ALIEN LIMIT CYCLES IN ABEL EQUATIONS

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Abstract. The aim of this paper is to study the existence of limit cycles for a family of generalized Abel equations $x' = A(t)x^m + B(t)x^n$, $m, n \geq 2$. Under certain assumptions, it is proved that there exists a non-trivial limit cycle. This limit cycle has the characteristic that it arises from neither a Hopf bifurcation nor a perturbation of periodic orbits in a period annulus around the centre at the origin.

1. Introduction

Let us consider the generalized Abel equation

$$x' = \frac{dx}{dt} = A(t)x^m + B(t)x^n, \quad (t, x) \in \mathbb{R}^2, \tag{1.1}$$

where A, B are 2π-periodic continuous functions. Here m and n are integer numbers such that $m > n \geq 2$. As usual, by periodic solution of (1.1) we mean a solution such that $x(0) = x(2\pi)$. Also, a limit cycle is a periodic solution isolated in the set of periodic solutions.

The number of limit cycles of Equation (1.1) has been profusely studied, partly because of the fact that some planar differential systems can be transformed into Equation (1.1), and limit cycles of such systems correspond to positive limit cycles of Equation (1.1). In this sense, this study is meant to be a tool with which to examine subcases of Hilbert's 16th problem. Planar systems that can be transformed to Equation (1.1) include not only quadratic and some cubic polynomial systems (see [10, 17, 18] and [11]), but also higher-order systems like so-called rigid systems for instance (see [4, 5, 6, 14, 15, 16]).

In particular, A. Lins-Neto [17] proved in the case $m = 3$ and $n = 2$ that, for any $k \in \mathbb{N}$, there exist periodic functions A, B such that Equation (1.1) has at least k limit cycles. This result was generalized by Gasull and Guillamon [14] to the $m > n \geq 2$ case. Note that $x(t) \equiv 0$ is always a periodic solution of (1.1) for $n, m \geq 1$. We shall call this solution the trivial solution.

To prove the existence of k limit cycles, Lins-Neto (see details in [17]) studied the family with parameter ϵ,

$$x' = B(t)x^2 + \epsilon A(t)x^3, \quad (t, x) \in [-1, 1] \times \mathbb{R}, \tag{1.2}$$

and $\int_{-1}^{1} B(t)\,dt = 0$. Note that in the $\epsilon = 0$ case, all orbits in a neighbourhood of the origin are closed, and hence the trivial solution is a centre of Equation (1.2).

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If $u(t, t_0, x, \epsilon)$ denotes the solution of Equation (1.2) determined by the initial condition $u(t_0, t_0, x, \epsilon) = x$ then, for every fixed ϵ, the limit cycles are isolated zeros of the so-called displacement map,

$$d(x, \epsilon) := u(1, -1, x, \epsilon) - x.$$

Solving the first variational equation with respect to ϵ, one gets two conditions. The first is $d(x, 0) \equiv 0$, and this means Equation (1.2) has, for $\epsilon = 0$, a centre at the origin. The second is

$$\frac{\partial d}{\partial \epsilon}(x, \epsilon)\bigg|_{\epsilon=0} = x^3 \int_{-1}^{1} \frac{A(t)}{1 - x\int_0^t B(s)\, ds}\, dt. \tag{1.3}$$

The perturbation theory of the Abel equation leads one to the problem of the vanishing of an appropriate Abelian integral. See [9, 13] and the references therein, for instance. Hence, by analogy with planar differential systems, the role of the Abelian integral in Hilbert's tangential problem (see [7]), but for the Abel equation (1.2), is played by the function $(\partial d/\partial \epsilon)(x, 0)$ defined in (1.3). We shall call this function the Abelian integral of Equation (1.2). Each of its simple zeros in the interior of the central domain at the origin, for small perturbations of ϵ, yields a limit cycle. In this setting, Lins-Neto proved that the functions A and B can be chosen in such a way that the Abelian integral has at least k simple zeros for each $k \in \mathbb{N}$.

In the planar case, there are situations in which some limit cycles appear when perturbing a Hamiltonian system, but they are not detected from the Abelian integral. In particular, there are perturbed planar systems whose Abelian integrals have no zeros but present limit cycles coming from a saddle loop. Such limit cycles in this context are termed alien limit cycles. Mimicking the analogous definition given in [12], we shall say that, when a limit cycle of the Abel equation (1.2) is not detected from the zeros of the Abelian integral (1.3), this limit cycle is an alien limit cycle.

In this paper, we shall address the topic of the limit cycles of the family of generalized Abel equations (or simply Abel equations)

$$x'(t) = (a_1A_1(t) + a_2A_2(t))\, x^m(t) + (a_3A_3(t) + a_4A_4(t))\, x^n(t), \tag{1.4}$$

where $t \in \mathbb{R}$, $a_1, a_2, a_3, a_4 \in \mathbb{R}$, m and n are integers such that $m > n \geq 2$, with the following assumptions:

(1) $A_1(t)\sin t,\ A_2(t)\sin 2t,\ A_3(t)\cos t,\ A_4(t)\sin 2t \geq 0$ for all $t \in \mathbb{R}$, and the inequality is strict if $t \neq k\pi/2,\ k \in \mathbb{Z}$.
(2) The functions $A_1(t)\sin t,\ A_2(t)\sin 2t,\ A_3(t)\cos t,\ A_4(t)\sin 2t$ are even at $t = k\pi/2$ for every $k \in \mathbb{Z}$ (we say a function $f(t)$ is even at $k\pi/2$ if $f(k\pi/2 + t) = f(k\pi/2 - t)$).

That is, in the family of generalized Abel equations (1.4), the function A_1 has "the same signs and symmetries" as a sine function, A_3 as a cosine, and A_2 and A_4 as the product of a sine times a cosine.

The cases in which some of the coefficients a_i of Equation (1.4) vanish have already been studied (see [1, 8]). In brief, if either $a_1 = 0$ or $a_3 = 0$, the equation has a centre at the origin. On the other hand, if $a_2 = 0$ or $a_4 = 0$, the equation's only limit cycle is the origin.

We shall study whether, for each fixed n, m, A_1, A_2, A_3, A_4, there exist coefficients a_1, a_2, a_3, a_4 such that Equation (1.4) has a non-trivial limit cycle. In particular, we shall prove that no limit cycle bifurcates from the period annulus at first-order bifurcation of the centres $a_1 = 0$ or $a_3 = 0$. Consequently, we consider the bifurcation proposed by Lins-Neto but for the family (1.4), i.e., one takes $A(t) = a_1A_1(t) + a_2A_2(t)$, $B(t) = a_3A_3(t) + a_4A_4(t)$ and considers the perturbation

$$x' = B(t)x^n + \epsilon A(t)x^m.$$

We will prove that, on the one hand, the corresponding Abelian integral has no isolated zeros, and therefore, to first order, no limit cycles bifurcate from the period annulus around the trivial solution, and, on the other hand, as shall be dealt with in detail below (see Theorem 1.1), Equation (1.4) has a non-trivial limit cycle. Hence, Equation (1.4) exhibits what we have called an alien limit cycle.

To state our main result, we introduce the following additional technical hypothesis (H):

$$\text{(H)} \qquad \frac{d}{dt}\left(\ln \frac{A_2(t)}{A_4(t)}\right) \in \mathbb{R}, \qquad \text{for all } t \in \mathbb{R}.$$

Note that if $t \neq k\pi/2$, $k \in \mathbb{Z}$, then hypothesis (H) follows from the preceding assumptions (1) and (2). Note also that, by assuming hypothesis (H), we are presupposing that

$$\frac{A_2(t)}{A_4(t)} \in (0, +\infty), \qquad \text{for all } t \in \mathbb{R}.$$

Our main theorem is the following:

Theorem 1.1. *Consider Equation* (1.4)*,*

$$x'(t) = (a_1A_1(t) + a_2A_2(t))\,x^m(t) + (a_3A_3(t) + a_4A_4(t))\,x^n(t),$$

with integers $m > n \geq 2$*, and that assumptions (1), (2) and condition (H) hold. Then there exist* $a_1, a_2, a_3, a_4 \in \mathbb{R}$ *such that* (1.4) *has non-trivial limit cycles.*

We must remark that, in our proof of the main result, hypothesis (H) will be required to allow a well-defined change of variables, but some numerical evidence leads us to conjecture that the hypothesis is not necessary to get the main result. Moreover, for some particular families of equations (1.4), we have obtained analytical proofs of the main result without using that hypothesis.

The problem of the existence of coefficients a_1, a_2, a_3, a_4 for which Equation (1.4) has non-trivial limit cycles has been studied by the authors. Under different assumptions for A_1, A_2, A_3 and A_4, we obtained sufficient conditions for the existence and non-existence of such coefficients (see [2] and references therein). In the case of existence, the limit cycle always arose by a Hopf bifurcation from the origin. In that sense, the family studied in the present paper is, as far as we know, the simplest one not to exhibit this behaviour (i.e., with a limit cycle not arising from a Hopf bifurcation). Moreover, in a forthcoming paper (see [3]), we shall prove that if A_1, A_2, A_3, A_4 are trigonometric monomials then the unique case with this behaviour is when assumptions (1) and (2) hold.

The paper is organized as follows. In Section 2, we study the first-order perturbations of the centres $a_1 = 0$ and $a_3 = 0$. We prove that the derivative of the Poincaré map around the trivial solution has definite sign. In Section 3 we prove the main result by computing the first non-zero term of the series expansion of the displacement map with respect to the perturbative parameters.

2. First-order perturbation

As noted above, when $a_1a_3 = 0$, Equation (1.4) has a centre at $x = 0$. In this section we shall study the first-order perturbation of this centre. By the changes of variables $t \to \pi/2 - t$ and $t \to \pi - t$, it is not restrictive to assume the centre variety is $a_3 = 0$, and that $a_1 < 0$. Furthermore, by the change of variables $x \to -x$, we can assume $x > 0$.

Let us prove that, in the centre case $a_3 = 0$, the derivative of the return map with respect to a_3, for $a_3 = 0$, has definite sign for all $x \neq 0$. Consequently, no limit cycle will bifurcate from the period annulus, using a_3 as the only perturbative parameter.

Denote by $u(t, t_0, x, a_1, a_2, a_3, a_4)$ the solution of (1.4) determined by the initial condition $u(t_0, t_0, x, a_1, a_3, a_3, a_4) = x$. For $t_0 = 0$, we write the solution as $u(t, x, a_1, a_2, a_3, a_4)$, or $u(t, x)$, or simply $u(t)$ when no possible confusion can arise with the initial condition or the values of the parameters.

Henceforth, as usual, the subscripts denote partial derivatives with respect to the variable.

To prove that $u_{a_3}(2\pi, x)$ has constant sign when $a_3 = 0$, we shall use the following property of the solutions.

Proposition 2.1. *If $a_3 = 0$ and $a_1 < 0$ then*

$$u(t, x) > u(\pi - t, x) \quad \text{for all } t \in [0, \pi/2), \; x > 0.$$

Proof. The function $u(\pi - t, x)$ is a solution of the differential equation

$$\begin{aligned} x' &= (-a_1 A_1(t) + a_2 A_2(t))\, x^m(t) + a_4 A_4(t) x^n(t) \\ &\geq (a_1 A_1(t) + a_2 A_2(t))\, x^m(t) + a_4 A_4(t) x^n(t), \;\; t \in [0, \pi/2], \end{aligned}$$

with the inequality being strict for $t \in (0, \pi/2)$. Therefore, $u(\pi - t, x)$ is an upper solution of (1.4) with $a_3 = 0$ (strict for $t \neq 0, \pi/2$), and $u(t, x)$ and $u(\pi - t, x)$ coincide at $t = \pi/2$, so that $u(t, x) > u(\pi - t, x)$ for all $t \in [0, \pi/2)$. □

Now we can prove that the derivative with respect to a_3 of the return map has definite sign.

Proposition 2.2. *Assume $a_3 = 0$ and $x > 0$ such that $u(t, x)$ is defined for $t \in [0, 2\pi]$. Then*

$$u_{a_3}(2\pi, x) > 0.$$

Proof. By the symmetries of the functions A_k, if $v(t) = u(-t)$ then one has

$$\begin{aligned} v'(t) &= (-a_1 A_1(-t) - a_2 A_2(-t))\, v^m(t) + (-a_3 A_3(-t) - a_4 A_4(-t))\, v^n(t) \\ &= (a_1 A_1(t) + a_2 A_2(t))\, v^m(t) + (-a_3 A_3(t) + a_4 A_4(t))\, v^n(t). \end{aligned}$$

Consequently, $u(-t, x, a_1, a_2, a_3, a_4) = u(t, x, a_1, a_2, -a_3, a_4)$. In particular, for $a_3 = 0$, the solution $u(t)$ is a 2π-periodic solution, and, using the variational equation, one gets

$$u_{a_3}(-\pi, x, a_1, a_2, 0, a_4) = -u_{a_3}(\pi, x, a_1, a_2, 0, a_4). \tag{2.5}$$

Since $u(\pi, x) = u(-\pi, u(2\pi, x))$ and using again the variational equation, one straightforwardly has that $u_x(-\pi, x) > 0$. It therefore holds that

$$\operatorname{sign}(u_{a_3}(2\pi, x)) = \operatorname{sign}(u_{a_3}(\pi, x)).$$

Consequently, it is sufficient to prove that $u_{a_3}(\pi, x)$ has definite sign.

Using expression (A.13) in Appendix A, one obtains

$$u_{a_3}(\pi, x) = u^m(\pi, x) \int_0^\pi A_3(t) \exp\left(\int_t^\pi (n - m) a_4 A_4(s) u^{n-1}(s, x)\, ds \right) dt.$$

Since $A_3(t)$ is odd at $t = \pi/2$, i.e., $A_3(\pi/2-t) = -A_3(\pi/2+t)$, one can split the previous integral into two, the first from 0 to $\pi/2$ and the other from $\pi/2$ to π. Then, after the change of variables $t \to \pi - t$, the previous equality continues as

$$\begin{aligned} u_{a_3}(\pi, x) =& u^m(\pi, x) \int_0^{\pi/2} A_3(t) \exp\left(\int_t^{\pi} (n-m) a_4 A_4(s) u^{n-1}(s, x)\, ds \right) dt \\ &- u^m(\pi, x) \int_0^{\pi/2} A_3(t) \exp\left(\int_{\pi-t}^{\pi} (n-m) a_4 A_4(s) u^{n-1}(s, x)\, ds \right) dt \\ =& u^m(\pi, x) e^{\int_{\pi/2}^{\pi} (n-m) a_4 A_4(s) u^{n-1}(s,x)\, ds} \int_0^{\pi/2} A_3(t) F(t, x)\, dt \end{aligned}$$

where

$$\begin{aligned} F(t, x) =& \exp\left(\int_t^{\pi/2} (n-m) a_4 A_4(s) u^{n-1}(s, x)\, ds \right) - \\ & \exp\left(- \int_{\pi/2}^{\pi-t} (n-m) a_4 A_4(s) u^{n-1}(s, x)\, ds \right). \end{aligned}$$

Now we prove that

$$F(t, x) < 0, \quad t \in (0, \pi/2),\ x > 0.$$

First, note that for all $t \in (0, \pi/2)$, $x > 0$,

$$\begin{aligned} \operatorname{sign}(F(t, x)) &= \operatorname{sign}\left(\int_t^{\pi/2} A_4(s) u^{n-1}(s, x)\, ds - \int_{\pi/2}^{\pi-t} A_4(s) u^{n-1}(s, x)\, ds \right) \\ &= \operatorname{sign}\left(\int_t^{\pi/2} A_4(s) \left(u^{n-1}(s, x) - u^{n-1}(\pi - s, x) \right) ds \right) \end{aligned}$$

But by Proposition 2.1, this last quantity is positive, so we conclude. □

3. Bifurcation of limit cycles

In this section, we prove the existence of parameters a_1, a_2, a_3, a_4 such that Equation (1.4) has at least one non-trivial limit cycle when condition (H) is satisfied. Specifically, we consider the centre given by $a_1 = a_3 = 0$, $a_4 = -1$, $a_2 = \epsilon > 0$. In this case, Equation (1.4) becomes

$$x' = -A_4(t) x^n + \epsilon A_2 x^m. \tag{3.6}$$

The limit cycle will bifurcate from the centre of Equation (3.6) for $\epsilon > 0$ sufficiently small after perturbing it by the terms having a_1, a_3. In particular, we shall prove that there exist initial conditions such that the sign of the displacement map is negative, and, since the stability at the origin is positive, as we shall prove in Proposition 3.1, by continuity of the vector field, non-trivial limit cycles must exist.

Firstly, we prove that the sign of the displacement map is given by the sign of $a_1 a_3$, when $a_1, a_3 \neq 0$ and $x_0 > 0$ is close to zero.

Proposition 3.1. *Assume that $a_1, a_3 \neq 0$. Then, for all $x > 0$, x close to zero,*

$$\operatorname{sign}(u(2\pi, x) - x) = \operatorname{sign}(a_1 a_3).$$

Proof. Denote

$$B(t) := a_1A_1(t) + a_2A_2(t), \quad A(t) := a_3A_3(t) + a_4A_4(t).$$

For every $x > 0$, x close to zero, the return map associated with Equation (1.4) is given by (see, for example, App. A of [1])

$$u(2\pi, x) = x + (m-n)\left(\int_0^{2\pi} B(t)\int_0^t A(s)\,ds\,dt\right)x^{n+m-1} + o(x^{n+m-1}).$$

If it is non-zero, the coefficient of the term of degree $n+m-1$ is called the first Lyapunov constant of the equation. Since $\int_0^t A_i(s)\,ds$ has opposite parity to A_i, and since the functions A_i are 2π-periodic, one has

$$\int_0^{2\pi} A_i(t)\int_0^t A_j(s)\,ds\,dt = 0, \quad i,j \in \{2,4\}.$$

Therefore, simplifying the first Lyapunov constant, one obtains

$$\int_0^{2\pi} B(t)\int_0^t A(s)\,ds\,dt = a_1a_3\int_0^{2\pi} A_1(t)\int_0^t A_3(s)\,ds\,dt.$$

From $A_3(t) = -A_3(\pi - t)$ and $A_3(t) = -A_3(\pi + t)$, one obtains

$$\int_0^t A_3(s)\,ds > 0, \qquad \int_0^t A_3(s)\,ds = -\int_0^{2\pi - t} A_3(s)\,ds, \quad t \in (0,\pi).$$

Then

$$A_1(t)\int_0^t A_3(s)\,ds > 0 \quad \text{for all } t \in (0,2\pi)\backslash\{\pi/2, \pi, 3\pi/2\},$$

and

$$\operatorname{sign}(u(2\pi, x) - x) = \operatorname{sign}(a_1a_3).$$

□

Now we prove that there exist $x_0 > 0$, a_1, a_2, a_3, a_4, $a_1a_3 > 0$ such that $u(2\pi, x_0) < u(0, x_0)$. Since Equation (1.4) has a centre at $x = 0$ for $a_1 = 0$ and for $a_3 = 0$ (by the symmetries of A_1, A_2, A_3, A_3, which imply symmetries in the solution u, see [1, Proposition 2.1] for more details), the series expansion of the solutions of Equation (1.4) in terms of a_1, a_3 is

$$u(2\pi, x_0) = x_0 + u_{a_1a_3}(2\pi, x_0)a_1a_3 + O(a_1^3, a_1^2a_3, a_1a_3^2, a_3^3). \tag{3.7}$$

Therefore, it is sufficient to prove that there exist x_0, a_1, a_2, a_3, a_4 such that $u_{a_1a_3}(2\pi, x_0) < 0$, where, as usual, the subscripts denote partial derivatives with respect to a_1, a_3.

To this end, we perturb the centre at $x = 0$ of Equation (3.6). Deriving with respect to a_1 in (2.5),

$$u_{a_1a_3}(\pi, x) = -u_{a_1a_3}(-\pi, x).$$

Now, as the sign of the Lyapunov constant (the first non-vanishing coefficient of the series expansion of the displacement map) does not depend on the initial point, then the sign of $u_{a_1a_3}(2\pi, x_0)$ is the same as the sign of $u_{a_1a_3}(\pi, x_0)$, so we shall prove that there exist $x_0, \epsilon > 0$ such that $u_{a_1a_3}(\pi, x_0) < 0$. The expression for $u_{a_1a_3}(\pi, x_0)$ is computed in Appendix A.

To begin, we study Equation (3.6) after the change of variable $x \to y$,

$$y = \frac{x}{\gamma(t,\epsilon)}, \quad \epsilon > 0,$$

where $\gamma(t,\epsilon)$ is the nullcline of Equation (3.6), i.e.,

$$\gamma(t,\epsilon)=\left(\frac{A_4(t)}{\epsilon A_2(t)}\right)^{\frac{1}{m-n}}.$$

With this change of variables, what we wish to do is to compare the solutions of Equation (3.6) with its nullcline, in such a way that we would be able to confine solutions to a band. For this purpose, we consider the solution $v(t)$ that in the new variable starts at $y=1$. In this way, $y>1$ means that the solution in the original variables is greater than the nullcline, while $y<1$ means that the solution is lesser than the nullcline.

Note that, by the condition (H), $1/\gamma(t,\epsilon)$ is a defined and strictly positive function for all $t\in[0,\pi/2]$. This change of variables transforms Equation (3.6) into the equation

$$\begin{aligned} y' &= \left(\ln\left(\frac{A_2(t)}{A_4(t)}\right)\right)'\frac{y}{m-n}+A_4(t)\gamma^{n-1}(t,\epsilon)\left(y^m-y^n\right) \\ &= \left(\frac{A_2'(t)}{A_2(t)}-\frac{A_4'(t)}{A_4(t)}\right)\frac{y}{m-n}+\epsilon^{-\frac{n-1}{m-n}}\frac{A_4^{\frac{m-1}{m-n}}(t)}{A_2^{\frac{n-1}{m-n}}(t)}\left(y^m-y^n\right). \end{aligned} \tag{3.8}$$

Let us denote by $v(t,t_0,y_0,\epsilon)$ the solution of Equation (3.8) determined by the initial condition $v(t_0,t_0,y_0,\epsilon)=y_0$. The following result proves that the solution $v(t,\pi/2,1,\epsilon)$ converges, for all $t\in[0,\pi/2]$, to the straight line $v=1$ as $\epsilon\to0^+$, with the convergence being uniform.

Proposition 3.2. *Let $v(t,t_0,y_0,\epsilon)$ be the solution of Equation* (3.8) *determined by the initial condition $v(t_0,t_0,y_0,\epsilon)=y_0$. Then*

$$\lim_{\epsilon\to0^+}v(t,\pi/2,1,\epsilon)=1,\quad \textit{uniformly on } t\in[0,\pi/2].$$

Proof. Fixed $\delta>0$, we shall prove that there exists $\epsilon^*>0$ such that

$$|v(t,\pi/2,1,\epsilon)-1|\le\delta,\quad \text{for all } t\in[0,\pi/2],\ \epsilon\in(0,\epsilon^*).$$

Define

$$\phi(t)=\frac{1}{m-n}\left(\frac{A_2'(t)}{A_2(t)}-\frac{A_4'(t)}{A_4(t)}\right),$$

and consider the linear differential equations

$$y'=\pm\left(1+|\phi(t)|\right)y,$$

with solutions

$$v_\pm(t,t_0,y_0)=y_0\exp\left(\pm\int_{t_0}^{t}1+|\phi(s)|\,ds\right),\quad t_0\in[0,\pi/2].$$

Fix $y_0\ge1$ and $t\le t_0$. Then $v_-(t,t_0,y_0)\ge1$ for all $t<t_0$, and

$$\begin{aligned} v_-'(t,t_0,y_0) &= -\left(1+|\phi(t)|\right)v_-(t,t_0,y_0) \\ &< \phi(t)v_-(t,t_0,y_0)+K\left(v_-^m(t,t_0,y_0)-v_-^n(t,t_0,y_0)\right), \end{aligned}$$

where

$$K=\epsilon^{-\frac{n-1}{m-n}}\frac{A_4^{\frac{m-1}{m-n}}(t)}{A_2^{\frac{n-1}{m-n}}(t)}>0.$$

That is, $v_-(t,t_0,y_0)$ is a lower solution of Equation (3.8) in the interval $(0,t_0)$, for every $\epsilon>0$.

If $y_0\le1$ then $v_+(t,t_0,y_0)\le1$ for all $0\le t\le t_0$. Analogously with the above, $v_+(t,t_0,y_0)$ is an upper solution of Equation (3.8) when $t\in(0,t_0)$, for every $\epsilon>0$.

For $\delta > 0$ small enough, there exist $0 < t_i < \pi/2$, $i = 1, 2, 3, 4$, such that

$$v_-(t_1, 0, 1 + \delta) = 1 + \delta/2, \quad v_+(t_2, 0, 1 - \delta) = 1 - \delta/2,$$
$$v_-(t_3, \pi/2, 1) = 1 + \delta/2, \quad v_+(t_4, \pi/2, 1) = 1 - \delta/2,$$

(see Figure 1).

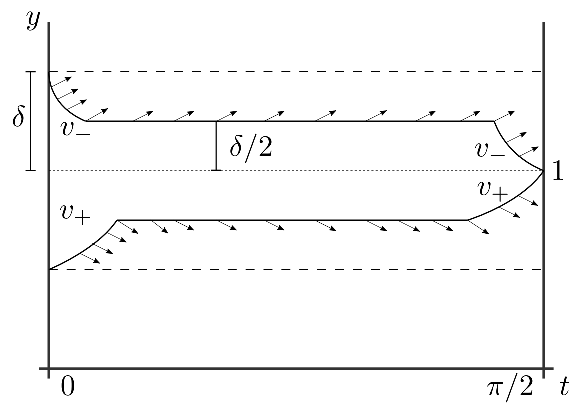

FIGURE 1. Scheme of the invariant region.

We evaluate the differential equation (3.8) on the straight line $y = 1 + \delta/2$, getting

$$y' = \left(1 + \frac{\delta}{2}\right)\phi(t) + A_4(t)\gamma^{n-1}(t, \epsilon)\left(\left(1 + \frac{\delta}{2}\right)^m - \left(1 + \frac{\delta}{2}\right)^n\right).$$

Now, for all $t \in [t_1, t_3]$, $\gamma^{n-1}(t, \epsilon) \to +\infty$ as $\epsilon \to 0^+$. Moreover, as $m > n$, $(1 + \delta/2)^m - (1 + \delta/2)^n > 0$, $A_2(t), A_4(t) > 0$ for all $t \in [t_1, t_3]$, and, by condition (H), $A_2'(t)/A_2(t) - A_4'(t)/A_4(t)$ is bounded. Consequently, there exists $\epsilon^* > 0$ such that if $0 < \epsilon < \epsilon^*$ then $y' > 0$ for $y = 1 + \delta/2$ and $t \in [t_1, t_3]$. That is, if we define the piecewise function

$$s(t) = \begin{cases} v_-(t, 0, 1 + \delta) & \text{if } t \in [0, t_1] \\ 1 + \delta/2 & \text{if } t \in [t_1, t_3] \\ v_-(t, \pi/2, 1) & \text{if } t \in [t_3, \pi/2], \end{cases}$$

then $s(t)$ is a lower solution of Equation (3.6) for $0 < \epsilon < \epsilon^*$. Therefore, for $\epsilon > 0$ small enough,

$$v(t, \pi/2, 1, \epsilon) \leq s(t) \leq 1 + \delta, \quad t \in [0, \pi/2].$$

Analogously, one proves that, for $\epsilon > 0$ small enough,

$$1 - \delta \leq v(t, \pi/2, 1, \epsilon), \quad t \in [0, \pi/2].$$

□

For every $\epsilon > 0$, let $x_1 = \gamma(\pi/2, \epsilon)$. Denote $x_0 = u(0, \pi/2, x_1, \epsilon)$. Henceforth, for the sake of simplicity, we shall denote $u(t, 0, x_0, \epsilon)$, the solution of Equation (3.6), simply by $u(t, \epsilon)$. Recall (see Proposition A.2 in Appendix A) that, for $a_1 = a_3 = 0$, $a_2 = \epsilon$, $a_4 = -1$, the derivative of the solution $u(t, \epsilon)$ with respect to a_1, a_3 at $t = \pi$ is

$$u_{a_1 a_3}(\pi, \epsilon) = 2(m - n)u^n(\pi, \epsilon)\int_0^{\pi/2} A_3(t)I_2(t)\int_t^{\pi/2}(F_1(s) + F_2(s))\,ds\,dt,$$

where

$$F_1(s) = u^{m-1}(s, \epsilon)A_1(s)$$

and

$$F_2(s) = -(m-1)\epsilon A_2(s)u^{2m-2}(s,\epsilon)I_4(s)\int_s^{\pi/2}\frac{A_1(\tau)}{I_4(\tau)}\,d\tau,$$

with

$$I_2(t) = \exp\left(\int_0^t (n-m)\epsilon A_2(s)u^{m-1}(s,\epsilon)\,ds\right),$$
$$I_4(t) = \exp\left(\int_0^t (m-n)A_4(s)u^{n-1}(s,\epsilon)\,ds\right).$$

Let us define

$$G_1(s,\epsilon) = \frac{\epsilon A_2(s)}{A_4(s)}u^{m-n}(s,\epsilon) = \left(\frac{u(s,\epsilon)}{\gamma(s,\epsilon)}\right)^{m-n},$$
$$G_2(s,\epsilon) = (m-n)A_4(s)u^{n-1}(s,\epsilon)I_4(s)\int_s^{\pi/2}\frac{A_1(\tau)}{I_4(\tau)}\,d\tau$$
$$= I_4'(s)\int_s^{\pi/2}\frac{A_1(\tau)}{I_4(\tau)}\,d\tau.$$

Then the function F_2 appearing in the integrand of the expression for $u_{a_1a_3}(\pi,\epsilon)$ can be written as

$$F_2(s) = -\frac{m-1}{m-n}u^{m-1}(s,\epsilon)G_1(s,\epsilon)G_2(s,\epsilon).$$

By Proposition 3.2, $G_1(s,\epsilon) \to 1$ as $\epsilon \to 0^+$ and the convergence is uniform on $[0,\pi/2]$. Now we prove that $G_2(s,\epsilon)$ converges to $A_1(s)$ as $\epsilon \to 0^+$, uniformly on compact subsets of $(0,\pi/2)$.

Proposition 3.3. *For every* $0 < \tau_1 < \tau_2 < \pi/2$ *and every* $\delta > 0$, *there exists* $\epsilon^* > 0$ *such that if* $0 < \epsilon < \epsilon^*$ *then*

$$|G_2(s,\epsilon) - A_1(s)| < \delta, \quad \textit{for all} \quad s \in [\tau_1,\tau_2].$$

Proof. Denote

$$h_4(s,\epsilon) = (m-n)A_4(s)u^{n-1}(s,\epsilon).$$

Then,

$$G_2(s,\epsilon) = \int_s^{\pi/2} A_1(\tau)h_4(s,\epsilon)e^{-\int_s^\tau h_4(\xi,\epsilon)\,d\xi}\,d\tau.$$

We shall split the above integral into two in such a way that one will tend uniformly to zero on the interval $[\tau_1,\tau_2]$, while the other will tend uniformly to $A_1(s)$ on the same interval. To this end, let us fix $0 < \tau_1 < \tau_2 < \pi/2$, $\delta > 0$, and take $\tau_3 = (\tau_2 + \pi/2)/2 < \pi/2$ and $s_1 > 0$ such that $\tau_2 + s_1 < \tau_3$. We then split the integral as follows:

$$\int_s^{\pi/2} A_1(\tau)h_4(s,\epsilon)e^{-\int_s^\tau h_4(\xi,\epsilon)\,d\xi}\,d\tau = \int_s^{s+s_1} A_1(\tau)h_4(s,\epsilon)e^{-\int_s^\tau h_4(\xi,\epsilon)\,d\xi}\,d\tau + \int_{s+s_1}^{\pi/2} A_1(\tau)h_4(s,\epsilon)e^{-\int_s^\tau h_4(\xi,\epsilon)\,d\xi}\,d\tau. \tag{3.9}$$

Firstly, let us prove that the second summand in (3.9) tends to zero as $\epsilon \to 0^+$, uniformly on $s \in [\tau_1,\tau_2]$. For $\tau \geq s + s_1$,

$$h_4(s,\epsilon)e^{-\int_s^\tau h_4(\xi,\epsilon)\,d\xi} \leq h_4(s,\epsilon)e^{-\int_s^{s+s_1} h_4(\xi,\epsilon)\,d\xi}.$$

In order to bound the above quantity, we first observe that

$$h_4(s,\epsilon)e^{-\int_s^{s+s_1} h_4(\xi,\epsilon)\,d\xi} \leq \frac{h_4(s,\epsilon)}{\min_{r\in[\tau_1,\tau_3]} h_4(r,\epsilon)}\left(\min_{r\in[\tau_1,\tau_3]} h_4(r,\epsilon)\right)e^{-s_1 \min_{r\in[\tau_1,\tau_3]} h_4(r,\epsilon)}.$$

We deal now with the first term in the previous expression. By Proposition 3.2, $u(s,\epsilon)/\gamma(s,\epsilon)\to 1$ as $\epsilon\to 0^+$, uniformly on $[0,\pi/2]$. Therefore, for any fixed $\delta>0$,

$$\begin{aligned}\lim_{\epsilon\to 0^+}\frac{h_4(s,\epsilon)}{\min_{r\in[\tau_1,\tau_3]}h_4(r,\epsilon)} &= \lim_{\epsilon\to 0^+}\frac{A_4(s)u^{n-1}(s,\epsilon)}{\min_{r\in[\tau_1,\tau_3]}A_4(r)u^{n-1}(r,\epsilon)}\\ &= \lim_{\epsilon\to 0^+}\frac{A_4(s)\gamma^{n-1}(s,\epsilon)\left(\frac{u(s,\epsilon)}{\gamma(s,\epsilon)}\right)^{n-1}}{\min_{r\in[\tau_1,\tau_3]}A_4(r)\gamma^{n-1}(r,\epsilon)\left(\frac{u(r,\epsilon)}{\gamma(r,\epsilon)}\right)^{n-1}}\\ &\leq \frac{A_4(s)\left(A_4(s)/A_2(s)\right)^{\frac{n-1}{m-n}}(1+\delta)}{\min_{r\in[\tau_1,\tau_3]}A_4(r)\left(A_4(r)/A_2(r)\right)^{\frac{n-1}{m-n}}(1-\delta)},\end{aligned}$$

which is a strictly positive function for all $s\in[\tau_1,\tau_2]$. Consequently, there exist $K,\epsilon^*>0$ such that if $0<\epsilon<\epsilon^*$ then

$$\frac{h_4(s,\epsilon)}{\min_{r\in[\tau_1,\tau_3]}h_4(r,\epsilon)}<K.$$

Also, from $h_4(r,\epsilon)=(m-n)A_4(r)v^{n-1}(r,\pi/2,1,\epsilon)\gamma^{n-1}(r,\epsilon)$, by again using Proposition 3.2 and the fact that $\gamma(t,\epsilon)\to\infty$ as $\epsilon\to 0^+$, one has

$$\lim_{\epsilon\to 0^+}\left(\min_{r\in[\tau_1,\tau_3]}h_4(r,\epsilon)\right)e^{-s_1\min_{r\in[\tau_1,\tau_3]}h_4(r,\epsilon)}=0.$$

Therefore,

$$\int_{s+s_1}^{\pi/2}A_1(\tau)h_4(s,\epsilon)e^{-\int_s^\tau h_4(\xi,\epsilon)\,d\xi}\,d\tau\leq\frac{K\min_{r\in[\tau_1,\tau_3]}h_4(r,\epsilon)}{e^{s_1\min_{r\in[\tau_1,\tau_3]}h_4(r,\epsilon)}}\int_0^{\pi/2}A_1(\tau)\,d\tau\to 0,$$

as $\epsilon\to 0^+$, uniformly on $s\in[\tau_1,\tau_2]$.

And secondly, let us consider the first summand in expression (3.9):

$$I_{21}=\int_s^{s+s_1}A_1(\tau)h_4(s,\epsilon)e^{-\int_s^\tau h_4(\xi,\epsilon)\,d\xi}\,d\tau=\int_s^{s+s_1}A_1(\tau)\,g(\tau,s,\epsilon)\,\psi(\tau,\epsilon)\,d\tau,$$

where

$$g(\tau,s,\epsilon)=\frac{h_4(s,\epsilon)}{h_4(\tau,\epsilon)},\quad \psi(\tau,\epsilon)=h_4(\tau,\epsilon)e^{-\int_s^\tau h_4(\xi,\epsilon)\,d\xi}.$$

Observe that

$$g(\tau,s,\epsilon)=\frac{A_4(s)\left(A_4(s)/A_2(s)\right)^{\frac{n-1}{m-n}}}{A_4(\tau)\left(A_4(\tau)/A_2(\tau)\right)^{\frac{n-1}{m-n}}}\frac{v(s,\pi/2,1,\epsilon)}{v(\tau,\pi/2,1,\epsilon)}.$$

Denote

$$g(\tau,s):=\lim_{\epsilon\to 0^+}g(\tau,s,\epsilon)=\frac{A_4(s)\left(A_4(s)/A_2(s)\right)^{\frac{n-1}{m-n}}}{A_4(\tau)\left(A_4(\tau)/A_2(\tau)\right)^{\frac{n-1}{m-n}}}.$$

Since $h_4(\tau,\epsilon)>0$, by the integral mean value theorem, there exists $\bar\tau\in(s,s+s_1)$ such that

$$I_{21}=A_1(\bar\tau)g(\bar\tau,s,\epsilon)\int_s^{s+s_1}\psi(\tau,\epsilon)d\tau.$$

Since $h_4(\bar\tau,\epsilon)\to+\infty$ as $\epsilon\to 0^+$, integrating and taking the limit as $\epsilon\to 0^+$,

$$\begin{aligned}\lim_{\epsilon\to 0^+}I_{21} &= \lim_{\epsilon\to 0^+}A_1(\bar\tau)g(\bar\tau,s,\epsilon)\left(1-e^{-\int_s^{s+s_1}h_4(\xi,\epsilon)\,d\xi}\right)\\ &= A_1(\bar\tau)g(\bar\tau,s),\quad \bar\tau\in(s,s+s_1).\end{aligned}$$

Observe that $\bar{\tau} \to s$ when $s_1 \to 0^+$ and $g(s,s) = 1$. Fix $\delta > 0$, and let s_1 be such that $|g(\tau, s) - 1| < \delta/(2\max_{s\in[\tau,\tau 2]} A_1(s))$ for all $s \in [\tau_1, \tau_2]$, $\tau \in [s, s_1]$. Now, let $\epsilon^* > 0$ such that for $0 < \epsilon < \epsilon^*$,

$$|G_2(s,\epsilon) - A_1(s)| \le |I_{21} - A_1(s)| + \left| \int_{s+s_1}^{\pi/2} A_1(\tau) h_4(s,\epsilon) e^{-\int_s^\tau h_4(\xi,\epsilon)\, d\xi}\, d\tau \right|$$
$$< |A_1(s)||1 - g(\bar{\tau}, s)| + \delta/2 < \delta.$$

□

The following result proves that, for $\epsilon > 0$ close to zero, the stability of the solution with initial condition $x_0 = u(0, \pi/2, \gamma(\pi/2, \epsilon), \epsilon)$ after the perturbation is the opposite of the stability at the origin.

Proposition 3.4. *Consider* (1.4) *with* $a_1 = a_3 = 0$, $a_4 = -1$, $a_2 = \epsilon > 0$. *There exists* $\epsilon > 0$ *such that*

$$u_{a_1 a_3}(\pi, \epsilon) < 0.$$

Proof. Recall that, by hypothesis (H), $\gamma(t, 1)$ is a strictly positive bounded function.

Fix $\delta > 0$. By Proposition 3.2, there exists $\epsilon^* > 0$ such that

$$\left| \frac{u(t,\epsilon)}{\gamma(t,\epsilon)} - 1 \right| < \delta, \quad \text{for all } 0 < \epsilon < \epsilon^*,\ t \in [0, \pi/2].$$

Multiplying the inequality by $\gamma(t, 1)$,

$$\left| \frac{u(t,\epsilon)}{\epsilon^{\frac{-1}{m-n}}} - \gamma(t,1) \right| < \delta\gamma(t,1) \quad \text{for all } 0 < \epsilon < \epsilon^*,\ t \in [0, \pi/2].$$

Consequently,

$$\gamma(t,1)(1-\delta) < \epsilon^{\frac{1}{m-n}} u(t,\epsilon) < \gamma(t,1)(1+\delta), \tag{3.10}$$

for all $0 < \epsilon < \epsilon^*$, $t \in [0, \pi/2]$.

Now, recall from expression (A.15) in Appendix A that

$$u_{a_1 a_3}(\pi, \epsilon) = 2(m-n) u^n(\pi, \epsilon) \int_0^{\pi/2} A_3(t) I_2(t) \int_t^{\pi/2} (F_1(s) + F_2(s))\, ds\, dt.$$

The idea of the proof is to split the above integral into three in such a way that each one can be bounded, and consequently to show that their sum is negative.

Fixed $0 < \tau_1 < \tau_2 < \pi/2$, we rewrite $u_{a_1 a_3}(\pi, \epsilon)$ as

$$u_{a_1 a_3}(\pi, \epsilon) = 2(m-n) u^n(\pi, \epsilon) \epsilon^{\frac{1-m}{m-n}} (J_1 + J_2 + J_3),$$

where

$$J_1 = \int_0^{\tau_1} A_3(t) I_2(t) \int_t^{\pi/2} \epsilon^{\frac{m-1}{m-n}} (F_1(s) + F_2(s))\, ds\, dt,$$
$$J_2 = \int_{\tau_1}^{\tau_2} A_3(t) I_2(t) \int_t^{\pi/2} \epsilon^{\frac{m-1}{m-n}} (F_1(s) + F_2(s))\, ds\, dt,$$
$$J_3 = \int_{\tau_2}^{\pi/2} A_3(t) I_2(t) \int_t^{\pi/2} \epsilon^{\frac{m-1}{m-n}} (F_1(s) + F_2(s))\, ds\, dt.$$

Bound of J_3**.** Let us recall that $F_2(s) \le 0$. Hence, by using (3.10), one gets

$$\epsilon^{\frac{m-1}{m-n}} (F_1(s) + F_2(s)) \le \epsilon^{\frac{m-1}{m-n}} u^{m-1}(s,\epsilon) A_1(s) \le \gamma^{m-1}(s,1)(1+\delta)^{m-1} A_1(s).$$

Therefore,

$$J_3 \le \int_{\tau_2}^{\pi/2} A_3(t) I_2(t) \int_t^{\pi/2} \gamma^{m-1}(s,1)(1+\delta)^{m-1} A_1(s)\, ds\, dt.$$

Since $I_2(t)$ is positive and strictly decreasing, and the rest of the terms are bounded, one has

$$\frac{J_3}{I_2(\tau_2)} \le J_{31} := \int_{\tau_2}^{\pi/2} A_3(t) \int_t^{\pi/2} \gamma^{m-1}(s,1)(1+\delta)^{m-1} A_1(s)\, ds\, dt.$$

In particular,

$$0 \le \lim_{\tau_2 \to \frac{\pi}{2}^-} \frac{J_3}{I_2(\tau_2)} \le \lim_{\tau_2 \to \frac{\pi}{2}^-} J_{31} = 0.$$

Bound of $J_2 + J_3$. For every $t \in [\tau_1, \tau_2]$, we split the interior integral in J_2 into two:

$$\begin{aligned} \int_t^{\pi/2} \epsilon^{\frac{m-1}{m-n}} \left(F_1(s) + F_2(s)\right) ds = & \int_t^{\tau_2} \epsilon^{\frac{m-1}{m-n}} \left(F_1(s) + F_2(s)\right) ds \\ & + \int_{\tau_2}^{\pi/2} \epsilon^{\frac{m-1}{m-n}} \left(F_1(s) + F_2(s)\right) ds. \end{aligned} \tag{3.11}$$

Arguing as in the J_3 case, the second summand can be bounded as

$$\int_{\tau_2}^{\pi/2} \epsilon^{\frac{m-1}{m-n}} \left(F_1(s) + F_2(s)\right) ds \le \int_{\tau_2}^{\pi/2} \gamma^{m-1}(s,1) \left(1+\delta\right)^{m-1} A_1(s)\, ds.$$

Now let us bound the first summand in (3.11). Since $A_1(s) > 0$ for all $s \in [\tau_1, \tau_2]$, then $\lim_{\epsilon \to 0^+} G_2(s,\epsilon) = A_1(s)$ is equivalent to

$$\lim_{\epsilon \to 0^+} \frac{G_2(s,\epsilon)}{A_1(s)}.$$

Now, by using the previous inequality and $G_1(s,\epsilon) \to 1$ as $\epsilon \to \infty$ uniformly on $[0, \pi/2]$, for every $\delta > 0$ there exists $\epsilon^* > 0$ such that if $0 < \epsilon < \epsilon^*$ then

$$\frac{F_2(s)}{u^{m-1}(s,\epsilon)} = -\frac{m-1}{m-n} G_1(s,\epsilon) G_2(s,\epsilon) \le -\frac{m-1}{m-n}(1-\delta)^2 A_1(s)$$

for every $s \in [\tau_1, \tau_2]$. Hence,

$$\frac{F_1(s) + F_2(s)}{u^{m-1}(s,\epsilon)} \le A_1(s) - \frac{m-1}{m-n}(1-\delta)^2 A_1(s).$$

Therefore, using inequality (3.10), one has

$$J_2 \le \int_{\tau_1}^{\tau_2} A_3(t) I_2(t) \left(J_{21} + J_{22} + J_{23}\right) dt,$$

where

$$J_{21} := \int_t^{\tau_2} \gamma^{m-1}(s,1)(1+\delta)^{m-1} A_1(s)\, ds,$$

$$J_{22} := -\int_t^{\tau_2} \frac{m-1}{m-n} \gamma^{m-1}(s,1)(1-\delta)^{m+1} A_1(s)\, ds,$$

$$J_{23} := \int_{\tau_2}^{\pi/2} \gamma^{m-1}(s,1)(1+\delta)^{m-1} A_1(s)\, ds.$$

Taking the limit as $\delta \to 0^+$ and $\tau_2 \to \pi/2$, and recalling that γ is well defined in the closed interval $[0, \pi/2]$, one gets

$$J_{21} \to \int_t^{\pi/2} \gamma^{m-1}(s,1) A_1(s)\, ds,$$

$$J_{22} \to -\int_t^{\pi/2} \frac{m-1}{m-n} \gamma^{m-1}(s,1) A_1(s)\, ds,$$

$$J_{23} \to 0.$$

Recall that $\lim_{s_2 \to \pi/2} J_{31} = 0$, so that there exist $\delta > 0$ and $\tau_2 \in (0, \pi/2)$ such that

$$\int_{\tau_1}^{\tau_2} A_3(t) \left(J_{21} + J_{22} + J_{23}\right)\, dt < -J_{31} - C,$$

where $C > 0$ is fixed. For instance,

$$C = \frac{1}{2} \frac{n-1}{m-n} \int_0^{\pi/2} A_3(t) \int_t^{\pi/2} A_1(s)\, ds\, dt.$$

Since $I_2(t)/I_2(\tau_2) \geq 1$ for all $t \in [\tau_1, \tau_2]$, then, for these δ, τ_2, there exists $\epsilon^* > 0$ such that if $0 < \epsilon < \epsilon^*$ then

$$\frac{J_2 + J_3}{I_2(\tau_2)} < -C.$$

Bound of J_1. We divide the inner integral of J_1 into three:

$$\int_t^{\pi/2} \epsilon^{\frac{m-1}{m-n}} \left(F_1(s) + F_2(s)\right)\, ds = \int_t^{\tau_1} \epsilon^{\frac{m-1}{m-n}} \left(F_1(s) + F_2(s)\right)\, ds$$
$$+ \int_{\tau_1}^{\tau_2} \epsilon^{\frac{m-1}{m-n}} \left(F_1(s) + F_2(s)\right)\, ds + \int_{\tau_2}^{\pi/2} \epsilon^{\frac{m-1}{m-n}} \left(F_1(s) + F_2(s)\right)\, ds.$$

Arguing as in the cases J_3 and J_2 by using inequality (3.10), one obtains for every $\delta > 0$ the existence of $\epsilon^* > 0$ such that $0 < \epsilon < \epsilon^*$ implies

$$\int_{\tau_2}^{\pi/2} \epsilon^{\frac{m-1}{m-n}} \left(F_1(s) + F_2(s)\right)\, ds \leq \int_{\tau_2}^{\pi/2} \gamma^{m-1}(s,1)(1+\delta)^{m-1} A_1(s)\, ds,$$

$$\int_{\tau_1}^{\tau_2} \epsilon^{\frac{m-1}{m-n}} \left(F_1(s) + F_2(s)\right)\, ds \leq \int_{\tau_1}^{\tau_2} \left((1+\delta)^{m-1} - \frac{m-1}{m-n}(1-\delta)^{m+1}\right) \gamma^{m-1}(s,1) A_1(s)\, ds,$$

$$\int_t^{\tau_1} \epsilon^{\frac{m-1}{m-n}} \left(F_1(s) + F_2(s)\right)\, ds \leq \int_t^{\tau_1} \gamma^{m-1}(s,1)(1+\delta)^{m-1} A_1(s)\, ds.$$

Therefore,

$$J_1 \leq \int_0^{\tau_1} A_3(t) I_2(t) \Bigg((1+\delta)^{m-1} \int_{\tau_2}^{\pi/2} \gamma^{m-1}(s,1) A_1(s)\, ds$$
$$+ \left((1+\delta)^{m-1} - \frac{m-1}{m-n}(1-\delta)^{m+1}\right) \int_{\tau_1}^{\tau_2} \gamma(s,1) A_1(s)\, ds$$
$$+ (1+\delta)^{m-1} \int_t^{\tau_1} \gamma^{m-1}(s,1) A_1(s)\, ds \Bigg)\, dt.$$

For $\delta > 0$ small enough, τ_1 close to zero, and τ_2 close to $\pi/2$, one obtains $J_1 < 0$.

In sum, it is possible to choose $\tau_1, \tau_2 \in (0, \pi/2)$ and $\delta > 0$ and $\epsilon^* > 0$ such that if $0 < \epsilon < \epsilon^*$,

$$u_{a_1 a_3}(\pi, \epsilon) = 2(m-n)u^n(\pi, \epsilon)\epsilon^{\frac{1-m}{m-n}}(J_1 + J_2 + J_3) < 0.$$

□

Proof of Theorem 1.1. Consider Equation (1.4), i.e.,

$$x'(t) = (a_1 A_1(t) + a_2 A_2(t))\, x^m(t) + (a_3 A_3(t) + a_4 A_4(t))\, x^n(t),$$

with $a_1 = a_3 = 0$, $a_2 = \epsilon$ and $a_4 = -1$. Using Proposition 3.4, one knows that there exist $\epsilon > 0$ and $x_0 \in \mathbb{R}$ such that

$$u_{a_1 a_3}(2\pi, x_0, \epsilon) < 0.$$

Then there exist a_1, a_3 small enough such that $a_1 a_3 > 0$ and

$$u(2\pi, x_0, \epsilon) - x_0 < 0.$$

From Proposition 3.1, one has that, for $x > 0$ sufficiently close to zero,

$$u(2\pi, x, \epsilon) - x > 0.$$

By continuity of the solutions with respect to the initial conditions, there exists $\xi \in \mathbb{R}$ such that

$$d(2\pi, \xi) = u(2\pi, \xi, \epsilon) - \xi = 0.$$

□

Appendix A. Derivatives of the perturbations of the centres

In this Appendix, we compute the derivatives with respect to the parameters a_1, a_3 of the solutions of Equation (1.4) for the central case $a_1 = a_3 = 0$.

Proposition A.1. *For every a_1, a_2, a_3, a_4,*

$$u_{a_3}(\pi)\Big|_{a_3=0} = u^n(\pi)\int_0^\pi A_3(t)\exp\left(\int_t^\pi (m-n)A(s)u^{m-1}(s)\,ds\right)dt \tag{A.12}$$

$$= u^m(\pi)\int_0^\pi A_3(t)\exp\left(\int_t^\pi (n-m)B(s)u^{n-1}(s)\,ds\right)dt, \tag{A.13}$$

$$u_{a_1}(\pi)\Big|_{a_1=0} = u^m(\pi)\int_0^\pi A_1(t)\exp\left(\int_t^\pi (n-m)B(s)u^{n-1}(s)\,ds\right)dt, \tag{A.14}$$

where $A(t) = a_1 A_1(t) + a_2 A_2(t)$ and $B(t) = a_3 A_3(t) + a_4 A_4(t)$.

Proof. Deriving Equation (1.4) with respect to a_3, and using that, by (1.4), $A(t)u^m = u' - B(t)u^n$, one obtains

$$\begin{aligned} u'_{a_3} &= \left(A(t)mu^{m-1}(t) + B(t)nu^{n-1}(t)\right)u_{a_3} + A_3(t)u^n(t) \\ &= \left(n\frac{u'(t)}{u(t)} + (m-n)A(t)u^{m-1}(t)\right)u_{a_3} + A_3(t)u^n(t) \end{aligned}$$

Since $u_{a_3}(0, x) = 0$, integrating over $[0, \pi]$, one obtains

$$\begin{aligned} u_{a_3}(\pi) &= \int_0^\pi A_3(t)u^n(t)\exp\left(\int_t^\pi \left(n\frac{u'(s)}{u(s)} + (m-n)A(s)u^{m-1}(s)\right)ds\right)dt \\ &= u^n(\pi)\int_0^\pi A_3(t)\exp\left(\int_t^\pi (m-n)A(s)u^{m-1}(s)\,ds\right)dt. \end{aligned}$$

This proves expression (A.12). Expression (A.13) follows in a similar way, but replacing $B(t)u^n$ by $u' - A(t)u^m$.

Since A_2, A_4 are π-periodic functions, odd at $t = k\pi/2$, $k \in \mathbb{Z}$, then, when $a_1 = a_3 = 0$, $u(t)$ is a π-periodic function, even at $t = k\pi/2$, $k \in \mathbb{Z}$.

If $a_1 = a_3 = 0$, then $A(s) = a_2 A_2(t)$ is a π-periodic odd function. Moreover, as $u(t)$ is even and π-periodic, then

$$I_2(t) := \exp\left(-\int_0^t (m-n)a_2 A_2(s)u^{m-1}(s)\,ds\right),$$

is even and π-periodic. Consequently,

$$\begin{aligned}
u_{a_3}(\pi) &= u^n(\pi)\int_0^{\pi} A_3(t)I_2(t))\,dt \\
&= u^n(\pi)\left(\int_0^{\pi/2} A_3(t)I_2(t)\,dt + \int_{\pi/2}^{\pi} A_3(t)I_2(t)\,dt\right) \\
&= u^n(\pi)\left(\int_0^{\pi/2} A_3(t)I_2(t)\,dt + \int_0^{\pi/2} A_3(\pi-t)I_2(\pi-t)\,dt\right) = \\
&= u^n(\pi)\left(\int_0^{\pi/2} A_3(t)I_2(t)\,dt - \int_0^{\pi/2} A_3(t)I_2(t)\,dt\right) = 0.
\end{aligned}$$

Arguing as above, one obtains

$$u_{a_1}(t)\Big|_{a_1=0} = u^m(t)\int_0^t A_1(s)\exp\left(\int_s^t (n-m)B(\tau)u^{n-1}(\tau)\,d\tau\right)\,ds.$$

□

Recall that, by definition,

$$I_4(t) = \exp\left(\int_0^t (n-m)a_4 A_4(s)u^{n-1}(s)\,ds\right),$$

so that I_4 is also a π-periodic even function.

Proposition A.2. *For* $a_1 = a_3 = 0$*,*

$$u_{a_1a_3}(\pi) = 2(m-n)u^n(\pi)\int_0^{\pi/2} A_3(t)I_2(t)\int_t^{\pi/2} F_1(s)+F_2(s)\,ds\,dt, \tag{A.15}$$

where

$$F_1(s) = u^{m-1}(s)A_1(s)$$

and

$$F_2(s) = -(m-1)a_2A_2(s)u^{2m-2}(s)I_4(s)\int_s^{\pi/2}\frac{A_1(\tau)}{I_4(\tau)}\,d\tau.$$

Proof. Deriving u_{a_3} with respect to a_1,

$$\begin{aligned}
\frac{u_{a_1a_3}(\pi)}{(m-n)u^n(\pi)} = &\int_0^{\pi} A_3(t)\left(\int_t^{\pi} A_1(s)u^{m-1}(s) + (m-1)A(s)u^{m-2}(s)u_{a_1}(s)\,ds\right) \\
&\exp\left(\int_t^{\pi}(m-n)A(s)u^{m-1}(s)\,ds\right)\,dt \\
&+ nu^{n-1}(\pi)u_{a_1}(\pi)\int_0^{\pi} A_3(t)\exp\left(\int_t^{\pi}(m-n)A(s)u^{m-1}(s)\,ds\right)\,dt.
\end{aligned}$$

Note that the second summand is $nu_{a_1}(\pi)u_{a_3}(\pi)/u(\pi) = 0$.

For $a_1 = a_3 = 0$,

$$u_{a_1a_3}(\pi) = (m-n)u^n(\pi)\int_0^\pi A_3(t)I_2(t)\int_t^\pi F(s)\,ds\,dt.$$

where

$$\begin{aligned} F(s) =& A_1(s)u^{m-1}(s) + (m-1)a_2A_2(s)u^{m-2}(s)u_{a_1}(s) \\ =& A_1(s)u^{m-1}(s) + (m-1)a_2A_2(s)u^{2m-2}(s)\int_0^s A_1(\tau)e^{\int_\tau^s (n-m)a_4A_4(\xi)u^{n-1}(\xi)\,d\xi}\,d\tau. \end{aligned}$$

Now, since $A_3(t) = -A_3(\pi - t)$ and $I_2(t) = I_2(\pi - t)$, then

$$\begin{aligned} \frac{u_{a_1a_3}(\pi)}{(m-n)u^n(\pi)} &= \int_0^{\pi/2} A_3(t)I_2(t)\int_t^\pi F(s)\,ds + \int_{\pi/2}^\pi A_3(t)I_2(t)\int_t^\pi F(s)\,ds\,dt. \\ &= \int_0^{\pi/2} A_3(t)I_2(t)\int_t^\pi F(s)\,ds - \int_0^{\pi/2} A_3(t)I_2(t)\int_{\pi-t}^\pi F(s)\,ds \\ &= \int_0^{\pi/2} A_3(t)I_2(t)\int_t^{\pi-t} F(s)\,ds, \end{aligned}$$

where $A_3(t), I_2(t) > 0$ for $t \in (0, \pi/2)$.

Now,

$$\begin{aligned} \int_t^{\pi-t} F(s)\,ds =& \int_t^{\pi-t} A_1(s)u^{m-1}(s)\,ds \\ &+ \int_t^{\pi-t} (m-1)a_2A_2(s)u^{2m-2}(s)I_4(s)\int_0^s \frac{A_1(\tau)}{I_4(\tau)}\,d\tau\,ds. \end{aligned}$$

By symmetries,

$$\int_t^{\pi-t} A_1(s)u^{m-1}(s)\,ds = 2\int_t^{\pi/2} A_1(s)u^{m-1}(s)\,ds,$$

and

$$\begin{aligned} &\int_t^{\pi-t} A_2(s)u^{2m-2}(s)I_4(s)\int_0^s A_1(\tau)/I_4(\tau)\,d\tau\,ds \\ &= \int_t^{\pi/2} A_2(s)u^{2m-2}(s)I_4(s)\int_0^s A_1(\tau)/I_4(\tau)\,d\tau\,ds \\ &+ \int_{\pi/2}^{\pi-t} A_2(s)u^{2m-2}(s)I_4(s)\int_0^s A_1(\tau)/I_4(\tau)\,d\tau\,ds \\ &= \int_t^{\pi/2} A_2(s)u^{2m-2}(s)I_4(s)\int_0^s A_1(\tau)/I_4(\tau)\,d\tau\,ds \\ &- \int_t^{\pi/2} A_2(s)u^{2m-2}(s)I_4(s)\int_0^{\pi-s} A_1(\tau)/I_4(\tau)\,d\tau\,ds \\ &= \int_t^{\pi/2} A_2(s)u^{2m-2}(s)I_4(s)\int_{\pi-s}^s A_1(\tau)/I_4(\tau)\,d\tau\,ds \\ &= -2\int_t^{\pi/2} A_2(s)u^{2m-2}(s)I_4(s)\int_s^{\pi/2} A_1(\tau)/I_4(\tau)\,d\tau\,ds. \end{aligned}$$

□

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