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Characterizing quasi-metric aggregation functions

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ABSTRACT

In this paper, we study those functions that allows us to combine a family of quasi-metrics, defined all of them on the same set, into a single one, which will be called quasi-metric aggregation functions. In particular, we characterize the quasi-metric aggregation functions and, in addition, we discuss a few of their properties. Moreover, a few methods to discard those functions that are useless as quasi-metric aggregation functions are introduced. Throughout the paper, different examples justify and illustrate the results presented. Finally, two possible fields where the developed theory can be useful are exposed.

KEYWORDS
Quasi-metric space; aggregation; subadditive function; monotone function; triangle triplet.

1. Introduction and preliminaries

The problem of fusing a collection of data (inputs) into a single one datum (output), which contains information on each of the inputs, plays an important role in many fields of research in applied sciences. For instance, it arises in a natural way, among others, in robotics, decision making, image processing, medical diagnosis, machine learning or pattern recognition. Many times in the aforesaid fields the information is coded as numerical data and such data must be merged in order to make a working decision. Hence, a lot of techniques to fuse these numerical inputs are based on the so-called aggregation functions (see Grabisch et al. (2009); Mesiar et al. (2018) for a deeper treatment of such class of functions).

A particular instance of data fusing consists in the case in which all inputs represent a distance between points of a set under consideration and the output must be interpreted, in a global sense, again as a distance. This kind of aggregation problems belong to a wide class in which the aggregation process only considers those aggregation functions that yield an output that remains the essential properties of the inputs.

In 1981, J. Borsík and J. Doboš made an study on those functions that merge a collection of metric spaces into a single one (see Borsík and Bobǒs (1981)). Such functions were called metric preserving functions. Let us recall that an n-metric preserving

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function is a function $F : [0, \infty)^n \to [0, \infty)$ such that, for each family of n metric spaces $\{(X_1, d_1), \ldots, (X_n, d_n)\}$, the function $D^n_F : \mathbf{X} \times \mathbf{X} \to [0, \infty)$, given by

$$D^n_F(\mathbf{x}, \mathbf{y}) = F(d_1(x_1, y_1), \ldots, d_n(x_n, y_n))$$

for each $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, is a metric on $\mathbf{X}$ (throughout the paper, $\mathbf{X}$ denotes the Cartesian product $X_1 \times \cdots \times X_n$).

The main goal of the original study made by Borsík and Doběs was given by the fact that they got a characterization of metric preserving functions. Next we introduce such a characterization. To this end, let us recall a few pertinent notions.

Fixed $n \in \mathbb{N}$ ($\mathbb{N}$ denotes the set of non-negative integer numbers), following Borsík and Boběs (1981), we will consider the partially ordered set $([0, \infty)^n, \preceq)$, where $\preceq$ denotes the point-wise order relation, i.e., given $\mathbf{a}, \mathbf{b} \in [0, \infty)^n$ we have $\mathbf{a} \preceq \mathbf{b} \Leftrightarrow a_i \leq b_i$ for each $i \in \{1, \ldots, n\}$. Of course, $\leq$ denotes the usual order relation on $[0, \infty)$. Besides, a function $F : [0, \infty)^n \to [0, \infty)$, is said to be:

1) monotone if $F(\mathbf{a}) \leq F(\mathbf{b})$ for each $\mathbf{a}, \mathbf{b} \in [0, \infty)^n$ with $\mathbf{a} \preceq \mathbf{b}$,
2) subadditive if $F(\mathbf{a}+\mathbf{b}) \leq F(\mathbf{a})+F(\mathbf{b})$ for each $\mathbf{a}, \mathbf{b} \in [0, \infty[^n$, where $+$ has been used for the usual addition on $[0, \infty)^n$ and $[0, \infty)$ simultaneously,
2) amenable if $F(\mathbf{a}) = 0 \Leftrightarrow \mathbf{a} = \mathbf{0}$.

In the light of the preceding notions a technique to induce metric aggregation functions was given by Borsík and Doběs as follows:

Proposition 1.1. *Let $n \in \mathbb{N}$ and let $F : [0, \infty)^n \to [0, \infty)$ be a function. If F is amenable, monotone and subadditive, then F is an n-metric preserving function.*

In the light of the preceding result, Borsík and Doboš asked if every metric preserving function is always amenable, subadditive and monotone. Nevertheless, there are metric preserving functions that are not monotone (see, Borsík and Boběs (1981); Doboš (1998)). This fact motivated that Borsík and Doboš gave a characterization of metric preserving functions by means of triangle triplets. Let us recall that, according to Borsík and Boběs (1981); Doboš (1998), a triplet $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in [0, \infty)^n$ forms an n-dimensional triangle triplet if $\mathbf{a} \preceq \mathbf{b} + \mathbf{c}$, $\mathbf{b} \preceq \mathbf{a} + \mathbf{c}$ and $\mathbf{c} \preceq \mathbf{b} + \mathbf{a}$. The aforesaid characterization can be stated as follows:

Theorem 1.2. *Let $n \in \mathbb{N}$ and let $F : [0, \infty)^n \to [0, \infty)$ be a function. Then the below assertions are equivalent:*

(1) F is an n-metric preserving function.
(2) F holds the following property:
 (2.1) F is amenable,
 (2.2) If $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is an n-dimensional triangle triplet, then $(F(\mathbf{a}), F(\mathbf{b}), F(\mathbf{c}))$ is a 1-dimensional triangle triplet.

From Theorem 1.2 one can deduce easily that every n-metric preserving function is also subadditive.

Later on, A. Pradera, E. Trillas and E. Castiñeira studied a particular case of the metric preserving problem. Concretely, they focused their efforts on the case in which all metrics of the family of metric spaces to be merged through a function are defined on the same nonempty set (see Pradera et al. (2000, 2002)). With this objective, they introduced the notion of metric aggregation function. According to

Pradera et al. (2000, 2002), given $n \in \mathbb{N}$, a function $F : [0, \infty)^n \to [0, \infty)$ is called an n-metric aggregation function if, for each non-empty set X and each family of metrics $\{d_1, \ldots, d_n\}$ on X, the function $D_F^n : X \times X \to [0, \infty)$ is a metric on X, where

$$D_F^n(x, y) = F(d_1(x, y), \ldots, d_n(x, y))$$

for each $x, y \in X$.

In Pradera et al. (2000, 2002), necessary conditions on a function F to be an n-metric aggregation function were proved (actually such conditions were provided for pseudo-metrics, although we just focus on the metric case because it is enough for the problem considered here). It must be stressed that the original work of Borsík and DobOš was extended by Pradera and Trillas to the case of pseudo-metric spaces in Pradera and Trillas (2002). Inspired by the study made by Pradera, Trillas and Castiñeira, G. Mayor and O. Valero posed the possibility of obtaining a characterization of metric aggregation functions in the spirit of Borsík and Doboš, i.e., in terms of triangle triplets. They showed that the answer to that question is affirmative in Mayor and Valero (2018). In order to state such a characterization, let us recall the notion of positive triangle triplet. Following Mayor and Valero (2018), the triplet $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in (0, \infty)^n$ forms an n-dimensional positive triangle triplet if it forms an n-dimensional triangle triplet. The aforementioned characterization can be stated as follows:

Theorem 1.3. *Let $n \in \mathbb{N}$. Consider a function $F : [0, \infty)^n \to [0, \infty)$. The following assertions are equivalent.*

(1) F is an n-metric aggregation function.
(2) F holds the following properties:
  (2.1) $F(\mathbf{0}) = 0$,
  (2.2) If $F(\mathbf{a}) = 0$, then $\min\{a_1, \ldots a_n\} = 0$,
  (2.3) If $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a positive n-dimensional triangle triplet, then $(F(\mathbf{a}), F(\mathbf{b}), F(\mathbf{c}))$ is a positive 1-dimensional triangle triplet.

Notice that, by Theorems 1.2 and 1.3, every metric preserving function is always a metric aggregation function. The converse is not true such as Example 7 in Mayor and Valero (2018) shows. In the light of the preceding result one can immediately deduce that every metric aggregation function is always positive subadditive. Notice that, on account of Mayor and Valero (2018), a function $F : [0, \infty)^n \to [0, \infty)$ is said to be positive subadditive provided that $F(\mathbf{a} + \mathbf{b}) \leq F(\mathbf{a}) + F(\mathbf{b})$ for each $\mathbf{a}, \mathbf{b} \in (0, \infty)^n$.

Motivated by the applications to Computer Science and Artificial Intelligence of quasi-metric spaces and by the fact that many quasi-metrics useful in the aforementioned fields can be obtained by aggregation of a collection of quasi-metric spaces through a function (see, among others, Hitzler and Seda (2009); Romaguera at al. (2012); Romaguera and Valero (2012); Schellekens (1995)), Mayor and Valero posed the metric preserving problem in the quasi-metric framework in Mayor and Valero (2010). Let us recall that, according to Deza and Deza (2016), a quasi-metric q on a non-empty set X is a function $q : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$ the following conditions are satisfied:

(q1) $q(x, y) = q(y, x) = 0 \Leftrightarrow x = y$;
(q2) $q(x, z) \leq q(x, y) + q(y, z)$.

In such a case, the pair (X, q) is called a quasi-metric space. Observe that every metric space (X, d) is a quasi-metric space such that $d(x, y) = d(y, x)$ for all $x, y \in X$.

Following Mayor and Valero (2010), given $n \in \mathbb{N}$, a function $\Phi : [0, \infty)^n \to [0, \infty)$ is an n-quasi-metric preserving function if, for each family of quasi-metric spaces $\{(X_1, q_1), \ldots, (X_n, q_n)\}$, the function $Q^n_\Phi : \mathbf{X} \times \mathbf{X} \to [0, \infty)$ is a quasi-metric on $\mathbf{X}$, where

$$Q^n_\Phi(\mathbf{x}, \mathbf{y}) = F(q_1(x_1, y_1), \ldots, q_n(x_n, y_n))$$

for each $\mathbf{x}, \mathbf{y} \in \mathbf{X}$.

Similar to the metric case, Mayor and Valero provided a complete description of quasi-metric preserving functions in terms of triangle triplets. Concretely they proved the following characterization.

Theorem 1.4. *Let $n \in \mathbb{N}$. Consider a function $F : [0, \infty)^n \to [0, \infty)$. The following assertions are equivalent.*

(1) F is an n-quasi-metric preserving function.
(2) F is amenable, subadditive and monotone.
(3) F holds the following properties:
  (3.1) F is amenable;
  (3.2) If $\mathbf{a}, \mathbf{b}, \mathbf{c} \in [0, \infty)^n$ and $\mathbf{a} \preceq \mathbf{b} + \mathbf{c}$, then $F(\mathbf{a}) \leq F(\mathbf{b}) + F(\mathbf{c})$.

In the light of Theorems 1.2 and 1.4, it is clear that every quasi-metric preserving function is a metric preserving function. However, the converse is not true such as Example 8 in Mayor and Valero (2010) shows. Notice that a notable difference between quasi-metric preserving functions and metric preserving functions is given by the fact that the former are always monotone.

Taking into account the exposed facts, in this paper we try to take a further step towards the quasi-metric aggregation study. Hence we consider a refinement of the problem studied in Mayor and Valero (2010), and we extend the characterization of metric aggregation functions given by Theorem 1.3 to the framework of quasi-metric spaces. To this end we introduce the notion of quasi-metric aggregation function and we characterize them in terms of triplets. We illustrate, by means of examples, differences between quasi-metric aggregation functions and the classical ones. Moreover, a complete description of the quasi-metric aggregation functions that preserve quasi-metrics is also given. Furthermore, based on Mayor and Valero (2018), we analyze some properties fulfilled by the functions in matter that will be crucial to discard those functions that cannot be used as quasi-metric aggregation functions. Finally, two possible fields where the developed theory can be useful are exposed.

2. Quasi-metric aggregation functions and their characterization

As pointed out in the preceding section we are interested in extending the characterization of metric aggregation functions, given by Theorem 1.3, to the context of quasi-metric spaces. Moreover, motivated by the fact that every quasi-metric preserving function is a metric preserving one, we discuss the relationship between quasi-metric aggregation functions and metric aggregation functions. In addition, the link with quasi-metric preserving functions is also explored. All this is illustrated with appropriate examples. To this end, we first introduce the notion of quasi-metric aggregation function following the spirit of Mayor and Valero (2018).

Let $n \in \mathbb{N}$. A function $F : [0, \infty)^n \to [0, \infty)$ will be called an n-quasi-metric aggregation function if, for each non-empty set X and each family of quasi-metrics $\{q_1, \ldots, q_n\}$ on X, the function $Q_F^n : X \times X \to [0, \infty[$ is a quasi-metric on X, where

$$Q_F^n(x, y) = F(q_1(x, y), \ldots, q_n(x, y))$$

for each $x, y \in X$.

The next example gives a simple, but illustrative, instance of a quasi-metric aggregation function.

Example 2.1. Let $n \in \mathbb{N}$ and fix a collection of coeficients $\{\alpha_2, \ldots, \alpha_n\}$ such that $\alpha_i \in (0, \infty)$ for all $i = 2, \ldots, n$. Define the function $F_s : [0, \infty)^n \to [0, \infty)$ given, for each $\mathbf{a} \in [0, \infty)^n$, by $F_s(\mathbf{a}) = \sum_{i=2}^{n} \alpha_i \cdot a_i$. A straightforward computation shows that F_s is an n-quasi-metric aggregation function.

Observe that every quasi-metric aggregation function is a metric aggregation function. Indeed, let $n \in \mathbb{N}$ and let $F : [0, \infty)^n \to [0, \infty)$ be an n-quasi-metric aggregation function. Consider a non-empty set X and a family of metrics $\{d_1, \ldots, d_n\}$ on it. By our assumption, the function Q_F^n is a quasi-metric on X, where $Q_F^n(x, y) = F(d_1(x, y), \ldots, d_n(x, y))$ for each $x, y \in X$. Besides, $Q_F^n(x, y) = Q_F^n(y, x)$ and so Q_F^n is a metric on X, since $d_i(x, y) = d_i(y, x)$ for each $i \in \{1, \ldots, n\}$. Therefore, F is an n-metric aggregation function.

With the aim of getting a characterization of quasi-metric aggregation functions we introduce the following lemmata which will be crucial later on.

Lemma 2.2. *Let $n \in \mathbb{N}$ and let $F : [0, \infty)^n \to [0, \infty)$ be an n-quasi-metric aggregation function. Then, $F(\mathbf{a}) \leq F(\mathbf{b}) + F(\mathbf{c})$ for each $\mathbf{a}, \mathbf{b}, \mathbf{c} \in [0, \infty)^n$ with $\mathbf{a} \preceq \mathbf{b} + \mathbf{c}$.*

Proof. Consider a set $X = \{x, y, z\}$ with all elements different. Next we show that for every $\mathbf{a}, \mathbf{b}, \mathbf{c} \in [0, \infty)^n$ with $\mathbf{a} \preceq \mathbf{b} + \mathbf{c}$, there exists a family of quasi-metrics $\{q_1^{a,b}, \ldots, q_n^{a,b}\}$ such that $q_i^{a,b}$ is defined on X and, in addition, $q_i^{a,b}(x, z) = a_i$, $q_i^{a,b}(z, y) = b_i$ and $q_i^{a,b}(y, z) = c_i$ for all $i \in \{1, \ldots, n\}$. To this end, given $\mathbf{a}, \mathbf{b}, \mathbf{c} \in [0, \infty)^n$, we distinguish two possible cases when each coordinate of $\mathbf{b}$ and $\mathbf{c}$ is considered:

Case 1. $\max\{b_i, c_i\} = 0$. Then we consider the quasi-metric $q_{i,1}^{a,b}$ defined as follows:

$$\begin{array}{l} q_{i,1}^{a,b}(x, y) = q_{i,1}^{a,b}(x, z) = q_{i,1}^{a,b}(y, z) = q_{i,1}^{a,b}(x, x) = q_{i,1}^{a,b}(y, y) = q_{i,1}^{a,b}(z, z) = 0; \\ q_{i,1}^{a,b}(y, x) = q_{i,1}^{a,b}(z, x) = q_{i,1}^{a,b}(z, y) = 1 \end{array}.$$

Case 2. $\max\{b_i, c_i\} \neq 0$. Then we consider the quasi-metric $q_{i,2}^{a,b}$ defined as follows:

$$\begin{array}{l} q_{i,2}^{a,b}(x, x) = q_{i,2}^{a,b}(y, y) = q_{i,2}^{a,b}(z, z) = 0; \\ q_{i,2}^{a,b}(x, z) = a_i; q_{i,2}^{a,b}(z, x) = \max\{b_i, c_i\}; \\ q_{i,2}^{a,b}(x, y) = q_{i,2}^{a,b}(z, y) = b_i; \\ q_{i,2}^{a,b}(y, z) = q_{i,2}^{a,b}(y, x) = c_i. \end{array}$$

Since F is an n-quasi-metric aggregation function, then the function $Q_F^n : X \times X \to$

$[0,\infty)$ given, for each $u, v \in X$, by

$$Q_F^n(u,v) = F(q_1^{a,b}(u,v), \ldots, q_1^{a,b}(u,v))$$

is a quasi-metric on X, where

$$q_i^{a,b}(u,v) = \begin{cases} q_{i,1}^{a,b}(u,v) & \text{if } \max\{b_i, c_i\} = 0 \\ q_{i,2}^{a,b}(u,v) & \text{if } \max\{b_i, c_i\} \neq 0 \end{cases}, \text{ for each } i \in \{1, \ldots, n\}.$$

It follows that

$$F(\mathbf{a}) = Q_F^n(x,z) \leq Q_F^n(x,y) + Q_F^n(y,z) = F(\mathbf{b}) + F(\mathbf{c})$$

as we claimed. □

The next results follow immediately from the preceding one.

Corollary 2.3. *Let $n \in \mathbb{N}$ and let $F : [0,\infty)^n \to [0,\infty)$ be an n-quasi-metric aggregation function. Then the following assertions hold:*

(1) F is subadditive.
(2) F is monotone.

Proof. By Lemma 2.2 we have that $F(\mathbf{a}) \leq F(\mathbf{b}) + F(\mathbf{c})$ for each $\mathbf{a}, \mathbf{b}, \mathbf{c} \in [0,\infty)^n$, with $\mathbf{a} \preceq \mathbf{b}+\mathbf{c}$. Taking $\mathbf{a} = \mathbf{b}+\mathbf{c}$ in the preceding inequality we obtain the subadditivity of F and taking $\mathbf{c} = \mathbf{0}$ we deduce the monotony of F. □

The below property will paly a central role in our subsequent discussion.

Lemma 2.4. *Let $n \in \mathbb{N}$ and let $F : [0,\infty[^n \to [0,\infty[$ be a subadditive function. Then the following assertions are equivalent:*

(i) There exists $i_0 \in \{1, \ldots, n\}$ satisfying the following: for each $\mathbf{a} \in [0,\infty)^n$ with $F(\mathbf{a}) = 0$ we have that $a_{i_0} = 0$;
(ii) If $\mathbf{a} \in [0,\infty)^n$ such that $F(\mathbf{a}) = 0$, then $\min\{a_1, \ldots, a_n\} = 0$.

Proof. It is obvious that *(i)* $\Rightarrow$ *(ii)*. So we only need to show that *(ii)* $\Rightarrow$ *(i)*. To this end, suppose for the purpose of contradiction that for each $i \in \{1, \ldots, n\}$ there exists $\mathbf{a}^i \in [0,\infty)^n$ such that $F(\mathbf{a}^i) = 0$ but $a_i^i > 0$. Since F is subadditive we obtain that $F(\mathbf{a}^1 + \cdots + \mathbf{a}^n) \leq F(\mathbf{a}^1) + \cdots + F(\mathbf{a}^n) = 0$. Thus, there exists $\mathbf{c} \in [0,\infty)^n$ with $\mathbf{c} = \mathbf{a}^1 \cdots + \mathbf{a}^n$ such that $F(\mathbf{c}) = 0$ and, however, $c_i > 0$ for each $i \in \{1, \ldots, n\}$, which contradicts *(ii)*. □

In the particular case of quasi-metric aggregation functions, by Lemma 2.4 and Corollary 2.3, we obtain the following result.

Lemma 2.5. *Let $n \in \mathbb{N}$ and let $F : [0,\infty)^n \to [0,\infty)$ be an n-quasi-metric aggregation function. Then the following assertions are equivalent:*

(i) There exists $i_0 \in \{1, \ldots, n\}$ satisfying the following: for each $\mathbf{a} \in [0,\infty)^n$ with $F(\mathbf{a}) = 0$ we have that $a_{i_0} = 0$;
(ii) If $\mathbf{a} \in [0,\infty)^n$ such that $F(\mathbf{a}) = 0$, then $\min\{a_1, \ldots, a_n\} = 0$.

In the light of the previous results, we are able to prove the promised characterization of quasi-metric aggregation functions.

Theorem 2.6. *Let $n \in \mathbb{N}$ and let $F : [0, \infty)^n \to [0, \infty)$ be a function. Then the following assertions are equivalent:*

(1) F is an n-quasi-metric aggregation function;
(2) F satisfies the following conditions:
  (2.1) $F(\mathbf{0}) = 0$;
  (2.2) If $F(\mathbf{a}) = 0$, then $\min\{a_1, \dots, a_n\} = 0$;
  (2.3) $F(\mathbf{a}) \leq F(\mathbf{b}) + F(\mathbf{c})$ for each $\mathbf{a}, \mathbf{b}, \mathbf{c} \in [0, \infty)^n$ with $\mathbf{a} \preceq \mathbf{b} + \mathbf{c}$.
(3) F satisfies the following conditions:
  (3.1) $F(\mathbf{0}) = 0$;
  (3.2) If $F(\mathbf{a}) = 0$, then $\min\{a_1, \dots, a_n\} = 0$;
  (3.3) F is monotone and subadditive.

Proof. (1) $\Rightarrow$ (2). By Lemmas 2.5 and 2.2 we have that F satisfies conditions (2.2) and (2.3), respectively. Next we show that $F(\mathbf{0}) = 0$. Indeed, consider the quasi-metric space $([0, \infty), q_u)$, where $q_u(x, y) = \max\{y - x, 0\}$. The fact that F is an n-quasi-metric aggregation function gives that the function Q_F^n is a quasi-metric on X, where $Q_F^n(x, y) = F(q_u(x, y), \dots, q_u(x, y))$ for all $x, y \in X$. It follows that, fixed $x \in [0, \infty)$, $F(\mathbf{0}) = F(0, \dots, 0) = F(q_u(x, x), \dots, q_u(x, x)) = Q_F^n(x, x) = 0$.

(2) $\Rightarrow$ (1). Consider a non-empty set X and a family $\{q_1, \dots, q_n\}$ of quasi-metrics on X. We will show that the function $Q_F^n : X \times X \to [0, \infty)$ is a quasi-metric.

Suppose that, given $x, y \in X$, $Q_F^n(x, y) = Q_F^n(y, x) = 0$. Then,

$$F(q_1(x, y), \dots, q_n(x, y)) = F(q_1(y, x), \dots, q_n(y, x)) = 0.$$

Since F satisfies condition (2.3) we deduce that F is subadditive. In addition F fulfills condition (2.2) and, thus, Lemma 2.4 guarantees that there exists $i_0 \in \{1, \dots, n\}$ such that $q_{i_0}(x, y) = q_{i_0}(y, x) = 0$. Thus $x = y$, since q_{i_0} is a quasi-metric on X. Besides, $Q_F^n(x, x) = 0$ for each $x \in X$, since F satisfies (2.1). So Q_F^n satisfies condition $(q1)$ required for quasi-metrics.

Next we prove that Q_F^n satisfies condition $(q2)$ for quasi-metrics. With this aim consider $x, y, z \in X$ and take $\mathbf{a}, \mathbf{b}, \mathbf{c} \in [0, \infty)^n$ such that

$$\begin{aligned} \mathbf{a} &= (q_1(x, z), \dots, q_n(x, z)), \\ \mathbf{b} &= (q_1(x, y), \dots, q_n(x, y)), \\ \mathbf{c} &= (q_1(y, z), \dots, q_n(y, z)). \end{aligned}$$

The fact that q_i is a quasi-metric on X for each $i \in \{1, \dots, n\}$ provides that $\mathbf{a} \preceq \mathbf{b} + \mathbf{c}$. Then condition (2.3) yields that

$$Q_F^n(x, z) = F(\mathbf{a}) \leq F(\mathbf{b}) + F(\mathbf{c}) = Q_F^n(x, y) + Q_F^n(y, z).$$

Therefore, Q_F^n is a quasi-metric on X and, thus, F is an n-quasi-metric aggregation function.

(2) $\Leftrightarrow$ (3). It is enough to take into account that F is monotone and subadditive if and only if F satisfies condition (2.3). $\square$

Theorem 2.6 allows us to provide a few examples of quasi-metric aggregation functions.

Example 2.7. Let $n \in \mathbb{N}$. The following functions $F : [0,\infty)^n \to [0,\infty)$ are quasi-metric aggregation functions where for all $\mathbf{a}, \mathbf{w} \in [0,\infty)^n$:

(1) $F(\mathbf{a}) = \sum_{i=1}^{n} w_i a_i$ with $\max\{w_1,\ldots,w_n\} > 0$. Notice that weighted arithmetic means, and thus the arithmetic mean, belong to this class of functions (see Grabisch et al. (2009)).
(2) $F(\mathbf{a}) = \max\{w_1 a_1,\ldots,w_n a_n\}$ with $\max\{w_1,\ldots,w_n\} > 0$.
(3) $F(\mathbf{a}) = \sum_{i=1}^{n} w_i a_{(i)}$ with $w_i \geq w_j$ for $i < j$ and $max\{w_1,\ldots,w_n\} > 0$, where $a_{(i)}$ is the ith largest of the $a_1,\ldots,a_n$. Notice that OWA operators with decreasing weights belong to this class of functions (see Grabisch et al. (2009); Recasens (2010)).
(4) $F(\mathbf{a}) = (\sum_{i=1}^{n}(w_i a_i)^p)^{\frac{1}{p}}$ for all $p \in [1,\infty[$ with $max\{w_1,\ldots,w_n\} > 0$. Notice that root-mean-powers with $p \geq 1$ belong to this class of functions (see Grabisch et al. (2009)).
(5) $F(\mathbf{a}) = \min\{c, \sum_{i=1}^{n} w_i a_i\}$ with $\max\{w_1,\ldots,w_n\} > 0$ and $c \in (0,\infty)$.
(6) $F(\mathbf{a}) = \begin{cases} 0 & \text{if } \min\{a_1,\ldots,a_n\} = 0 \\ c & \text{otherwise} \end{cases}$, with $c \in (0,\infty)$.

It is worth mentioning that Aumann functions are instances of quasi-metric aggregation functions. Let us recall that, according to Pokorný (1996), an Aumann function is a monotone and subadditive function $F : [0,\infty[^n \to [0,\infty[$ such that $F(a,0,\ldots,0) = F(0,a,\ldots,0) = F(0,0,\ldots,0,a) = a$ for all $a \in [0,\infty)$. Clearly, by Theorem 1.3, positive Aumann functions are metric aggregation functions. Let us recall that the concept of positive Aumann function was introduce in Mayor and Valero (2018) by replacing the monotony and the subadditivity by positive monotony and positive subadditivity, respectivaly, in the definition of Aumann function. We can conclude that positive Aumann functions are not quasi-metric aggregation functions. Indeed, it is easy to verify that the function provided in Example 2.10 below is a positive Aumann function which is not a 2-quasi-metric aggregation function.

In the light of Theorems 1.4 and 2.6 we immediately obtain that every quasi-metric preserving function is a quasi-metric aggregation function. Nevertheless, the converse is not true. Certainly the function F_s introduced in Example 2.1 satisfies all assumptions in the statement of Theorem 2.6 and, hence, it is an n-quasi-metric aggregation function. However, $F_s(1,0,\ldots,0) = 0$ and so F_s is not amenable. Consequently, Theorem 1.4 ensures that F is not an n-quasi-metric preserving function.

We have pointed out before that each quasi-metric aggregation function is a metric aggregation function. The next example shows that the converse of such an affirmation is not true, in general.

Example 2.8. Consider the function $F : [0,\infty)^2 \to [0,\infty)$ given by

$$F(x,y) = \begin{cases} 0 & \text{if } x = 0, y \in [0,1[\\ y & \text{if } x = 0, y \in [1,\infty) \\ 0 & \text{if } x \in [0,1[, y = 0 \\ x & \text{if } x \in [1,\infty), y = 0 \\ x+y & \text{if } x, y \in]0,\infty) \end{cases}.$$

It is clear that F verifies all conditions in statement of Theorem 1.3 and, thus, that

F is a 2-metric aggregation function. Nevertheless, F does not satisfy condition (2.3) in Theorem 2.6. Indeed, $(1,0) \preceq (\frac{1}{2},0) + (\frac{1}{2},0)$ but

$$F(1,0) = 1 > 0 = F\left(\frac{1}{2},0\right) + \left(\frac{1}{2},0\right).$$

So, F is not a 2-quasi-metric aggregation function.

The following example shows another instance of a metric aggregation function, which is not a quasi-metric aggregation function.

Example 2.9. Let $F : [0,\infty[\to [0,\infty)$ be the function given by $F(0,0) = 0$ and

$$F(a,b) = \begin{cases} 2 & \text{if } first(a,b) \in]0,1[\\ 1 & \text{if } first(a,b) \in [1,\infty[\end{cases},$$

where $(a,b) \neq (0,0)$ and $first(a,b)$ denotes the first value of (a,b) different from 0. According to Mayor and Valero (2018), F is a 2-metric aggregation function. Clearly F is not positive monotone (and so it is not monotone) because $(\frac{1}{2},\frac{1}{2}) \preceq (1,1)$ but $F(\frac{1}{2},\frac{1}{2}) = 2 > F(1,1) = 1$. Thus, F does not satisfy condition (3.3) in Theorem 2.6 and, hence, F is not a 2-quasi-metric aggregation function.

In view of the characterization of quasi-metric aggregation functions provided by Theorem 2.6, one can wonder whether new equivalent conditions to those given in the aforementioned result are obtained when either the monotony is weakened to positive monotony or subadditivity is weakened to positive subadditivity and, in addition, the remaining conditions continue the same. Nonetheless, the answer to the posed question is negative such as the next examples show.

Example 2.10. Consider $F : [0,\infty[^2\to [0,\infty)$ given by

$$F(x,y) = \begin{cases} y & \text{if } x = 0, y \in [0,\infty) \\ x & \text{if } x \in [0,\infty[, y = 0 \\ \frac{x+y}{2} & \text{if } x,y \in]0,\infty) \end{cases}.$$

It is not hard to check that F is positive monotone, subadditive and, in addition, it satisfies conditions (2.1) and (2.2) in Theorem 2.6. However, F is not monotone, since $(0,7) \preceq (1,7)$ but $F(0,7) = 7 > 4 = F(1,7)$. Whence, by Theorem 2.6, we deduce that F is not a 2-quasi-metric aggregation function.

Example 2.11. Let $F : [0,\infty)^3 \to [0,\infty)$ be the function given by

$$F(x,y,z) = \begin{cases} (x+y)^2 & \text{if } x,y \in \left[0,\frac{1}{2}\right[, z = 0 \\ (x+z)^2 & \text{if } x,z \in \left[0,\frac{1}{2}\right[, y = 0 \\ (y+z)^2 & \text{if } y,z \in \left[0,\frac{1}{2}\right[, x = 0 \\ x+y+z & \text{otherwise} \end{cases}.$$

One can easily verify that F is monotone, positive subadditive and, in addition, it satisfies conditions (2.1) and (2.2) in Theorem 2.6. However, F is not subadditive, since if we take $\mathbf{a} = \mathbf{b} = \left(\frac{1}{4}, \frac{1}{4}, 0\right)$ we obtain

$$F(\mathbf{a}+\mathbf{b}) = F\left(\frac{1}{2}, \frac{1}{2}, 0\right) = \frac{1}{2} + \frac{1}{2} = 1 > \frac{1}{2} = \left(\frac{1}{4} + \frac{1}{4}\right)^2 + \left(\frac{1}{4} + \frac{1}{4}\right)^2 = F(\mathbf{a}) + F(\mathbf{b}).$$

Thus, F is not a 3-quasi-metric aggregation function by condition (3.3) in Theorem 2.6.

Taking into account the characterization, provided by Theorem 1.2, of metric preserving functions, it seems natural to discuss the relationship between this kind of functions and the quasi-metric aggregation functions. In this direction, Example 2.9 provides a 2-metric preserving function (compare Example 8 in Mayor and Valero (2010) and Example 10 in Mayor and Valero (2018)) which is not a 2-quasi-metric aggregation function. Moreover, Example 2.1 gives an instance of quasi-metric aggregation function that it not amenable and, hence, it is not metric preserving.

On account of Deza and Deza (2016), given a quasi-metric space (X, q), a quasi-metric q^{-1} and a metric d_q can be induced on X from q as follows: $q^{-1}(x, y) = q(y, x)$ and $d_q(x, y) = \max\{q(x, y), q(y, x)\}$ for all $x, y \in X$. Notice that, given a family of quasi-metrics $\{q_1, \ldots, q_n\}$ on X, one can get, on the one hand, the quasi-metric induced by aggregation of $\{q_1, \ldots, q_n\}$ and the quasi-metric induced by aggregation of $\{q_1^{-1}, \ldots, q_n^{-1}\}$. Moreover, every quasi-metric aggregation function is always a metric aggregation function and, thus, one can try to discern what is the relationship between the metric induced by the aggregation of the family $\{d_{q_1}, \ldots, d_{q_n}\}$ and the metric induced on X by the quasi-metric obtained via aggregation of $\{q_1, \ldots, q_n\}$. The next result clarify the posed questions.

Proposition 2.12. *Let $n \in \mathbb{N}$ and let $F : [0, \infty)^n \to [0, \infty)$ be an n-quasi-metric aggregation function. Then the following assertions hold for all $x, y \in X$:*

(1) $(Q_F^n)^{-1}(x, y) = F(q_1^{-1}(x, y), \ldots, q_n^{-1}(x, y))$.
(2) $d_{Q_F^n}(x, y) \leq F(d_{q_1}(x, y), \ldots, d_{q_n}(x, y)) \leq 2d_{Q_F^n}(x, y)$.

Proof. (1). Let $x, y \in X$. Then

$$\begin{aligned}&(Q_F^n)^{-1}(x, y) = Q_F^n(y, x) = \\ &F(q_1(y, x), \ldots, q_n(y, x)) = F(q_1^{-1}(x, y), \ldots, q_n^{-1}(x, y))\end{aligned}.$$

(2). Let $x, y \in X$. On the one hand, the monotony of F gives that

$$\begin{aligned}&d_{Q_F^n}(x, y) = \max\{F(q_1(x, y), \ldots, q_n(x, y)), F(q_1^{-1}(x, y), \ldots, q_n^{-1}(x, y))\} \leq \\ &F(d_{q_1}(x, y), \ldots, d_{q_n}(x, y)).\end{aligned}$$

On the other hand, the subadditivity of F yields that

$$F(d_{q_1}(x, y), \ldots, d_{q_n}(x, y)) \leq F(q_1(x, y), \ldots, q_n(x, y)) + F(q_1^{-1}(x, y), \ldots, q_n^{-1}(x, y)),$$

since $d_{q_i}(x,y) \leq q_i(x,y) + q_i^{-1}(x,y)$ for all $i \in \{1,\ldots,n\}$. Moreover, we have that

$$F(q_1(x,y),\ldots,q_n(x,y)) + F(q_1^{-1}(x,y),\ldots,q_n^{-1}(x,y)) \leq 2d_{Q_F^n}(x,y).$$

□

We end this section discussing another question that arises in a natural way. When we consider a quasi-metric space (X,q) and the family of quasi-metrics $\{q_1,\ldots,q_n\}$ on a non-empty set X such that $q_i = q$ for all $i \in \{1,\ldots,n\}$, then it seems interesting to know what is the relationship between the quasi-metric generated by aggregation of $\{q_1,\ldots,q_n\}$ and q. In order to give an answer to such a question, let us recall two appropriate notions included in Grabisch et al. (2009). A function $F : [0,\infty)^n \to [0,\infty)$ has $p \in [0,\infty)$ as an idempotent element provided that $F(p,\ldots,p) = p$. As usual, we will say that F is idempotent if each element of $[0,\infty)$ is an idempotent element of F. Moreover, $F : [0,\infty)^n \to [0,\infty)$ is called internal if $\min\{a_1,\ldots,a_n\} \leq F(\mathbf{a}) \leq \max\{a_1,\ldots,a_n\}$ for all $\mathbf{a} \in [0,\infty)^n$.

The next theorem clarifies the issue under discussion.

Theorem 2.13. *Let $n \in \mathbb{N}$ and let $F : [0,\infty)^n \to [0,\infty)$ be an n-quasi-metric aggregation function. Then the following assertions are equivalent:*

(1) F is internal.
(2) F is idempotent.
(3) $q(x,y) = F(q(x,y),\ldots,q(x,y))$ for every quasi-metric space (X,q) and for all $x,y \in X$.
(4) $d(x,y) = F(d(x,y),\ldots,d(x,y))$ for every metric space (X,d) and for all $x,y \in X$.

Proof. (1) ⇔ (2). Theorem 2.6 gives that every quasi-metric aggregation function is monotone. Proposition 2.63 in Grabisch et al. (2009) warranties that every monotone function is idempotent if and only if it is internal.

(2) ⇒ (3). Let (X,q) be a quasi-metric space and $x,y \in X$. Obviously if F is idempotent, then $q(x,y) = F(q(x,y),\ldots,q(x,y))$.

(3) ⇒ (4). Let (X,d) be a metric space and $x,y \in X$. Since every metric d is a quasi-metric we deduce that $d(x,y) = F(d(x,y),\ldots,d(x,y))$.

(4) ⇒ (2). It is clear that every n-quasi-metric aggregation function is an n-metric aggregation function. According to (Mayor and Valero 2018, Theorem 19), an n-metric aggregation function such that $d(x,y) = F(d(x,y),\ldots,d(x,y))$ for every metric space (X,d) and for all $x,y \in X$ is indempotent. □

3. Discarding functions as quasi-metric aggregation functions

In this section we explore a little more about quasi-metric aggregation functions in such a way that a few useful methods to discard those functions that are useless as quasi-metric aggregation functions are presented.

Next we discuss whether a quasi-metric aggregation function can have absorbent elements. To this end, recall that, given $n \in \mathbb{N}$, a function $F : [0,\infty)^n \to [0,\infty)$ has

$u \in [0, \infty)$ as an absorbent (or annihilator) element in its ith coordinate provided that

$$F(a_1, \dots, a_{i-1}, u, a_{i+1}, \dots, a_n) = u$$

for each $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n \in [0, \infty)$ (see Grabisch et al. (2009)). The next results provide information about the matter under consideration when the function has idempotent elements.

Proposition 3.1. *Let $n \in \mathbb{N}$ and let $F : [0, \infty)^n \to [0, \infty)$ be an n-quasi-metric aggregation function. Then F has not $u \in [0, \infty)$ as an absorbent element in at least two variables whenever F has an idempotent element $p \in (0, \infty)$ with $p > 2u$.*

Proof. Suppose, without loss of generality, that F has an absorbent element $u \in [0, \infty)$ in the first two variables. By Theorem 2.6 we have that F is subadditive. Then we deduce that

$$2u = F(u, p-u, p-u, \dots, p-u) + F(p-u, u, u, \dots, u) \geq F(p, \dots, p) = p > 2u,$$

which is impossible. $\square$

As a consequence of the preceding result we obtain the following property.

Corollary 3.2. *Let $n \in \mathbb{N}$. If $F : [0, \infty)^n \to [0, \infty)$ is an idempotent n-quasi-metric aggregation function, then F has not $u \in [0, \infty)$ as an absorbent element in at least two variables.*

The preceding results assure that those functions with an absorbent element in at least two variables are not useful to be quasi-metric aggregation functions when idempotency is enjoyed.

On account of (Mayor and Valero 2018, Proposition 20), Proposition 3.1 and Corollary 3.2 are just true for metric aggregation functions whenever $u \in (0, \infty)$. This fact shows, one more time, that quasi-metric aggregation functions and metric aggregation functions enjoy, in general, different properties.

In the particular case in which the quasi-metric aggregation is not idempotent we have that 0 cannot become an absorbent element in at least two variables. Indeed, on the one hand, we have that every quasi-metric aggregation function is always a (subaddtitive) metric aggregation function. On the other hand, in (Mayor and Valero 2018, Proposition 23) it was proved that every subadditive metric aggregation function has not 0 as an absorbent element in at least two variables. This reasoning allows us to discard all functions having 0 as an absorbent element in at least two variables as quasi-metric aggregation functions. Moreover, in (Mayor and Valero 2018, Proposition 22), it was proved that every conjunctive n-metric aggregation function has 0 as an absorbent element in at least two variables. So we conclude that any quasi-metric aggregation function cannot be conjunctive. Recall that, following Grabisch et al. (2009), a function $F : [0, \infty)^n \to [0, \infty)$ is conjunctive whenever $F(\mathbf{a}) \leq \min\{a_1, \dots, a_n\}$ for each $\mathbf{a} \in [0, \infty)^n$. This reasoning allows us to discard all conjunctive functions as quasi-metric aggregation functions.

Now, we focus our attention on the analysis of the existence of neutral elements of quasi-metric aggregation functions. According to Grabisch et al. (2009), given $n \in \mathbb{N}$, a function $F : [0, \infty)^n \to [0, \infty)$ has $e \in [0, \infty)$ as a neutral element if $F(\mathbf{a_i e}) = a_i$ for each $a_i \in [0, \infty[$ and each $i \in \{1, \dots, n\}$, where $\mathbf{a_i e}$ denotes the element of

$[0,\infty[^n$ that consist of a_i in the ith coordinate and e in the rest of coordinates. In (Mayor and Valero 2018, Proposition 26), it was proved that every subadditive n-metric aggregation function has not any neutral element in $(0,\infty)$. Whence we deduce that any quasi-metric aggregation function has not neutral elements in $(0,\infty)$. Hence we can discard as quasi-metric aggregation function all functions with neutral elements in $(0,\infty)$.

Notice that the unique possible neutral element of a quasi-metric aggregation function is 0. However, the class of those quasi-metric functions with 0 as neutral element matches up with Aumann functions. Furthermore, in Mayor and Valero (2018) it was proved that every subadditive and monotone metric aggregation function F with 0 as neutral element is disjunctive, where, following Grabisch et al. (2009), a function $F:[0,\infty)^n \to [0,\infty)$ is said to be disjunctive provided that $\max\{a_1,\ldots,a_n\} \leq F(\mathbf{a})$ for all $\mathbf{a} \in [0,\infty)^n$. Moreover, the aforementioned metric aggregation functions are always majorized by a disjunctive function and, in addition, they always majorize a conjunctive function. In particular, $\frac{1}{n}\sum_{i=1}^n a_i \leq F(\mathbf{a}) \leq \sum_{i=1}^n a_i$ for all $\mathbf{a} \in [0,\infty)^n$. Consequently, every quasi-metric aggregation function with 0 as neutral element enjoys all the previously indicated properties. So those functions with 0 as neutral element that do not satisfy any of the previously listed properties must be rejected as quasi-metric aggregation function.

Finally, taking into account the exposed facts we have the next surprising result.

Proposition 3.3. *Let $n \in \mathbb{N}$ and let $F:[0,\infty)^n \to [0,\infty)$ be an n-quasi-metric aggregation function with* 0 *as a neutral element. Then, F fulfills the following inequality for all $\mathbf{a},\mathbf{b} \in [0,\infty)^n$:*

$$\max\{F(\mathbf{a})-F(\mathbf{b}),0\} \leq \sum_{i=1}^n \max\{a_i-b_i,0\}. \tag{1}$$

Proof. Let $\mathbf{a},\mathbf{b} \in [0,\infty)^n$ and take $\mathbf{c} \in [0,\infty)^n$ with $c_i = \max\{a_i-b_i,0\}$ for each $i \in \{1,\ldots,n\}$. Then $\mathbf{a} \leq \mathbf{b}+\mathbf{c}$. Thus, by Theorem 2.6, we have that

$$F(\mathbf{a})-F(\mathbf{b}) \leq F(\mathbf{c}) = F(\max\{a_1-b_1,0\},\ldots,\max\{a_n-b_n,0\}).$$

Since $F(\max\{a_1-b_1,0\},\ldots,\max\{a_n-b_n,0\}) \leq \sum_{i=1}^n \max\{a_i-b_i,0\}$ we deduce that $F(\mathbf{a})-F(\mathbf{b}) \leq \sum_{i=1}^n \max\{a_i-b_i,0\}$. The last inequality implies the next one $\max\{F(\mathbf{a})-F(\mathbf{b}),0\} \leq \sum_{i=1}^n \max\{a_i-b_i,0\}$. □

Notice that inequality (1) is a quasi-metric Lipschitz condition with constant 1, since $q_l(a,b) = \max\{a-b,0\}$ is a quasi-metric on $[0,\infty)$.

In the light of Proposition 3.3 those functions with 0 as a neutral element which do not satisfy inequality (1) can not be selected as quasi-metric aggregation function.

From Proposition 3.3 we get the following:

Corollary 3.4. *Let $n \in \mathbb{N}$ and let $F:[0,\infty)^n \to [0,\infty)$ be an n-quasi-metric aggregation function with* 0 *as a neutral element. Then, F fulfills the following inequality for all $\mathbf{a},\mathbf{b} \in [0,\infty)^n$:*

$$|F(\mathbf{a}) - F(\mathbf{b})| \leq \sum_{i=1}^{n} |a_i - b_i| . \tag{2}$$

Proof. By Proposition 3.3 we have that

$$\max\{F(\mathbf{a}) - F(\mathbf{b}), 0\} \leq \sum_{i=1}^{n} \max\{a_i - b_i, 0\}$$

and

$$\max\{F(\mathbf{a}) - F(\mathbf{b}), 0\} \leq \sum_{i=1}^{n} \max\{b_i - a_i, 0\}.$$

It follows that

$$|F(\mathbf{a}) - F(\mathbf{b})| = \max\left\{\max\{F(\mathbf{a}) - F(\mathbf{b}), 0\}, \max\{F(\mathbf{a}) - F(\mathbf{b}), 0\}\right\} \leq$$

$$\max\left\{\textstyle\sum_{i=1}^{n} \max\{a_i - b_i, 0\}, \sum_{i=1}^{n} \max\{b_i - a_i, 0\}\right\} \leq$$

$$\textstyle\sum_{i=1}^{n} |a_i - b_i| .$$

□

4. Two possible fields for applications

We end the paper describing two scenarios, that arise in applied fields, where the exposed theory could be helpful. Concretely we illustrate that in many cases the distances used in Asymptotic Complexity Analysis of algorithms and in Location Analysis can be constructed as an aggregation of quasi-metrics defined all of them on the same non-empty subset. So the use of quasi-metric aggregation functions could be useful to construct a mathematical framework under which many of the specific cases exposed in the literature can be unified under the same general framework and, in addition, the new approach could allows us to discern what method of quasi-metric aggregation is the most appropriate in each problem under study.

4.1. *Asymptotic Complexity Analysis*

In 2003, L.M. García-Raffi et al. introduced the theory of polynomial complexity spaces with the aim of developing a general mathematical framework suitable for asymptotic complexity analysis of algorithms (García-Raffi et al. (2003)). Let us recall that, fixed a polynomial $P(n)$ such that $P(n) > 0$ for all $n \in \mathbb{N}$, the polynomial complexity space is the quasi-metric space $(\mathcal{C}_{P(n)}, d_{\mathcal{C}_{P(n)}})$, where

$$\mathcal{C}_{P(n)} = \{f : \mathbb{N} \to [0, +\infty) : \sum_{n=1}^{+\infty} 2^{-P(n)} f(n) < +\infty\},$$

and the quasi-metric $d_{\mathcal{C}_{P(n)}}$ is given by

$$d_{\mathcal{C}_{P(n)}}(f, g) = \sum_{n=1}^{+\infty} 2^{-P(n)} \max\{g(n) - f(n), 0\}.$$

The utility of the polynomial complexity space in asymptotic complexity analysis of algorithms is based on the fact that the numerical value $d_{\mathcal{C}_{P(n)}}(f, g)$ can be understood as a measure of the progress made in lowering of complexity when an algorithm with running time represented by g is replaced by another one with running time represented by f. Indeed,

$$d_{\mathcal{C}_{P(n)}}(f, g) = 0 \Leftrightarrow g(n) \leq f(n) \text{ for all } n \in \mathbb{N}$$

and, hence, the running time of computing represented by g is more "efficient" than the algorithm whose running time of computing is represented by f on inputs size when $f \neq g$. Thus when the running time of computing of an algorithm, represented by g, is not known with precision, the fact that $d_{\mathcal{C}_{P(n)}}(f, g) = 0$ guarantees that f provides an asymptotic upper bound of g and, thus, that the algorithm under consideration will take at most $f(n)$ time, when the size of the input is n, to solve the problem for which it has been designed. Observe that the asymmetry of $d_{\mathcal{C}_{P(n)}}$ is crucial in order to get f as an asymptotic upper bound of g. Indeed, a pseudo-metric will be able to provide information about the efficiency but it would not be useful to state which algorithm is more efficient of both.

The applicability of the complexity space to the asymptotic analysis of algorithms has been illustrated providing new techniques, by means of fixed point techniques, to specify asymptotic upper bounds for those algorithms whose running time of computing satisfies a recurrence equation. We refer the reader to García-Raffi et al. (2002); Llull-Chavarría and Valero (2009); Romaguera and Schellekens (1999); Romaguera and Valero (2008); Romaguera at al. (2011) for a detailed treatment of the topic. In García-Raffi et al. (2003), a variant of the polynomial complexity space was introduced in order to provide a mathematical framework to perform an appropriate description of the running time of computing of exponential time algorithms and, thus, to develop suitable fixed point techniques to provide asymptotic upper bounds. In this case the new complexity space was called supremum polynomial complexity space and it was given as the quasi-metric space $(\mathcal{C}_{P(n)}, d_{\infty,\mathcal{C}_{P(n)}})$, where

$$d_{\infty,\mathcal{C}_{P(n)}}(f, g) = \sup_{n \in \mathbb{N}} \left\{2^{-P(n)} \max\{g(n) - f(n), 0\}\right\}.$$

According to García-Raffi et al. (2004), the asymptotic behaviour of the running time of algorithms can be discussed through finite approximations $d_{\mathcal{C}_{m,P(n)}}(f, g)$ of the numerical value $d_{\mathcal{C}_{P(n)}}(f,g)$, where

$$d_{\mathcal{C}_{m,P(n)}}(f,g)=\sum_{i=1}^{m}2^{-P(n)}\max\{g(n)-f(n),0\}$$

for any $m\in\mathbb{N}$. Observe that $d_{\mathcal{C}_{m,P(n)}}$ is a quasi-metric on $\mathcal{C}_{m,P(n)}$, where $\mathcal{C}_{m,P(n)}$ denotes the set all functions belonging to $\mathcal{C}_{P(n)}$ whose domain is restricted to $\{1,\ldots,m\}$. Moreover, notice that

$$d_{\mathcal{C}_{P(n)}}(f,g)=0\Leftrightarrow d_{\mathcal{C}_{m,P(n)}}(f,g)=0 \text{ for all } m\in\mathbb{N}.$$

Therefore, the fixed point techniques developed to specify asymptotic upper bounds for those algorithms whose running time of computing satisfies a recurrence equation can be rewritten in terms of non-asymptotic criteria involving fixed point arguments based on the use of the quasi-metrics $d_{\mathcal{C}_{m,P(n)}}$. Of course, the same happens for the case of $d_{\infty,\mathcal{C}_{P(n)}}$, where now the finite approximations are yielded by the quasi-metrics $d_{\max,m,\mathcal{C}_{m,P(n)}}$ given by

$$d_{\max,m,\mathcal{C}_{m,P(n)}}(f,g)=\max_{1\leq i\leq m}\left\{2^{-P(n)}\max\{g(n)-f(n),0\}\right\}.$$

It is clear that the quasi-metrics $d_{\mathcal{C}_{m,P(n)}}$ and $d_{\max,m,\mathcal{C}_{m,P(n)}}$ can be obtained by aggregation, through the quasi-metric aggregation functions given by (1) and (2) in Example 2.7, of the family of quasi-metrics $\{q_1,\ldots q_m\}$ on $[0,\infty)$, where $q_i=q_u$ and $w_i=2^{-P(i)}$ for all $i\in\{1,\ldots,m\}$.

In the light of the preceding fact, it seems natural to incorporate quasi-metric aggregation functions in the asymptotic complexity analysis of algorithms via developing general polynomial complexity spaces in such a way that the exposed mathematical frameworks can be retrieved as particular case and, in addition, with the aim of, on the one hand, exploring what quasi-metric aggregation function is the most appropriate for developing measures of the progress made in lowering of complexity and, on the other hand, developing non-asymptotic criteria fixed point methods best adapted to each family of recurrences that (may) arise in the study of algorithms.

4.2. *Location Analysis*

A location problem consists of looking for a new facilities of a company to provide a service for a set of customers. So this is a relevant topic in Logistics because of location of facilities and allocation of customers to the facilities provide constraints in the distribution process and its cost and efficiency. Thus the main problem in Location Analysis is a Decision Making problem in which the company wants to decide how to place new facilities, taking into account the customers allocation, in such a way that the facilities are placed in an optimum way, i.e., reducing the cost or maximizing the customer satisfaction. For a fuller treatment of the topic we refer the reader to Sule (2001).

Two typical problems in Location Analysis are the Weber problem (or minsum problem) and the Rawls problem (or minmax problem). In the first one, the target is to place n ($n\in\mathbb{N}$) facilities in n locations minimizing the global cost, which is usually described in terms of time, money, number of trips, etc. Of course, the demand of the customer is associated to each facility location and, thus, each location contributes to the objective in a different way or with a different weight. Examples where this kind of problem arises in a natural way when the facility to be located is a distribution center or a center for energy production. The target for the second problem is again to place n facilities but this time minimizing the maximum cost. Typical examples of this problem are those where the facility to be placed is an emergency service like fire or police station and an ambulance service.

From a mathematical viewpoint, the exposed problems can be stated as follows:

We have to choose a location for a facility among a collection of them X (the set of facilities X can be discrete or continuous) in such a way that the selection of a facility is influenced by the cost of the interaction between the facility and a collection of destinations (that represent the customers) normally finite $A = \{a_1, \ldots, a_n\}$. The aim is to determine the location that minimizes the global cost interaction C. Normally, the cost is measured as a function of the distance between the facility and the destination. Hence, given a location $x \in X$, the interaction cost between x and destination i is provided as a function $c_i(x, a_i) = c_i(d(x, a_i))$. This kind of cost functions are known as transportation cost functions. According to Drezner and Wesolowsky (1989); Plastria (1992, 1995, 2009); Tamir (1992) in many real problems the transportation cost depends on quasi-metrics. This is the case when we consider problems where there are involved one-way paths, rush-hour traffic, navigation in presence of wind, fuel cost, time travel, etc. Observe that the global transportation cost can be understood as a global distance from the facility to all destinations.

A typical cost is proportional to the distance, that is $c_i(x, a_i) = w_i q(x, a_i)$ for all $i \in \{1, \ldots, n\}$, and the global cost is obtained by means of aggregation of each individual transportation cost. Thus the problem under consideration is reduced to the following optimization problem:

$$Min_{x \in X} C(x),$$

where

$C(x) = \sum_{i=1}^{n} w_i q(x, a_i)$ for the Weber problem,
$C(x) = \max\{w_1 q(x, a_1), \ldots, w_n q(x, a_n)\}$ for the Rawls problem.

Notice that in the expression of the global transportation costs can be used, at the same time, different quasi-metric depending on the nature of the cost under consideration and, thus, one would obtain:

$$Min_{x \in X} C(x),$$

where

$C(x) = \sum_{i=1}^{n} w_i q_i(x, a_i)$ for the Weber problem,
$C(x) = \max\{w_1 q_1(x, a_1), \ldots, w_n q_n(x, a_n)\}$ for the Rawls problem.

In Nickel et al. (2005), it has been pointed out that the both preceding problems are a particular case of a more general one where the aggregation of costs is made by means of an OWA operator. Thus a unified framework based on OWAs is presented and a deep discussion about how solve such problems is carried out in Nickel et al. (2005).

In the light of the exposed facts, it appears natural to consider quasi-metric aggregation functions in Location Analysis with the aim of finding out what they can contribute to develop best adapted global transportation costs functions and new optimization criteria.

5. Conclusions

We have continued the study of the problem of aggregation of distances. Concretely, we have introduced the notion of quasi-metric aggregation function and we have provided a characterization of such a notion in terms of (triangle) triplets. Moreover, the relationship with metric aggregation functions has been also discussed and a few differences have been shown. Moreover, we have analyzed some properties fulfilled by the functions under consideration which play a central role in order to discard those functions that are useless as quasi-metric aggregation functions. Appropriate and illustrative examples have been given. Finally, two possible fields where the developed theory can be useful have been exposed.

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