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Indistinguishability Operators Applied to Task Allocation Problem in Multi-Agent Systems

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Featured Application: Indistinguishability can provide a new family of response functions that could be applied to implement a new generation of swarm-like methods to carry out missions, like for example, allocate tasks to a set of robots in different environments. Actually, this paper provides a first implementation of this kind of systems.

Abstract: In this paper we show an application of indistinguishability operators to model response functions. Such functions are used in the mathematical modeling of the task allocation problem in multi-agent systems when the stimulus, perceived by the agent, to perform a task is assessed by means of the response threshold model. In particular, we propose this kind of operators to represent a response function when the stimulus only depends on the distance between the agent and a determined task, since we prove that two celebrated response functions used in the literature can be reproduced by appropriate indistinguishability operators when the stimulus only depends on the distance to each task that must be carried out. Despite nowadays there is not a systematic method to generate response functions, this paper provides, for the first time, a theoretical foundation to generate them and study their properties. To validate the theoretical results, the aforementioned indistinguishability operators have been used to simulate under MATLAB the allocation of a set of tasks in a multi-robot system with fuzzy Markov chains.

Keywords: task allocation; multi-agents; multi-robot; swarm; response function; *t*-norm; additive generator; indistinguishability operator; distance.

1. Introduction

The distribution of a determined number of tasks among a group of agents is a problem intensely studied in different fields, as Economics or Robotics (we refer the reader, for instance, to [1–3] for a deeper treatment of the topic). It consists in allocating a collection of labours on an amount of agents in the most efficient way, i.e., in such a way that the best agent is selected to perform each one of the labour to be carried out. This problem, commonly referred to as task allocation problem, is still an open issue in real environments where the agents have a limited number of resources to obtain the optimal allocation. One of those challenging environments is the one formed by two or more autonomous robots that perform cooperatively a common mission, from now on referenced as multi-robot systems.

Among all the methods proposed to address the task allocation problem, this paper focuses on swarm methods, which are inspired by insect colonies where an intelligent behaviour emerges from the interaction of very simple skills running on each agent. Concretely, this work focuses on the so-called Response Threshold Method (RTM for short). In these methods each involved agent has associated a task response threshold and a task stimulus. The task stimulus value indicates how much attractive is the task for the agent and its threshold is a parameter of the system. Thus, an agent starts the execution of a task following a probability function, referenced as response function and denoted as P, that depends on both aforementioned values. As the probability of executing a task only depends on the current task, or state, the decision process can be modelled as a probabilistic Markov chain. This classical probabilistic approach presents some well known disadvantages (see [4]), for instance problems with the selection of the probability function (response function) when more than two tasks are considered, asymptotic converge to a system's stable state, and so on. Due to the inconveniences, in [4] it was proposed a new possibilistic theoretical formalism for a RTM. The RTM is implemented considering transitions possibilities (response functions) instead of transitions probabilities (response functions) and possibilistic Markov chains (also known as fuzzy Markov chains) instead the classical probabilistic ones. The theoretical and empirical results demonstrated, among other advantages, that fuzzy Markov chains applied to task allocation problems require a very few number of steps to converge to a stable state.

As will be proved for the first time in this paper, the most widely used response functions are a specific kind of mathematical functions called indistinguishability operators. This work demonstrates that the indistinguishability operators, in general, are useful in modeling the response probability function in response threshold task allocation problems in those cases in which the stimulus of each agent of the group only depends on the distance to each task that must be carried out. So, the aim of this paper from a theoretical point of view is twofold. On the one hand, we will use a well-known technique to induce indistinguishability operators in order to provide a few examples that could be useful to model of response functions. On the other hand, we will show that two popular response functions exposed in [5] can be reproduced from indistinguishability operators. Thus, this paper provides a new systematic method to generate response functions.

This paper extends the previous work in [4,6] in order to apply the indistinguishability operators as possibility transition functions to allocate tasks in a multi-robot system with fuzzy Markov chains. As tested in the aforementioned references, each robot uses a fuzzy Markov chain to decide the next task to execute. This paper also extends these previous works with new simulations performed under MATLAB in order to test the system's behaviour in environments with tasks placed in clusters or groups. The results show that, on the one hand, the convergence time does not depend on the tested indistinguishability operator used as possibility transition function. On the other hand, the simulations also show that the placement of the objects clearly impacts on the system performance. In all cases, the fuzzy Markov chains always outperform their probabilistic counterpart.

The paper is organized as follows: in Section 2 the main concepts on Response Threshold Methods are reviewed. In Section 3 we recall all pertinent aspects of indistinguishability operators necessary to our subsequent discussion. In Section 4, first we illustrate the well-kby means ofnown technique for generating indistinguishability operators from distances by means of two illustrative examples. After this, we show that a celebrated response function that appear in [5] can be retrieved from an appropriate indistinguishability operator and the mentioned technique. Furthermore, inspired by the so-called the classical exponential response function given in [7], we introduce a new indistinguishability operator. The numerical values admit the same interpretation as the aforesaid paradigmatic exponential response function. We also show that the such an exponential response function can be exactly retrieved from the introduced indistinguishability operator by means of a known technique to induce distances from indistinguishability operators. Section 5 reviews the basics of possibilistic theory and fuzzy Markov chains. In Section 6 it will be posed the possibilistic multi-robot task allocation problem. Then, in Section 7 we will show the experimental results. Finally, Section 8 presents the conclusions and future work.

2. Swarm task allocation: Response Threshold

In this section we will introduce the main concepts of classical (probabilistic) Response Threshold Methods (RTMs) and we will motivate that their response functions can be assimilated to indistinguishability operators. It must be recall that classical RTMs are modelled using probabilistic Markov chains.

As was mentioned in Section 1, one way to model the probability transition function of the aforementioned Markov chain is by means of the so-called stimulus and response thresholds. Concretely, the stimulus expresses the need perceived by the agent to develop a task and the threshold determines the tendency of an agent to respond to an stimulus intensity and, therefore, to make the task. In [5], it was proposed a method, based on response functions, to model the aforesaid probability when the response threshold is fixed over time. In the aforesaid reference, the probability response function $P(s,\theta)$ can be defined by

$$P(s,\theta) = \frac{s^n}{s^n + \theta^n}, \tag{1}$$

where s denotes for each agent the intensity of a stimulus to carry out a particular task and θ denotes the threshold for each agent and task. Notice that, according to [5], $n > 1$ (with $n\mathbb{N}$) determines the steepness of the threshold. Of course, the numerical value $P(s,\theta)$ can be interpreted as follows: On the one hand, values of the stimulus intensity much smaller than threshold (denoted by $s << \theta$), implies response values (probabilities of engaging task perfomance) close to 0. On the other hand, stimulus intensity much greater than the threshold (denoted by $s >> \theta$), means probability of engaging task performance close to 1.

Other authors have used response functions of type (1) in order to model probability of engaging tasks performance in multi-robot task allocation. We can find an instance in [8], where it was proposed a mathematical model to assign particular events to individual robots in such a way that each robot is limited to one task at time. Concretely, they assume that each robot senses the need to handle the closest task. In this direction, the stimulus produced by a task e for a robot r was taken as the inverse of the distance between the task and the robot, i.e. $s = \sigma(r,e) = \frac{1}{d(r,e)}$. Then, the probability response function is formulated as follows:

$$P(s,\theta) = \frac{\sigma(r,e)^n}{\sigma(r,e)^n + \theta^n}. \tag{2}$$

After exploring different thresholds, As was pointed out in [8], the best performance was achieved with the inverse of the expected distance between tasks D, i.e. $\theta = \frac{1}{D}$. In this case, $s << \theta$, or equivalently $d(r,a) >> D$, implies low response to engage the task and $s >> \theta$, or equivalently $d(r,e) << D$, implies high motivation to take on the task.

A straightforward computation yields that (2) can be transformed into the response function

$$P(s,D) = \frac{D^n}{D^n + d(r,e)^n}, \tag{3}$$

Notice that expression (3) maintains the essential properties of the response function (2). It must be stressed that this kind of response functions have been recently applied to possibilistic multi-robot task allocation problems (see [4] for more details).

Expression (3) has motivated this paper, since as we will show in Section 4, $P(s,D)$ is an indistinguishability operator (see Section 3).

3. Preliminaries on Indistinguishability Operator

The concept of *triangular norm*, briefly t-norm, appeared in the literature as a tool to manage the triangle inequality in the construction of metric spaces which take as values a probability distribution instead of a positive real number. Since then, they have played an essential role in Fuzzy Logic and many authors have contributed to the development of this kind of binary operation. Our basic reference for t-norms and all related notions is [9].

Let us recall that a t-norm is a function $T : [0,1]^2 \to [0,1]$ such that for all $x, y, z \in [0,1]$ the following four axioms are satisfied:

(T1) $T(x,y) = T(y,x)$; (Commutativity)
(T2) $T(x, T(y,z)) = T(T(x,y),z)$; (Associativity)
(T3) $T(x,y) \geq T(x,z)$; where $y \geq z$ (Monotonicity)
(T4) $T(x,1) = x$. (Boundary Condition)

An interesting subclass of t-norms in our subsequent study are the so-called Archimedean which are defined as follows:

A t-norm is called Archimedean if for each $x, y \in]0,1[$ there exists $n \in \mathbb{N}$ such that $x^{(n)} < y$, where $x^{(n)} = T(x, \ldots, x)$ n-times.

Archimedean t-norms are exactly those t-norms that satisfy $T(x,x) < x$ for each $x \in]0,1[$ whenever T is, in addition, continuous. Two well-known examples of continuous Archimedean t-norms are the usual product T_P and the Luckasievicz t-norm T_L, where $T_P(x,y) = x \cdot y$ and $T_L(x,y) = \max\{x + y - 1, 0\}$ for all $x, y \in [0,1]$. An example of continuous t-norm which is non-Archimedean is the minimum t-norm T_M, i.e., $T_M(x,y) = \min\{x,y\}$ for all $x, y \in [0,1]$. These t-norms are the most commonly used in Fuzzy Logic.

A concept related to a t-norm, which will play an important role in this paper, is the notion of pseudo-inverse and additive generator. The notion of pseudo-inverse is given as follows:

Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing continuous function provided that $f(1) = 0$. Then the *pseudo-inverse* $f^{(-1)} : [0,\infty] \to [0,1]$ of f is defined as follows:

$$f^{(-1)}(y) = f^{-1}(\min\{f(0), y\}) = \max\{0, f^{-1}(y)\}. \tag{4}$$

Moreover, given a t-norm T, a strictly decreasing continuous function $f_T : [0,1] \to [0,\infty]$ is said to be an additive generator of T provided that $f_T(1) = 0$ and

$$T(x,y) = f_T^{(-1)}(f_T(x) + f_T(y)).$$

Note that in this case, the t-norm is continuous.

It is known that each t-norm with an additive generator is Archimedean. However, the converse of the former assertion is not true in general. The next result states that continuous t-norms always admit an additive generator.

Theorem 1. *A mapping $T : [0,1] \times [0,1] \to [0,1]$ is a continuous Archimedean t-norm if and only if there exists a continuous additive generator f_T of T.*

Let us note that if T is a t-norm and f_T is an additive generator of T, then this additive generator multiplied by a positive constant is again an additive generator of T, i.e., if f_T is an additive generator of T, then the function $f_{T,\theta}$ is again an additive generator of T, where $f_{T,\theta}(x) = \theta \cdot f_T(x)$ for all $x \in [0,1]$ and each $\theta \in]0,\infty[$. Furthermore, if $f_T^{(-1)}$ is the pseudo-inverse of an additive generator f_T, then it is easy to verify that the pseudo-inverse of the additive generator $f_{T,\theta}$ is given by $f_{T,\theta}^{(-1)}(y) = f_T^{(-1)}(\frac{y}{\theta})$ for all $y \in [0,\infty]$.

We have introduced some details about t-norms that will be necessary later on. Now, we are able to recall the concept of *T-indistinguishability operator*. This concept was introduced in 1982 by E. Trillas as a way to measure the degree of equivalence, in Fuzzy Logic, between the elements of a set X (see [10,11]).

Let X be a nonempty set and let T be a t-norm, we will say that a fuzzy set $E : X \times X \to [0,1]$ is a T-indistinguishability operator if it satisfies for each $x, y, z \in X$ the following:

(E1) $E(x,x) = 1$; (Reflexivity)
(E2) $E(x,y) = E(y,x)$; (Symmetry)
(E3) $E(x,z) \geq T(E(x,y), E(y,z))$. (Transitivity)

A T-indistinguishability operator E is said to separate points provided that $E(x,y) = 1 \Leftrightarrow x = y$ for all $x, y \in X$. The notion of indistinguishability operators is essentially interpreted as a measure of similarity (in contrast to dissimilarity modelled by pseudo-metrics). Thus, $E(x,y)$ matches up with the degree of indistinguishability between the objects x and y. In fact, the greater $E(x,y)$ the most similar are x and y. In such a way that when $x = y$, then the measure of similarity is exactly $E(x,x) = 1$.

Since Trillas introduced the notion of indistinguishability operator, many authors have contributed to the development of a theory in which this concept plays an essential role. We focus our attention on a method to construct T-indistinguishability operators from distances and vice-versa (for a detailed treatment of the topic we refer the reader to [12–14]).

The next proposition provides a technique that allows us to construct distances from indistinguishability operators.

Theorem 2. *Let X be a nonempty set and let T^* be a t-norm with additive generator $f_{T^*} : [0,1] \to [0,\infty]$. Let $d_E : X \times X \to [0,\infty]$ be the function defined by*

$$d_E(x,y) = f_{T^*}(E(x,y))$$

for all $x, y \in X$. If T is a t-norm, then the following assertions are equivalent:

1) *$T^* \leq T$ (i.e., $T^*(x,y) \leq T(x,y)$ for all $x, y \in X$).*
2) *For any T-indistinguishability operator E on X that separates points the function d_E is a distance on X.*

The following result developes a technique that allows to induce indistinguishability operators from distances.

Theorem 3. *Let d be a distance on a nonemprty set X and let T be a continuous Archimedean t-norm with additive generator f_T. Then, the fuzzy set $E_T : X \times X \to [0,1]$, given by $E_T(x,y) = f_T^{(-1)}(d(x,y))$ for all $x, y \in X$, is a T-indistinguishability operator that separates points on X.*

Recall that a distance on a nonempty set X is a fucntion $d : X \times X \to [0,\infty]$ satisfying the following axioms for all $x, y, z \in X$:

(d1) $d(x,y) = 0 \Leftrightarrow x = y$;
(d2) $d(x,y) = d(y,x)$;
(d3) $d(x,z) \leq d(x,y) + d(y,z)$.

4. T-indistinguishability operators, distances and response functions

In this section we will apply Theorems 2 and 3 for some particular t-norms in order to construct two indistinguishability operators that allow to reproduce two celebrated response functions that appear in [5]. In this direction we first provide a few examples of T-indistinguishability operators that separate points with the aim of illustrating such a technique, due to the lack of this sort of examples in the literature.

4.1. Examples

We begin applying the aforesaid construction to the Luckasievicz t-norm T_L which is continuous and Archimedean (see Section 3).

A T_L-indistinguishability operator

Let d be a distance on a nonempty set X and consider the Luckasiewicz t-norm T_L. It is clear that the function $f_{T_L} : [0,1] \to [0,\infty]$, given by $f_{T_L}(x) = 1 - x$ for all $x \in [0,1]$, is an additive generator of T_L. Applying (4), an easy computation shows that the pseudo-inverse $f_{T_L}^{(-1)}$ of the additive generator f_{T_L} is given by

$$f_{T_L}^{(-1)}(y) = \max\{0, 1 - y\}$$

for all $y \in [0,\infty]$. Then, using the construction of Theorem 3, i.e. $E_{T_L}(x,y) = f_{T_L}^{(-1)}(d(x,y))$ for all $x, y \in X$, we obtain a T_L-indistinguishability operator on X that separates points, which has the following expression:

$$E_{T_L}(x,y) = \begin{cases} 1 - d(x,y), & \text{if } 0 \leq d(x,y) < 1; \\ 0, & \text{elsewhere,} \end{cases}$$

for all $x, y \in X$.

A T_P-indistinguishability operator

Let d be a distance on a nonempty set X and consider the product t-norm T_P. It is clear that the function $f_{T_P} : [0,1] \to [0,\infty]$, given by $f_{T_P}(x) = -\log(x)$ for all $x \in [0,1]$, is an additive generator of T_P. The pseudo-inverse $f_{T_P}^{(-1)}$ of f_{T_P} is given by

$$f_{T_P}^{(-1)}(y) = \mathrm{e}^{-y}$$

for all $y \in [0,\infty]$. Then, using the construction of Theorem 3, i.e., $E_{T_P}(x,y) = f_{T_P}^{(-1)}(d(x,y))$ for all $x, y \in [0,1]$, we obtain the following expression:

$$E_{T_P}(x,y) = \mathrm{e}^{-d(x,y)}$$

for all $x, y \in X$. As in the above example, Theorem 3 ensures that E_{T_P} is a T_P-indistinguishability operator on X that separates points.

After presenting these two easy, but illustrative, preceding examples, which are constructed by means of the most commonly continuous Archimedean t-norms used in Fuzzy Logic, we will continue showing that the response function P, given by (3), is an indistinguishability operator which opens a wide range of potential applications from a mixed framework based on indistinguishability operators and distances to task allocation problems in multi-agent systems.

4.2. *P as an indistinguishability operator*

Consider the family of t-norms $(T^{\lambda}_{Dom})_{\lambda \in [0,\infty]}$ due to Dombi. Recall, according to [9], that such a family of t-norms is given by:

$$T^{\lambda}_{Dom}(x,y) = \begin{cases} T_D(x,y), & \text{if } \lambda = 0; \\ T_M(x,y), & \text{if } \lambda = \infty; \\ \frac{1}{1+\left(\left(\frac{1-x}{x}\right)^{\lambda}+\left(\frac{1-y}{y}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}, & \text{elsewhere,} \end{cases}$$

where T_D is the drastic t-norm (see [9]).

According to [9], the t-norm T^{λ}_{Dom} is continuous and Archimedean for each $\lambda \in]0,\infty[$. Moreover an additive generator of T^{λ}_{Dom} is given by

$$f_{T^{\lambda}_{Dom}}(x) = \left(\frac{1-x}{x}\right)^{\lambda}$$

for all $x \in [0,1]$ and for each $\lambda \in]0,\infty[$. It is not hard to verify that the pseudo-inverse of this additive generator $f_{T^{\lambda}_{Dom}}^{(-1)}$ is given by

$$f^{(-1)}_{T^{\lambda}_{Dom}}(y) = \frac{1}{1+y^{\frac{1}{\lambda}}}$$

for all $y \in [0,\infty]$ and for each $\lambda \in]0,\infty[$.

Taking into account the preceding facts we are able to prove that P is in fact an indistinguishability operator constructed from a Dombi t-norm. To this end, assume that d is a distance on X and let T^{λ}_{Dom} be a Dombi t-norm for an arbitrary $\lambda \in]0,\infty[$. By Theorem 3 we obtain a T^{λ}_{Dom}-indistinguishability operator $E_{T^{\lambda}_{Dom}}$ that separates points by means of $E_{T^{\lambda}_{Dom}}(x,y) = f^{(-1)}_{T^{\lambda}_{Dom}}(d(x,y))$ for all $x,y \in X$. It follows that

$$E_{T^{\lambda}_{Dom}}(x,y) = \frac{1}{1+d(x,y)^{\frac{1}{\lambda}}}$$

for all $x,y \in X$.

Next fix $n \in \mathbb{N}$ and take $\lambda = \frac{1}{n}$. Then we have that $f_{T^{\frac{1}{n}}_{Dom},\theta}$ is also an additive generator of $T^{\frac{1}{n}}_{Dom}$, where

$$f_{T^{\frac{1}{n}}_{Dom},\theta}(x) = \theta \cdot f_{T^{\frac{1}{n}}_{Dom}}(x) = \theta \cdot \left(\frac{1-x}{x}\right)^{\frac{1}{n}}$$

for all $x \in X$ and for each $\theta \in]0,\infty[$.

Now, we will apply Theorem 3 through $f_{T^{\frac{1}{n}}_{Dom},\theta}$. Since the pseudo-inverse $f^{(-1)}_{T^{\frac{1}{n}}_{Dom},\theta}$ of $f_{T^{\frac{1}{n}}_{Dom},\theta}$ is given by

$$f^{(-1)}_{T^{\frac{1}{n}}_{Dom},\theta}(y) = f^{(-1)}_{T^{\frac{1}{n}}_{Dom}}\left(\frac{y}{\theta}\right) = \frac{1}{1+\left(\frac{y}{\theta}\right)^{n}} = \frac{\theta^n}{\theta^n + y^n}.$$

Therefore, $E_{T^{\frac{1}{n}}_{Dom}}$ is a $T^{\frac{1}{n}}_{Dom}$-indsitinguishability operator with

$$E_{T^{\frac{1}{n}}_{Dom}}(x,y) = f^{(-1)}_{T^{\frac{1}{n}}_{Dom},\theta}(d(x,y)) = \frac{\theta^n}{\theta^n + d(x,y)^n} \tag{5}$$

for all $x,y \in X$. Since P given by (3) matches up with the preceding one we conclude that the response function P is a $T^{\frac{1}{n}}_{Dom}$-indsitinguishability operator that separates points.

4.3. An exponential response function and indistinguishability operators

In [7] (see also [5]) an exponential response function was introduced in order to model honey bee division of labour by means of response thresholds. In particular, the exponential response function of an agent taken under consideration was the following:

$$P_{exp}(s,\theta) = 1 - \mathrm{e}^{-\frac{s}{\theta}}, \tag{6}$$

where s denotes the intensity of the stimulus for an agent to carry out a task and θ is the threshold. Note that, as in the case of response function (1), the probability of engaging task performance is small for $s << \theta$, and is close to 1 for $s >> \theta$.

Our final goal of this section is twofold. On the one hand, we provide an example of indistinguishability operator which exhibits a behavior similar to response function (6) and, thus, it could be used in task allocation problems. On the other hand, we are able to retrieve exactly, from the generated indistinguishability operator and through the technique stated in the statement of Theorem 2, the response function (6) when it depends on the distance between an agent and a task.

Next consider the family of t-norms $(T^{\lambda}_{AA})_{\lambda\in[0,\infty]}$ introduced by Aczél and Alsina. Recall, according to [9], that such a family is given as follows:

$$T^{\lambda}_{AA}(x,y) = \begin{cases} T_D(x,y), & \text{if } \lambda = 0; \\ T_M(x,y), & \text{if } \lambda = \infty; \\ \mathrm{e}^{-\left((-\log x)^{\lambda}+(-logy)^{\lambda}\right)^{\frac{1}{\lambda}}}, & \text{elsewhere.} \end{cases}$$

Following [9], for each $\lambda \in]0,\infty[$ we have that T^{λ}_{AA} is continuous and Archimedean. Moreover, an additive generator of T^{λ}_{AA} is given by

$$f_{T^{\lambda}_{AA}}(x) = (-\log(x))^{\lambda}$$

for all $x \in [0,1]$. A straightforward computation shows that the pseudo-inverse $f^{(-1)}_{T^{\lambda}_{AA}}$ of $f_{T^{\lambda}_{AA}}$ is given as follows:

$$f^{(-1)}_{T^{\lambda}_{AA}}(y) = e^{-\left(y^{\frac{1}{\lambda}}\right)},$$

for all $y \in [0,\infty]$ and for each $\lambda \in]0,\infty[$.

In the light of the exposed facts we are able to introduce the announced indistinguishability operator. To this end, assume that d is a distance on a nonempty set X and let T^{λ}_{AA} be an Aczél-Alsina t-norm for $\lambda \in]0,\infty[$. Applying Theorem 3 we obtain the T^{λ}_{AA}-indistinguishability operator $E_{T^{\lambda}_{AA}}$ given by

$$E_{T^{\lambda}_{AA}}(x,y) = e^{-\left(d(x,y)^{\frac{1}{\lambda}}\right)}$$

for all $x,y \in X$. Notice that $E_{T^{\lambda}_{AA}}$ separates points.

Now, following similar arguments to those given in Subsection 4.2 we obtain, for each $n \in \mathbb{N}$ and $\theta \in]0,\infty[$, the following indistinguishability operator from the preceding one:

$$E_{T^{\frac{1}{n}}_{AA}}(x,y) = e^{-\frac{d(x,y)^n}{\theta^n}}. \tag{7}$$

Of course, one can observe that the last operator presented involves the same elements of function (3). Indeed, this operator depends simultaneously on the distance $d(x,y)$, on a threshold parameter θ and it contains the non-linearity constant n. Besides, the nature of this indistinguishability operator is an exponential function as response function (6). Nevertheless, a slightly difference between them must be stressed with the aim of interpreting the operator given in (7) as a response function. It is clear that in (7), s must be considered as the inverse of the distance with the aim of the indistinguishability operator in (7) can be interpreted as a response function. Indeed, s must be understood as the inverse of the distance in order to preserve the essence of the impact of the stimulus s in the expression of a response function. Thus we have that the operator given by (7) acts as response function since it satisfies the following: $d(x,y) >> \theta$ implies probability response close to 0 and $d(x,y) << \theta$ returns a probability response close to 1. It follows that the idea of “an agent is high motivated for performing closer tasks” is preserved. The fact that the operator in (7) can be interpreted as a response function inspires that several families of indistinguishability operators can be proposed and tested with a large number of experiments in order to be compared with previous response functions used in the literature and, thus, to determine if indistinguishability operators are an appropriate mathematical tool for task allocation problems.

Finally, we show that, in addition, the indistinguishability operator given by (7) allows to retrieve exactly the response function (6). Hence, indistinguishability operators can still be used for generating response function even in those cases in which the stimulus s cannot be understood as the inverse of the distance. Indeed, note that $T_L \leq T^1_{AA} = T_P$ and, thus, Theorem 2 guarantees that the function given by $d_{E_{T^1_{AA}}} = f_{T_L}(E_{T^1_{AA}}) = 1 - E_{T^1_{AA}}$ is a distance on X. But such a function matches up with the exponential response function given by 6.

5. Possibilistic Markov chains: theory

As was proved in [4], possibilistic Markov chains provide a lot of advantages and outperform its probabilistic counterpart when they are applied to task allocation problems. This section summarizes the main theoretical concepts of possibilistic (fuzzy) Markov chains.

Following [15,16] we can define a possibility Markov (memoryless) process as follows: let $S = \{s_1, \ldots, s_m\}$ ($m \in \mathbb{N}$) denote a finite set of states. If the system is in the state s_i at time τ ($\tau \in \mathbb{N}$), then the system will move to the state s_j with possibility p_{ij} at time $\tau + 1$. Let $x(\tau) = (x_1(\tau), ..., x_m(\tau))$ be a fuzzy state set, where $x_i(\tau)$ is defined as the possibility that the state s_i will occur at time τ for all $i = 1, \ldots, m$. Notice that $\bigvee_{i=1}^{m} x_i(\tau) \leq 1$ where $\vee$ stands for the maximum operator on $[0, 1]$. In the light of the preceding facts, the evolution of the fuzzy Markov chain in time is given by

$$x_i(\tau) = \bigvee_{j=1}^{m} p_{ji} \wedge x_j(\tau - 1),$$

where $\wedge$ stands for the minimum operator on $[0, 1]$. The preceding expression admits a matrix formulated as follows:

$$x(\tau) = x(\tau - 1) \circ M = x(0) \circ M^{\tau}, \tag{8}$$

where $M = \{p_{ij}\}_{i,j=1}^{m}$ is the fuzzy transition matrix, $\circ$ is the matrix product in the max-min algebra $([0, 1], \vee, \wedge)$ and $x(\tau) = (x_1(\tau), \ldots, x_m(\tau))$ is the possibility distribution at time τ.

Taking into account the preceding matrix notation and following [15], a possibility distribution $x(\tau)$ of the system states at time τ is said to be stationary, or stable, whenever $x(\tau) = x(\tau) \circ M$. During the experiments, explained in Section 7, each state will be a task to execute and, therefore, m will stand for the number of tasks.

One of the main advantages of the possibilistic Markov chains with respect to their probabilistic counterpart is given by the fact that under certain conditions, provided in [17] by J. Duan, the system converges to a stationary state in at most $m - 1$ steps.

6. Possibilsitic Multi-robot Task Allocation

In this section we will see how to use possibilistic Markov chains for developing a RTM in order to allocate a set of robots to tasks using the aforementioned indistinguishability operators (see (1) and (7)). Although the implementation proposed in this section only considers robots, it can be easily extended to more generic multi-agent scenarios.

Formally, the problem to solve could be defined as follows: Let $l, m \in \mathbb{N}$. Denote by R the set of robots with $R = \{r_1, ..., r_l\}$ and by T the set of tasks to carry out with $T = \{t_1, ..., t_m\}$. Both, tasks and robots are placed in an environment.

According to the classical RTMs (see Section 2), for each robot r_k and for each task t_j, a stimulus $s_{r_k, t_j} \in \mathbb{R}$ that represents how suitable t_j is for r_k is defined. Besides, a threshold value θ is assigned to each robot r_k. Thus, a robot r_k, allocated at task t_i, will select a task t_j to execute with a possibility $E_{T^{\frac{1}{n}}_{Dom}}(t_i, t_j)$ according to a fuzzy Markov decision chain. In the following, the stimulus of each robot r_k to transit from task t_i to task t_j only depends on the distance between the tasks which will be denoted by $d(t_i, t_j)$. So, the stimulus of each robot r_k to transit from t_i to task t_j is given as follows:

$$s_{t_i, t_j} = \begin{cases} \frac{1}{d(t_i, t_j)} & \text{if } d(t_i, t_j) \neq 0 \\ \infty & \text{if } d(r_k, t_j) = 0 \end{cases}. \tag{9}$$

This stimulus s_{r_k,t_j} allows us to obtain, by means of the indistinguishability operator (5), the following possibilistic transition function (response function),

$$p_{ij} = E_{T_{Dom}^{\frac{1}{n}}}(t_i, t_j) = \frac{\theta^n}{\theta^n + d(t_i, t_j)^n}. \tag{10}$$

If the same stimulus (distance) is applied to the indistinguishability operator (7), then the following possibilistic transition function (response function), is obtained:

$$p_{ij} = E_{T_{AA}^{\frac{1}{n}}}(t_i, t_j) = e^{-\frac{d(t_i,t_j)^n}{\theta^n}}. \tag{11}$$

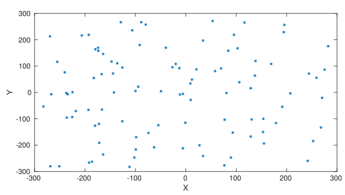
From now on, we will reference the response function given by (11) as Exponential Possibility Response Function (EPRF for short) and the response function given by (10) as Original Possibility Response Function (OPRF for short). Furthermore, as the transition functions EPRF and OPRF are indistinguishability operators, we will use both terms, transition possibility and indistinguishability operator equally.

As was proved in [6] (see also [4]), when either response functions (or indistinguishability operators), (10) or (11), are used as a possibility transition, the obtained fuzzy Markov chain holds the Duan's convergence requirements. Therefore, we can ensure that the system converges to a stationary state in at most $m - 1$ steps. It must be recall that, in general, the convergence of the probabilistic Markov chains is only guaranteed asymptotically.

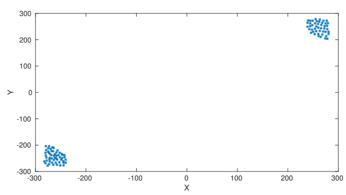
7. Experimental Results

In this section we will show the experiments carried out to compare the number of steps required to converge to a stationary state using probabilistic and possibilistic Markov chains induced from the indistinguishability operators given in (5) and (7).

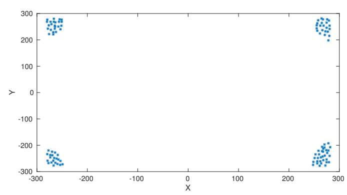
The robots must perform the task according to the stimulus defined in Section 6 under different configurations of the system: different position of the objects, parameters of the possibility response functions (θ and the power n) and number of tasks. All the experiments have been carried out using MATLAB with different synthetic environments. Figure 1 shows an example of the 3 types of environment used during the experiments depending on the position of the tasks: randomly placed (Figure 1(a)), tasks grouped into 2 clusters (Figure 1(b)) and grouped into 4 clusters (Figure 1(c)). This section extends the previous work given in [6] in order to considering clustered tasks.



(a) Tasks placed randomly.



(b) Tasks arranged into 2 clusters.



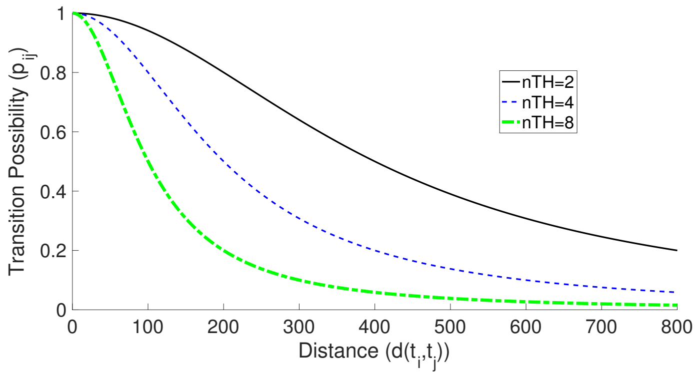
(c) Tasks arranged into 4 clusters.

Figure 1. Environments with 100 tasks used for the experiments. Blue dots represent the position of the tasks or objects.

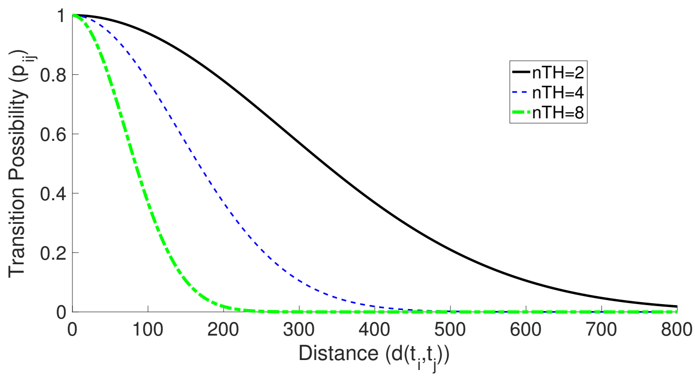
As pointed out in Section 2, the threshold value θ must depend on the position of the tasks. During the performed experiments the θ will depend on the maximum distance between tasks as follows:

$$\theta = \frac{d_{max}}{nTH}, \tag{12}$$

where d_{max} is the maximum distance between two objects and nTH is a parameter of the system. In our simulations d_{max} is constant and equals to 800.5 units. In order to see the impact of the parameter nTH on the transition possibility (p_{ij}) from a task t_i to the task t_j, Figure 2 shows the values of p_{ij} using the indistinguishability operators OPRF (Figure 2(a)) and EPRF (Figure 2(b)) with $nTH = 2, 4, 8$ and the power value $n = 2$. It It should be noted that, if the distance is equal to 0 ($d(t_i, t_j) = 0$) then $t_j = t_i$ and $p_{ij} = pii$ is the possibility of remain in the current task.



(a) p_{ij} with the the indistinguishability operator OPRF.



(b) p_{ij} with the the indistinguishability operator EPRF.

Figure 2. Transition possibility p_{ij} with $nTH = 2, 4, 8$ and power value $n = 2$.

Whichever possibility response function is used, (10) or (11), the possibilistic transition matrix for each robot, M, must be transformed into a probabilistic matrix for the possibilistic and probabilistic Markov chains results to be comparable. To make this conversion we use the transformation proposed in [18], where each element of M is normalized (divided by the sum of all the elements in its row) meeting the conditions of a probability distribution.

7.1. Experiments with randomly placed objects

This section focuses on experiments with tasks placed randomly, as can be seen in Figure 1(a). All the experiments have been performed with 500 different environments, with different number of tasks

($m = 50, 100$) and different values of the power n in the expression of the indistinguishability operators (10) and (11). The threshold θ values under consideration are obtained from (12) setting $nTH = 2, 4, 8$.

In [6] it was shown that OPRF and EPRF in these randomly generated environments needed the same number of steps to converge. This number of steps does not depend on neither θ nor the power value n. These simulations also show that only a 50% of the 500 environments could converge to a stationary state when probabilistic Markov chains are used. In contrast, with fuzzy Markov chains all the experiments converge and they required less than 25 steps. This shows that fuzzy Markov chains with indistinguishability operators always outperform their probability counterpart. Figure 3 shows the mentioned percentage of experiments that, using probabilistic Markov chains, do converge with 100 randomly placed tasks.

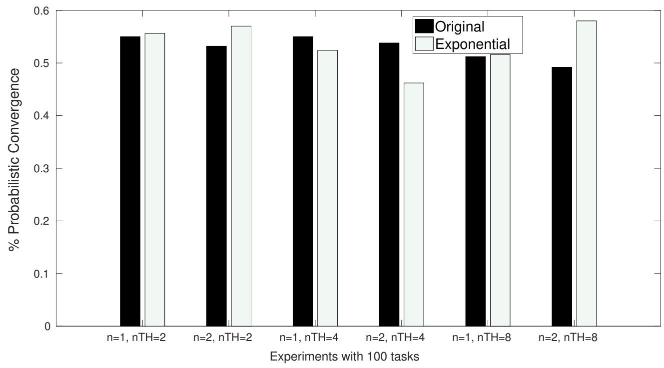


Figure 3. Percentage of experiments that converge with 100 tasks using probabilistic Markov process.

7.2. Clustered environments

This section shows the results obtained using environments with the tasks arranged in the groups or clusters, shown in Figures 1(b) and 1(c). As occurred in Section 7.1 the obtained results are very similar whichever indistinguishability operator is applied, OPRF or EPRF. Therefore, even if the tasks are arranged into clusters, both indistinguishability operators present an equivalent behavior and they are not affected by its parameters.

Figure 4 shows the number of iterations required to converge with fuzzy Markov chains with 2 and 4 clusters of tasks and different number of tasks ($m = 20, 40, 60, 80, 100, 120$). As can be observed, the number of clusters have a great impact on the system. For all cases, the environment with 4 groups needs a lower number of iterations to converge compared to the environment with 2 clusters. From these results, we can see that, with fuzzy Markov chains, the number of steps to converge to stable state depends only on the placement of tasks and not on the parameters of possibility transition function. Recall that the number of iterations required to converge is the same whichever indistinguishability operators is under consideration.

Figure 5 shows the number of steps required to converge with several number of tasks ($m = 20, 40, 60, 80, 100, 120$), a single environment with 2 clusters of tasks, $nTH = 2$ and probabilistic Markov chains. When the number of iterations is equal to 500 means that the chain does not converge. Figure 5(a) shows this number of steps with the indistinguishability operator OPRF and Figure 5(b) with the indistinguishability operator EPRF, when the evolution of the process is modelled as a probbilistic Markov chain. These results show that the value n value has a great impact on the results and that, in general, the exponential transition requires a greater number of steps to converge compared to its original counterpart. Furthermore, the indistinguishability operator has a great impact on the number of steps required to converge when probabilistic Markov chains are considered. It must be recalled that with possibilistic MArkov chains both indistinguishability operators provide very similar results and

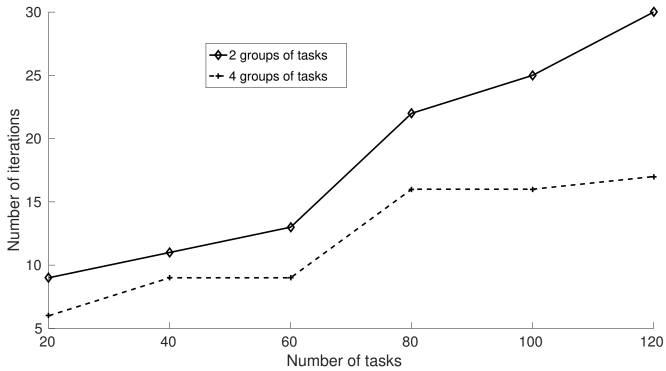


Figure 4. Number of iteration required to converge with fuzzy Markov chains for environments with 2 and 4 cluster (groups) of tasks.

it must be stressed that with possibilistic Markov chains, in general, the convergence is not guaranteed in a finite amount of steps.

8. Conclusion and Further Work

We have shown that the two most famous response functions given in literature, are retrieved as a particular cases from appropriate indistinguishability operators. This fact opens a wide range of potential applications from a mixed framework based on indistinguishability operators and distances to task allocation problems in multi-agent systems. We have applied the mentioned indistinguishability operators to allocate tasks to a group of robots according to a fuzzy Markov chain. We have shown that the results are very similar whichever indistinguishability operator is applied and, thus, that both present an equivalent behaviour. The simulations extend the results previously obtained in [6] to analyse environments where the tasks are arranged in groups or clusters. The results show that the number of iteration to converge with fuzzy Markov chains only depend on the placement of tasks in the environment and that they are not affected by the reminder of parameters of the system, whichever indistinguishability operator is applied. In contrast, when probabilistic Markov chains were used, this number of steps also depends on the indistinguishability operator. The theoretical and empirical obined results in this paper open a wide range of potential applications from a mixed framework based on indistinguishability operators and distances to task allocation problems in multi-agent systems when fuzzy Markov chains are under consideration.

We plan to propose several families of indistinguishability operators to perform a large number of experiments in order to compare the new results with those provided by the task allocation methods that implement the two aforesaid response functions. An implementation of these methods on real robots is also under consideration.

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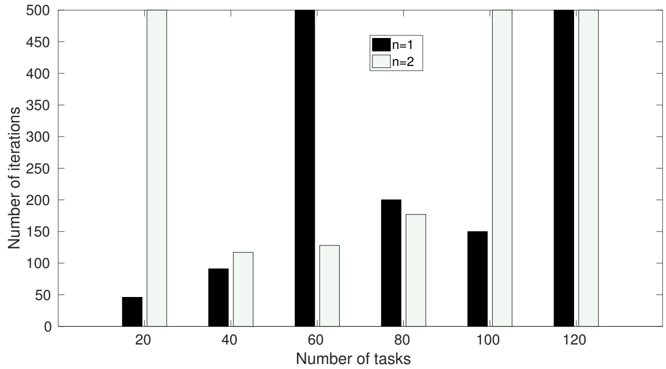
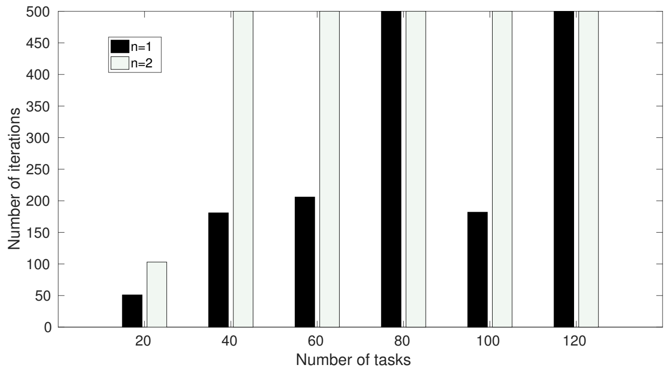
Abbreviations

The following abbreviations are used in this manuscript:

EPRF	Exponential Possibility Response Function
OPRF	Original Possibility Response Function
RTM	Response Threshold Method



(a) Results obtained from OPRF indistinguishability operator.



(b) Results obtained from EPRF indistinguishability operator.

Figure 5. Number of iteration required to converge with probabilistic Markov chain with different values of n power ($n = 1, 2$), $nTH = 2$, several number of tasks and 2 clusters of tasks. 500 iterations means no convergence.

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