, we have that the function $d_E : X \times X \to [0,\infty]$ defined by

$$d_E(x,y) = \sup\{t \in]0, f_*(0)[: M_E(x,y,t) \leq f_*^{(-1)}(t)\} = f_*(E(x,y)),$$

is a pseudo-metric on X. In addition, applying the aforementioned theorem, we have that d_E is a metric on X if and only if $(M_E,*)$ is a fuzzy metric on X.

□

According to Theorem 4.1 we infer that the pseudo-metric d_{M,f_*} will not take the value ∞ whenever the t-norm is nilpotent (notice that the t-norm is continuous and Archimedean). Contrarily, d_{M,f_*} can take the value ∞, when a continuous Archimeden t-norm $*$ is strict because in that case each additive generator f_* satisfies $f_*(0) = \infty$. In order to guarantee the finiteness of the pseudo-metric we provide a necessary condition through the next result.

Corollary 4.5. *Let $(M,*)$ be a fuzzy pseudo-metric on X such that for each $x,y \in X$ there exists $t_0 \in]0,\infty[$ satisfying $M(x,y,t_0) > 0$, where $*$ is a strict continuous Archimedean t-norm. Then the function $d_{M,f_*} : X \times X \to [0,\infty]$ defined as*

$$d_{M,f_*}(x,y) = \sup\{t \in]0,\infty[: M(x,y,t) \leq f_*^{(-1)}(t)\},$$

is a pseudo-metric on X such that $d_{M,f_*}(x,y) < \infty$ for each $x,y \in X$, where f_* is an additive generator of $*$. Moreover, d_{M,f_*} is a metric on X if and only if $(M,*)$ is a fuzzy metric on X.

Proof. By Theorem 4.1 de deduce that d_{M,f_*} is a pseudo-metric on X. Then we only need to show that $d_{M,f_*}(x,y) < \infty$ for each $x,y \in X$. To this end, consider $x,y \in X$ and we will see that $\sup\{t \in]0,\infty[: M(x,y,t) \leq f_*^{(-1)}(t)\} < \infty$.

On the one hand, by our hypothesis, given $x,y \in X$ there exists $t_0 \in]0,\infty[$ such that $M(x,y,t_0) > 0$. Furthermore, since $M_{x,y}$ is increasing then $M(x,y,t) > 0$ for all $t \in [t_0,\infty[$.

On the other hand, $f_*^{(-1)}$ is decreasing and continuous. Then, for each $\epsilon \in]0,\infty[$ there exists $t_\epsilon \in]0,\infty[$ such that $f_*^{(-1)}(t) < \epsilon$ for all $t \in [t_\epsilon,\infty[$. In particular, if we take $\epsilon = M(x,y,t_0) \in]0,\infty[$ there exists $t_\epsilon \in]0,\infty[$ such that $f_*^{(-1)}(t) < M(x,y,t_0)$ for all $t \in [t_\epsilon,\infty[$.

Therefore, $M(x,y,t) > f_*^{(-1)}(t)$ for all $t \in [t_1,\infty[$, where $t_1 = \max\{t_0,t_\epsilon\}$. So

$$\sup\{t \in]0,\infty[: M(x,y,t) \leq f_*^{(-1)}(t)\} \leq t_1.$$

Hence, $d_{M,f_*}(x,y) < \infty$ as we claimed. □

The following example shows that we cannot delete the additional condition on the fuzzy pseudo-metric imposed in the preceding corollary to construct a pseudo-metric which does not take the value ∞.

Example 4.6. Define the fuzzy set M_0 on $\mathbb{R} \times \mathbb{R} \times]0,\infty[$ as follows

$$M_0(x,y,t) = \begin{cases} 0, & \text{if } x \neq y \\ 1, & \text{if } x = y \end{cases},$$

for all $t \in]0, \infty[$. It is not hard to check that $(M_0, *_P)$ is a fuzzy metric on $\mathbb{R}$ and, obviously, the additional condition imposed in Corollary 4.5 is not fulfilled. Notice that $*_P$ is a continuous Archimedean t-norm which is strict.

Next we show that $d_{M,f_{*_P}}$ can take the value ∞. Indeed, let $x, y \in \mathbb{R}$ with $x \neq y$. Then $M(x, y, t) = 0$ for all $t \in]0, \infty[$ and so

$$0 = M(x, y, t) \leq f_{*_P}^{(-1)}(t) = \mathsf{e}^{-t}$$

for all $t \in]0, \infty[$. Therefore,

$$d_{M,f_{*_P}}(x, y) = \sup\{t \in]0, \infty[: M(x, y, t) \leq \mathsf{e}^{-t}\} = \infty.$$

5. Conclusions

We have addressed the problem of establishing whether there is a relationship between the last ones and fuzzy (pseudo-)metrics, inspired by the duality relationship between indistinguishability operators and (pseudo-)metrics. Thus, we have yielded a method for generating fuzzy (pseudo-)metrics from (pseudo)-metrics and vice-versa. In such methods we have made use of the pseudo-inverse of the additive generator of a continuous Archimedean t-norm. From our new methods we have derived a new technique to generate non-strong fuzzy (pseudo-)metrics from (pseudo-)metrics. We have illustrated the aforementioned methods by means of appropriate examples. Finally, we have shown that the classical duality relationship between indistinguishability operators and (pseudo)-metrics can be retrieved as a particular case of our results when continuous Archimedean t-norms are under consideration.

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