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Modelos de campos escalares para la inflación cosmológica.

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Paraules clau del treball:
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Contents

1 Introduction

1.1 Objectives.

The aims of this TFG are: understanding a new scenario for the very early Universe, the study of different models for this scenario and checking how these models adjust to the current experimental data. This topic deals not only with general relativity concepts, but also with particle physics and quantum field theory ideas in order to build a theory able to explain how the universe evolved during an inflationary epoch. Then, it will be calculated analytical expressions and values for some specific simple models, and they will be compared with experimental measurements of this quantities.

1.2 Basic concepts of General Relativity

Cosmology is the field of physics which studies the past origin, the present and future evolution of the universe. Considering the four known fundamental interactions of nature (Electromagnetic, Strong, Weak and Gravitational), gravity is the one who dominates at long distances and, therefore, is the main interaction at large scales in the universe. Thus, we will start reviewing the principal concepts of the current gravitational theory.

The fundamental equations of Newton's theory of gravity are:

$$m_I \vec{\mathbf{a}} = -m_g \nabla \Phi, \tag{1}$$

$$\Delta \Phi = 4\pi G \rho, \tag{2}$$

where m_I and m_g are the inertial and gravitational mass, respectively, G is the gravitational constant, Δ is the Laplace operator and Φ is the gravitational potential.

At the beginning of the last century, Einstein realized that the set of equations (1) and (2) are incompatible with special relativity. This motivated him to research a new theory of gravity, which he presented in 1916 as the general theory of gravity[1], commonly called General Relativity (GR) theory. This theory is based on the principles of special relativity and the equivalence principle. The equivalence principle asserts that all freely falling bodies experience the same acceleration in a gravitational field, the main idea behind this fact is that gravity is universal, it affects all particles in the same way. In other words, the inertial and gravitational masses must be the same for all freely falling bodies.

Einstein[2] soon understood that with these assumptions gravity could not continue being treated as a classical force, but like a property of spacetime configuration. Thus, gravity is a consequence of the geometry of spacetime and it has to be described in the language of differential geometry. The next sections provide a brief introduction to differential geometry in order to better comprehend GR theory.

1.2.1 Differential geometry and the Einstein equation

The basic tool to describe curved spaces is the metric tensor($g_{\mu\nu}$), which is a two times covariant tensor which permits to generalize the idea of distance and from which all geometric properties of spacetime can be derived. The path length of an infinitesimal squared displacement is called the line element and it is defined as:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{3}$$

[1] If one is interested, one can look at the original paper [7].
[2] For more historical details, look [6].

The next step is to generalize the concept of curvature[3]. Curvature can be understood through the concept of "connection" which defines "parallel transport" of vectors, which is a way to relate vectors from different tangents spaces of nearby points. This connection in GR is called Christoffel symbol, and it is related to the metric by:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}). \tag{4}$$

The Christoffel symbol is not a tensor because it does not follow the transformation law of a tensor. This connection permits to define a generalization of the partial derivative, the covariant derivative of a vector field V^ν is:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^{\nu}_{\mu\sigma} V^\sigma. \tag{5}$$

This mathematical formalism permits GR theory to describe the paths followed by freely falling particles, the geodesics, which are a generalization of the concept of straight lines in a curved geometry. The parameterized curve $x^\nu(\lambda)$ is a geodesic if it satisfies:

$$\frac{d^2 x^\nu}{d\lambda^2} + \Gamma^{\nu}_{\mu\sigma} \frac{dx^\mu}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0, \tag{6}$$

which is known as the geodesic equation and it is a generalization of equation (1) with $\mathbf{a} = \mathbf{0}$ or which is the same $\nabla\Phi$=0.

The Riemann tensor is the mathematical object that describes spacetime curvature. The Riemann curvature tensor is related with the connection and the metric by

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda} \Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda} \Gamma^{\lambda}_{\mu\sigma}. \tag{7}$$

From (7) one can derive the Ricci tensor, the Ricci scalar (or curvature scalar) and the Einstein tensor; which are given by:

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}, \tag{8}$$

$$R = R^{\mu}_{\mu} = g^{\mu\nu} R_{\mu\nu}, \tag{9}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}. \tag{10}$$

This set of equations describe completely the curvature of spacetime. However, describing gravitational interaction requires a generalization of Newton's law of gravity (2), in other words, it is necessary to find an equation which relates the geometry of the spacetime with its matter distribution, this matter configuration is characterized by the energy momentum tensor[4] ($T_{\mu\nu}$). This is the Einstein equation which is given by:

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = 8\pi G T_{\mu\nu}, \tag{11}$$

where the second term in the left hand side of (11) is the metric multiplied by the cosmological constant (Λ), which is a new fundamental constant and it is supposed to have a small value nowadays. It is chosen for simplicity to write all past and future equations in natural units where $\hbar = c = 1$.

Looking at equations (11) and (2) some parallelisms can be seen, for example, the right hand side of both represent the distribution of matter. The major difference comes from substituting the classical gravitational potential by a tensorial quantity which depends directly on the geometry of the spacetime.

[3] For more details, look [1], or [2].

[4] More about the properties of the energy momentum tensor will be explained in next sections.

1.3 Description of the evolution of the universe

General relativity can be used to describe the evolution and structure of the universe. On sufficiently large scales the universe seems to be homogeneous and isotropic, this is the cosmological principle. The condition of homogeneity and isotropy permits to define a line element given by:

$$ds^2 = -d\tau^2 + a^2(\tau)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{12}$$

This is the Friedmann-Robertson-Walker (FRW) line element, where τ is the proper time; r,θ,ϕ are polar comoving coordinates, $a(t)$ is the scale factor, which measures the size of the universe, and the constant k indicates the spatial curvature, with its possible values: +1 for spherical geometry, 0 for flat geometry, and -1 for hyperboloid geometry.

It is important to note some features about this metric. Homogeneity and isotropy of the FRW model of the universe imply that the scale factor must depend only on time. The coordinate r is a dimensionless comoving coordinate with the motion of the universe, note that this is important because the metric depends on the coordinate frame of the observer and, of course, the universe does not look homogeneous and isotropic for all the observers; it only looks so for a special group of observers: the comoving observers, which follow the motion of the universe[5].

Proper time is the time which would mark the clock of a comoving observer, and mathematically, for a Lorentzian metric (-,+,+,+,...,+) is given by:

$$\tau = \int \sqrt{-g_{\mu\nu}u^\mu u^\nu}dt, \tag{13}$$

where $u^\mu = \frac{dx^\mu}{dt}$ represents the four-velocity.

The matter distribution of the universe is the other component necessary to solve the Einstein equation. If it is supposed that the majority of the ordinary energy-mass of the universe is concentrated in galaxies, that galaxies look like grains of dust on cosmic scales and that velocities of galaxies are small so that the pressure of the dust is negligible; then can be taken the energy momentum tensor to be of the perfect fluid to a good approximation. The general perfect fluid form for $T^{\mu\nu}$ is:

$$T^{\mu\nu} = \rho u^\mu u^\nu + P(g^{\mu\nu} + u^\mu u^\nu), \tag{14}$$

where ρ is the density, P is the pressure and $g^{\mu\nu}$ is the inverse of the metric.
Now, if comoving coordinates are chosen, then the fluid will be at rest with respect to this frame and $T_{\mu\nu} = diag(\rho, -P, -P, -P)$. Then substituting this expression for $T_{\mu\nu}$ in the Einstein equation, calculating the Einstein tensor from the metric and ignoring the cosmological constant term, one obtains[6]:

$$H^2 = \frac{8\pi}{3m_P^2}\rho - \frac{k}{a^2}, \tag{15}$$

$$\dot{\rho} + 3H(\rho + P) = 0. \tag{16}$$

The overdot means time derivative, and H is the Hubble parameter defined as
$H = \dot{a}(t)/a(t)$ and m_P is the Planck mass[7]. These are the Friedmann and fluid continuity

[5]This comoving frame would be used later to make calculations easier.
[6]For more details in the calculations see [1], [2] or [3]
[7]See its definition in section 1.3.2.

equations, respectively, and they are the basic equations describing the dynamics of the universe. They can be combined to form the so called acceleration equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3m_{Pl}^2}(\rho + 3P). \tag{17}$$

From these equations it is observed that the universe cannot be static for normal matter since that $\rho > 0$ and $P \geq 0$, thus equation (17) implies that $\ddot{a} < 0$. This is the normal behaviour of the scale factor in our current universe, however, as will be seen in next sections, this is not the case at very early times of the universe, concretely, during inflation[8] it is assumed an accelerated expansion with $\ddot{a} > 0$.

Cosmological redshift data confirm that the universe is currently expanding ($\dot{a} > 0$). This phenomenon is based on the fact that the light which travels from other galaxies to our own, suffers a stretching of its wavelength because the scale factor, which measures the physical distance between galaxies, is increasing with time. This longer wavelength produces a spectrum with its spectral lines shifted toward the red.

In the FRW model of the universe, there are special cases for the dynamics of the universe depending on the matter content of it. They are the radiation-dominated, matter dominated and vacuum-dominated models. For each one of these, there is a different equation of state according to their single properties, which are summarized in Table 1.

Cosmological models	**Equation of state**	**Energy conservation**	**Scale factor**
Matter-dominated	$P_M = 0$	$\rho_M \times a^3 = constant$	$a(t) \sim t^{2/3}$
Radiation-dominated	$P_R = \frac{\rho_R}{3}$	$\rho_R \times a^4 = constant$	$a(t) \sim t^{1/2}$
Vacuum-dominated	$P_\Lambda = -\rho_\Lambda$	$\rho_\Lambda = constant$	$a(t) \sim e^{Ht}$

Table 1 – Different filled universe models.

These different models for the cosmological fluid can be applied to describe different stages of the universe. At very early times of the universe radiation was the dominant form of matter-energy; however, since the **recombination era** until now, matter has become the dominant material of the Universe. Furthermore, it is expected that in the future the universe will be vacuum-dominated due to the fact that the universe is expanding and creating more and more spacetime; thus, there is more and more vacuum.

One usually introduces the critical density, as the density which the universe would have if it was exactly flat (k=0); from (15) we have:

$$\rho_c(t) = \frac{3m_P^2 H^2}{8\pi}. \tag{18}$$

It is also useful to define the dimensionless density parameter Ω as:

$$\Omega = \frac{\rho}{\rho_c}. \tag{19}$$

An analysis of the dynamical equations leads to the conclusion that the universe must have been expanding faster in the past than it is nowadays. If the expansion rate would have always been the same, then at the time $T = \dot{a}/a = H^{-1}$ the scale factor would have been zero. However, if

[8] More details on inflation will be explained later. Now it is important to clarify that the scale factor could not have had always the same behaviour during the evolution of the universe.

the expansion was faster, the time at which $a(t)$ was zero would be even closer to the present time. Assuming homogeneity and isotropy, GR predicts that at some time in the past the universe was in a singular state at which a=0. This event is referred to as the Big bang.

The big bang could be interpreted as an extremely dense and hot state of the universe, and it is no sense in asking physically what happened before the Big bang. GR theory predicts the beginning of the universe at the Big bang. Some key events after Big bang can be summarized as follow[9]:

- 10^{-12}s – 3min . **Particle era**[10]: this epoch begins once the electroweak interaction has been decoupled, the universe contains leptons and quarks, which condense into baryons.
- 3 – 20min. **Nucleosynthesis**: universe is cool enough so that nuclei can be formed. At that point, basically the most light elements nuclei of the present universe are formed.
- 10^4 years. Matter density overcomes radiation density and becomes the energy density dominating the universe.
- 10^5 years. **Recombination/Decoupling era**: the electrons combine with protons. Recombination is important because when photons interact with free electrons, they increase vastly their mean free path and the universe becomes effectively transparent for photons. These photons form the Cosmic Microwave Background Radiation (CMBR), which is released during this epoch.
- 10^{10} years. The **present**.

1.3.1 Particle horizons

The main idea behind the concept of particle horizons is that the light emitted from some point of the spacetime cannot reach all the points of the spacetime, because the light can only travel a finite distance since it was emitted. Physically, this idea can be understood in Figure 1, where a spacetime diagram is shown. The particle horizon distance is the distance that a photon, which follows null geodesics ($ds^2 = 0$), could have reached in a time τ_0 since it was emitted.

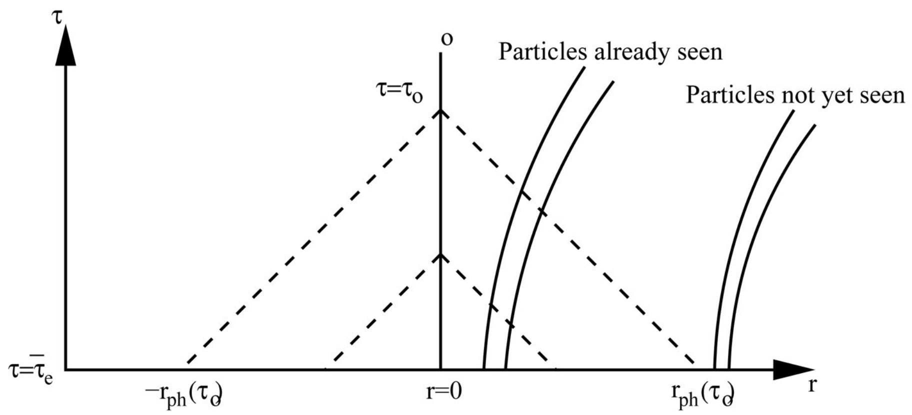

Fig. 1 – Spacetime diagram. Dashed lines inclined ± 45º represent the light cone surface which separates events causally disconnected from the observer O. In this figure, r_{ph} represents the particle horizon radius. This picture is taken from *http : //ned.ipac.caltech.edu/level5/Sept03/Trodden/Trodden2_5.html*.

From Figure 1 one can see that not all comoving observers are visible for the observer O.

[9]These references of time can change according to different authors, which follow determined criteria put the names of the distinct epochs. Here we have follow the criteria of [13].

[10]This name refers to a period of the universe which contains the hadron and lepton epoch together.

Mathematically particle horizons can be calculated using (12) and the fact that photons travel along null paths ($ds^2 = 0$), for light emitted at a time t_0 which arrives at a point at time t, as:

$$R_H(t) = a(t) \int_{t_0}^{t} \frac{dt'}{a(t')}. \tag{20}$$

The particle horizon radius R_H can be also computed for photons, using (12), assuming a radial trajectory and flat space(k=0), one obtains:

$$R_H(t) = a(t) \int_{r_0}^{r} (1 - kr^2)^{-1/2} dr = a(t)r. \tag{21}$$

This distance represents the horizon distance and defines a boundary between causally connected regions of space, points separated a distance greater than the given by (21) are causally disconnected.

Now, we can compute the particle horizon assuming $a(t) \propto t^n$ with $n < 1$, using equation (20) and we obtain:

$$R_H(t) \propto t^n \int_0^t \frac{dt'}{(t')^n} = \frac{n}{1-n} H^{-1} \sim H^{-1}, \tag{22}$$

where the symbol "$\sim$" indicates that irrelevant numerical factors has been ignored.

1.3.2 Limitations of the Big bang model

It can be thought that the big bang is a consequence of assuming homogeneity and isotropy, however, it can be proved that singularities are general characteristics of cosmological models[11]. Nevertheless, GR theory cannot be assumed valid at close time of the Big bang singularity, when quantum gravity effects are supposed to be relevant. The scale at which quantum gravity effects are important is defined by the Planck scale, which can be estimated from a combination of Planck constant, Newton constant and the speed of light. These are: the Planck mass (m_P), the Planck length (l_P), the Planck time (t_P) and the Planck energy (E_P).

$$m_P = \left(\frac{\hbar c}{G}\right)^{1/2} = 2.18 \times 10^{-8} kg, \tag{23}$$

$$l_P = \left(\frac{\hbar G}{c^3}\right)^{1/2} = 1.62 \times 10^{-36} m, \tag{24}$$

$$t_P = \left(\frac{\hbar G}{c^5}\right)^{1/2} = 5.39 \times 10^{-44} s, \tag{25}$$

$$E_P = \left(\frac{\hbar c^5}{G}\right)^{1/2} = 1.9544 \times 10^{9} J, \tag{26}$$

In natural units, all four quantities have the relation $m_P = l_P^{-1} = t_P^{-1} = E_P = 1.22 \times 10^{19} GeV$. In conclusion, quantum gravity is expected to become important when particles have a mass greater than m_P, or when dealing with times shorter than t_P, lengths smaller than l_P or energies higher than E_P.

[11] For detailed explanation see Chapter 9 of [1].

1.4 Successes and Problems of the Big bang model

1.4.1 Successes of Big bang model

The main idea of the Big bang model is the prediction that in the past all matter in the universe was in an extremely hot and dense state, afterwards the Universe began to expand and cool down. This model has succeeded making predictions like:

1. The expansion of the Universe.
2. The existence of the Cosmic Microwave Background (CMB).
3. The synthesis of light elements (**Nucleosynthesis**). The big bang model accounts properly for the relative abundances of light elements in the universe.
4. The age of the Universe, which is compatible with experimental evidences.

These are the most important successes of the big bang model. However, there are some questions or problems which that model cannot answer by its own.

1.4.2 Problems of the Big bang model

Flatness Problem

This problem pertains to one type of problems of the Big Bang model known as fine-tuning problems. It is a problem related to the necessity of adjusting at high precision the density parameter (Ω) at early times in order to have a value for Ω according to the present value. Experimental data are consistent with a nearly flat universe[12], and with $\Omega \simeq 1$. Equation (15) can be rewritten in terms of Ω in order to calculate its time evolution:

$$\mid \Omega(t) - 1 \mid = \frac{\mid k \mid}{a^2 H^2}. \tag{27}$$

From (27) it is important to note that during the expansion a^2H^2 decreases if one assumes a decelerating expansion ($\ddot{a} < 0$), therefore Ω increasingly deviates from one. Furthermore, we can take results for $a(t)$ from Table 1 for the matter- and radiation-dominated universe, the definition of H and equation (27) to compute Ω. The results of these calculations are shown in Table 2.

	$\mid \Omega(t) - 1 \mid$
Matter-dominated	$\sim t^{2/3}$
Radiation-dominated	$\sim t$

Table 2 – Density parameter at different stages of the universe.

These relations show that the value of Ω must be highly fine-tuned at early epochs to reproduce the flatness of the current observed universe. The big bang model does not offer a natural mechanism to explain why the Universe appears to be so flat.

Horizon Problem

The CMBR over all the sky seems to have the same spectrum as a black body with temperature variations[13] of 10^{-5}. These small variations indicate that the CMBR is nearly in a state of thermal equilibrium. However, it is not possible for the microwave photons from opposite directions

[12] For more details look at [8].
[13] For more details on the CMBR look at Chapter 6 from [3].

of the sky to be in causal contact with each other because light has not had enough time since the big bang to travel to such far regions.
One natural explanation would be that the different regions of the sky have interacted in order to get this thermal equilibrium. However, the big bang model discards this interpretation, in order to understand why; it is necessary to remember the horizon distance given by equation (20). In the case of CMBR, the distance before the releasing of microwave radiation is much smaller than the present horizon distance, this means:

$$\int_{t_*}^{t_{dec}} \frac{dt}{a(t)} \ll \int_{t_0}^{t_{dec}} \frac{dt}{a(t)}, \tag{28}$$

where t_0, t_*, t_{dec} are the present time, a time close to the big bang and the time of the beginning of the decoupling era, respectively.
In addition, the horizon size at the decoupling era corresponds nowadays to a distance in the sky of no more than 2°, so that they were causally disconnected at the time of decoupling[14]. Hence, the fact that the big bang model cannot explain the high degree of homogeneity of the universe is one of the main drawbacks of the model.

Monopole Problem

Generally speaking, the effect of a phase transition depends on the termodynamic properties of the system considered. In the cosmological context, phase transitions are typically modelled by a scalar field ϕ, which in quantum field theory represents spin-0 bosons; and its potential $V(\phi)$, which determines the temporal dependence of ϕ. These phase transitions can be associated with the cooling of the universe and lead to different configurations of the scalar field, depending on the symmetry being broken, called topological defects.
Grand Unified Theories (GUT's) predict the creation of topological defects arising from the symmetry breaking at early times in the Universe. Some of these relics are:

1. Domain Walls
   The symmetry consists in only discrete states. This defect contains two connected regions of distinct phases separated by walls with certain energy.

2. Strings
   These are linear defects, where the "phase" of ϕ changes by multiples of 2π around the string. They are described by some energy per unit length. They could explain the large-scale structure of the universe; however, they are disfavoured by the lack of experimental observations.

3. Monopoles
   These are point defects with a characteristic mass, where the scalar field points radially away from the defect. They can have a magnetic configuration and, hence, they can be analogous to the magnetic monopoles.

All these topological defects are supposed to be created at very early times of the universe and diluted by the cosmological expansion. The big bang model permits the creation of magnetic monopoles in some symmetry breakdowns, such as the electroweak breakdown, this production of monopoles would become them the dominant material of the Universe[15].

However, magnetic monopoles have never been observed. Thus, this leads to a contradiction between theory and observations. It is thought that if the concentration of defects was relevant it would have had direct effects in, for example, the curvature of spacetime, the galaxy formation or the value of the Hubble parameter.

[14] For more details on the angular distance of the CMBR look at [8].
[15] For more details on topological defects look Chapter 10 from [4]

2 Introduction to Inflation

In the last section, some crucial problems of the big bang model have been outlined. In order to solve them, Guth and others introduced at the beginning of the 80's the idea of inflation[16]. The main idea behind inflation is that at very early times of the universe, there was a moment when the universe suffered a large accelerated expansion. This period of a great accelerated expansion is expected to last from 10^{-35} seconds to 10^{-33} or 10^{-32} seconds approximately[17], after the big bang. Inflation has to be seen not like a concrete model[18], but like a scenario with the possibility of choosing different models to explain the inflationary period. As it will be discussed in sections 2.1, 2.2 and 2.3, this accelerated expansion can solve in a natural way the cosmological problems exposed before.

2.1 Inflation as a solution to the Big bang problems

Inflation can be defined as a period of accelerated expansion, where $\ddot{a} > 0$. From equations (15) and (16) it is easy to check, assuming k=0, that the previous condition implies $p < -\rho/3$. Another consequence of the first condition is that:

$$\frac{d}{dt}\left(\frac{1}{aH}\right) = -\frac{\ddot{a}}{(\dot{a})^2} < 0. \tag{29}$$

Equation (29) affirms that the characteristic length of the universe, measured in comoving coordinates, decreases during inflation. This could seem a contradiction, however, what is happening is that although the universe expands very fast, its characteristic length or scale with respect to the expansion is becoming smaller. Therefore, inflation has to begin at very early times of the universe, last a short period of time and then come to an end followed by a conventional behaviour of the Universe. This new scenario does not contradict the big bang model, it just complements it, to solve some of its deficiencies.

2.1.1 Solution to the Flatness Problem

Historically, inflation arose from trying to solve this problem. Recalling equation (27), a decelerating expansion was a central problem in the big bang model since aH always decreases, and, therefore, Ω is shifted away from 1. Nevertheless, inflation ensures the opposite situation, this is to say that, during inflation the right hand side of equation(27) will decrease and therefore, Ω is driven to 1. This can be seen using equation (27) and calculating:

$$\frac{d}{dt}\left(\frac{1}{(aH)^2}\right) = \frac{d}{dt}\left(\dot{a}^{-2}\right) = -2\frac{\ddot{a}}{(\dot{a})^3} < 0, \tag{30}$$

where the conditions $\dot{a} > 0$ and $\ddot{a} > 0$ have been used.
Hence, inflation implies that the curvature term becomes very small and, thus, guarantees that the universe becomes effectively flat.

2.1.2 Solution to the Horizon Problem

Inflation provides two arguments in order to solve this problem. The first argument is more qualitative but conceptually very important. Combining equations (20) and (21) the particle horizon for a flat radiation-dominated universe can be written as:

$$H^{-1} = R_H = a(t)r. \tag{31}$$

[16]Some key articles of the beginning of inflation can be found in [5].

[17]This temporal references are not exact, there is no way to known exactly when happened, although it is known that it has to be after the Planck time and last a short period of time.

[18]For more reasons supporting the idea of inflation look at [11].

Isolating the comoving radius r from equation (31) and using equation (29), one obtains:

$$\frac{d}{dt}(r(t)) < 0. \tag{32}$$

This condition means that during inflation the comoving radius is decreasing. In the comoving frame, which is at rest with respect to the expansion, one observes that the particle horizon is shrinking. This fact is often visualized with the example of an expanding balloon like in the Figure 2 and Figure 3.

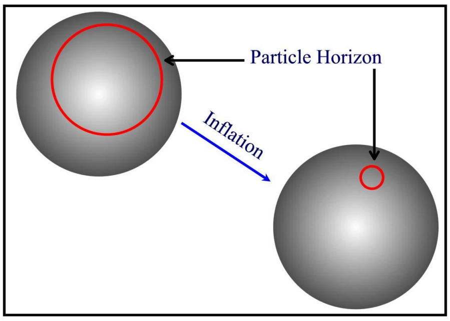

Fig. 2 – This figure shows the expansion in comoving coordinates. In this coordinates one observer do not realize that universe is expanding, although for him the particle horizon is seen to contract. Both figures have been taken from [11].

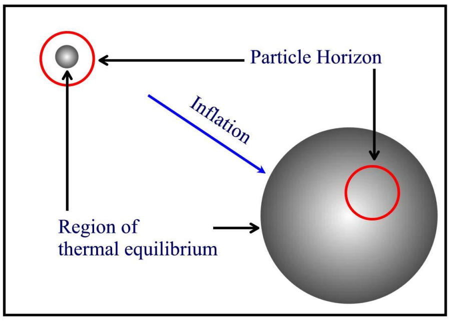

Fig. 3 – This figure shows that after inflation regions that have reached thermal equilibrium can be expanded outside the horizon. After inflation regions do not expand faster than the horizon and thus the horizon can "catch up" with them.

A comoving frame has the advantage that the horizon distance and the Hubble length remains approximately constant; and by definition the particle horizon must move at the speed of light. Besides, if the previous conditions imply that spacetime has to be larger than the particle horizon, then the spacetime background has to expand faster than the speed of light, consequently the expansion has to take place at super-luminal velocities. Once inflation ends, the spacetime returns to a subluminal rate of expansion so that the particle horizon can reach it. This fact does not contradict any statement of special relativity because what is expanding is spacetime, thus, no information is transmitted.

Quantitatively, the horizon problem would be solved if:

$$a(t_{dec}) \int_{t_{infl}}^{t_{dec}} \frac{dt}{a(t)} \gg a(t_0) \int_{t_{dec}}^{t_0} \frac{dt}{a(t)}, \tag{33}$$

where t_{infl} indicates the beginning of inflation, t_{dec} is the time of decoupling and t_0 is the present time.

Making some reasonable assumptions inequality (33) can be estimated numerically: first, the time differences between the integral limits are so big that the lower limits can be set to zero. Moreover, in the first integral, during inflation, as we will see in section 2.5, the scale factor can be approximated by $a(t) \sim e^{Ht}$. In the second integral, the scale factor can be assumed to be the matter-dominated of the Table 1, $a(t) \sim t^{2/3}$. If one puts $t_{dec} = \Delta t$, then one can write:

$$e^{H\Delta t} \int_0^{\Delta t} \frac{dt}{e^{H\Delta t}} \gg t_0^{2/3} \int_0^{t_0} \frac{dt}{t^{2/3}}. \tag{34}$$

Calculating the integrals we obtain:

$$H^{-1}\left(e^{H\Delta t} - 1\right) \gg 3t_0 = 2H^{-1}. \tag{35}$$

Inequality (35) holds if Δt has a determined value, in other words, if inflation lasts a Δt which satisfies the inequality. Therefore, with these two arguments inflation solves the horizon problem.

2.1.3 Solution to the Monopole Problem

Considering the solution to the flatness problem, the monopole problem can be solved by the same mechanism. The problem with monopoles were that the big bang conditions could led to the production of unwanted relics, which, however, have not been observed experimentally. This problem is solved because the production of monopoles is diluted by the accelerated expansion explained before[19].

Note that the previous explication holds provided that, at the end of the inflationary period, the energy density is not high enough so that thermal effects can recreate these relics. At the end of inflation the energy density dedicated to expand the Universe has to be transformed in conventional matter-energy density, this process of conversion of energy is known as reheating[20], and, as its name indicates, it supposes an increment of the temperature of the Universe after inflation.

2.2 Description of the Dynamics of Inflation

As seen in section 2.1, inflation requires an exotic equation of state ($P < -\rho/3$). The standard model known with an equation of state with negative pressure, looking at Table 1, is the vacuum-dominated universe model and it implies and exponential increment of the scale factor. Thus, vacuum or non conventional matter have to be present during the inflationary epoch.

During inflation, the energy density of the universe is assumed to be contained in a scalar field and in its potential. At the end of inflation, this scalar field releases its energy in order to reheat the universe and permit particle creation.

2.2.1 Scalar fields and field theory

In standard models, inflation is governed by a scalar field (ϕ) and its potential ($V(\phi)$).

Scalar fields are used to describe the spontaneous symmetry breaking of vacuum states in some systems and are supposed to represent spin-0 bosons, which are invariant under a change of coordinates. Therefore, during inflation the scalar field represents vacuum energy of the universe, and; at the end of it, is the source of latent heat for the reheating epoch. As in the case of the scale factor, the scalar field in an isotropic and homogeneous universe is a function only of time.

In field theory, the fundamental quantity is the Langragian density ($\mathcal{L}$), which is a generalization of the classical Langragian (L). The relation between Lagrangian and the $\mathcal{L}$ is:

$$L = \int d^4x\mathcal{L}. \tag{36}$$

In the case of a scalar field the Lagrangian density is given by[21]

$$\mathcal{L} = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi), \tag{37}$$

where ϕ is the scalar field, commonly called inflaton, $V(\phi)$ its potential[22]. The first term of the $\mathcal{L}$ represents the kinetic contribution of the scalar field, its kinetic energy; and the second one

[19]The explanation of the dilution of the concentration of unwanted relics is a little more complicated than the solution of the flatness problem. For more details look at [4] and [11].

[20]A complete understanding of reheating requires the introduction of scalar fields, which would be made in section 2.2. Furthermore, reheating could be a topic for a TFG in itself, that is why in this work, reheating is only going to be commented briefly. For more details on the reheating look at [14].

[21]For a complete explanation of this expression look at [3] or [4].

[22]Note that in (37) partial derivatives appear instead of covariant derivatives, but in this case is the same because ϕ is a scalar quantity

represents the potential energy of the scalar field.

This Lagrangian density appears in the action(S) as:

$$S = \int d^4x\sqrt{-g}\mathcal{L} = \int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\mu\phi - V(\phi)\right], \tag{38}$$

where g is the determinant of the metric. Applying the principle of least action to equation (38), one obtains the Euler-Lagrange equations[23] for the Lagrangian density as:

$$\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial\phi} - \frac{d}{dx^\mu}\left(\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial\left(\partial_\mu\phi\right)}\right) = 0. \tag{39}$$

Substituting $\mathcal{L}$ from equation (37) in equation (39) and after some calculations one obtains:

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(g^{\mu\nu}\sqrt{-g}\frac{\partial\phi}{\partial x^\nu}\right) + \frac{dV(\phi)}{d\phi} = 0. \tag{40}$$

This is the dynamical equation of the scalar field during inflation.

Using the field theory formalism, the Einstein equation can also be deduced, making a variation of the Einstein-Hilbert action[24]:

$$S_H = \int \sqrt{-g}R d^n x, \tag{41}$$

where R is the Ricci scalar and g is the determinant of the metric.
In the Einstein equation, the matter dependence of the gravitational interaction is represented by the energy momentum tensor. It can be proved that the $T^{\mu\nu}$ is a conserved quantity in a matter field theory. Energy momentum tensor conservation is mathematically expressed as

$$\nabla^\mu T_{\mu\nu} = 0, \tag{42}$$

in this equation ∇^μ means covariant derivative.
Using the fact that $T^{\mu\nu}$ is a conserved quantity and considering the Lagrangian density of equation (37); one can derive an expression for the energy momentum tensor ($T^{\mu\nu}$) given by[25]

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\mathcal{L}. \tag{43}$$

2.3 Dynamical equations of the inflaton

Before deriving the equations of motion for the scalar field, it is important to note that the homogeneity condition during inflation holds because the physical and comoving gradients are related by:

$$\nabla_{physical} = \frac{1}{a(t)}\nabla_{comoving}, \tag{44}$$

thus, all gradients can be neglected, as during inflation the scale factor undergoes an extreme growth in its value.

Using the FRW metric given by (12), and restricting it to the flat space case, the metric is: $g^{\mu\nu} = diag(-1, a^{-2}, a^{-2}, a^{-2})$. With this metric, the factor $\sqrt{-g}$ is $\sqrt{-(-a^6)} = a^3$, and then equation (40) can be written as:

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2\phi}{a^2} + \frac{dV(\phi)}{d\phi} = 0. \tag{45}$$

[23] For a more complete deduction look [3].
[24] See [3] for a more detailed explanation.
[25] The deduction of this expression for $T^{\mu\nu}$ is based on Noether's theorem. For more details look [3].

Using the fact that ϕ is only a function of time, the equation of motion for the inflaton is:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \tag{46}$$

where the overdot represents time derivatives and the prime ($'$) derivatives with respect to ϕ.

The $T^{\mu\nu}$ of a perfect fluid is given by equation (14), this equation can be written in comoving coordinates as: $T^{\mu\nu} = (\rho, P, P, P)$. Then, one can compare the $T^{\mu\nu}$ of a perfect fluid in comoving coordinates with the result of computing (43) for the different components. For the 00-component one obtains:

$$\rho = T^{00} = \partial^0\phi\partial^0\phi - g^{00}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right). \tag{47}$$

After substituting the metric and evaluating the derivatives, the final result is:

$$T^{00} = \rho = \frac{1}{2}(\dot{\phi})^2 + \frac{(\nabla\phi)^2}{2a^2} + V(\phi). \tag{48}$$

The same procedure can be followed to calculate an expression for the pressure considering that in this case P satisfies the relation:

$$P = \frac{a^4}{3}\left(T^{11} + T^{22} + T^{33}\right), \tag{49}$$

computing the components of the tensor using equation (43) one obtains:

$$P = \frac{1}{2}(\dot{\phi})^2 - \frac{(\nabla\phi)^2}{6a^2} - V(\phi). \tag{50}$$

If the gradients of equations (48) and (50) are neglected, then one obtains,

$$\rho = \frac{1}{2}(\dot{\phi})^2 + V(\phi), \tag{51}$$

$$P = \frac{1}{2}(\dot{\phi})^2 - V(\phi). \tag{52}$$

Now substituting equations (51) and (52) in the Friedmann, in the fluid continuity and in the acceleration equations (equations (15), (16) and (17) respectively) one obtains,

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \tag{53}$$

$$H^2 = \frac{8\pi}{3m_P^2}\left[V(\phi) + \frac{1}{2}(\dot{\phi})^2\right], \tag{54}$$

$$\dot{H} + H^2 = \frac{8\pi}{3m_P^2}\left[(\dot{\phi})^2 - V(\phi)\right]. \tag{55}$$

Equation (53) is the same expression as the equation (46), we have obtained them by two different ways. On the other hand, in obtaining equation (54) has been neglected the curvature term from (15) because during inflation it becomes rapidly negligible. Equations (53) and (54) are the basic equations of motion for the scalar field.

2.4 Slow-Roll Approximation

From the pressure and density equations (51) and (52), it can be seen that:

$$\text{if} \quad \dot{\phi}^2 \ll V(\phi) \quad \Rightarrow \quad P \simeq -\rho, \tag{56}$$

hence, one has the equation of state of a vacuum-dominated universe, and inflation can take place. The assumption that $\dot{\phi}^2 \ll V(\phi)$ significantly simplifies the equations of motion of the scalar field and is known as Slow-Roll Approximation (SRA). Thus, during the SRA period of inflation, the variations with respect to the time of the inflaton are negligible, as can be seen in Figure 4.

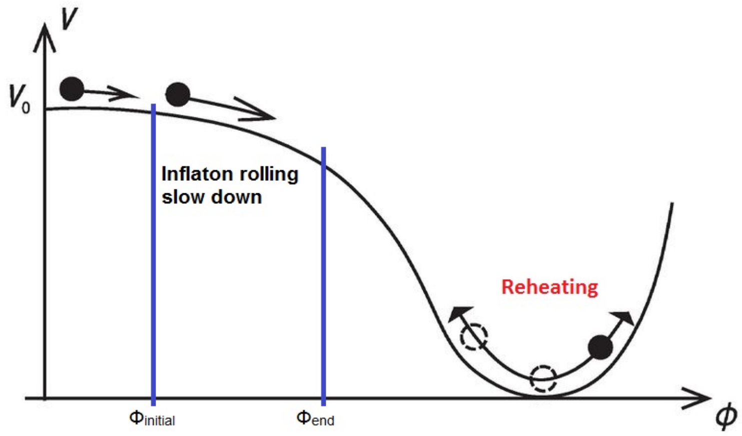

Fig. 4 – This figure illustrates a conventional behaviour of the scalar field with some potential. First, the inflaton begins to roll down slowly to a minimum and during this period the kinetic term is negligible. However, as long as the inflaton reach the minimum the $\ddot{\phi}$ term of the equation of motion becomes larger and larger and the inflaton field rolls down rapidly. When the inflaton reaches the minimum, the inflationary period ends and the inflaton oscillates around the minimum radiating its energy and reheating the universe.

From an equivalent point of view, if inflation take place ($\ddot{a} > 0$), then, from (17) we obtain the condition $P < -\frac{1}{3}\rho$, and then using equations (51) and (52) one obtains that the potential must dominate over the kinetic term:

$$\ddot{a} > 0 \quad \Rightarrow \quad P < -\frac{1}{3}\rho \quad \Rightarrow \quad (\dot{\phi})^2 \ll V(\phi). \tag{57}$$

Thus, using this approximation the equations of motion of the inflaton can be rewritten as:

$$3H\dot{\phi} = -V'(\phi), \tag{58}$$

$$H^2 = \frac{8\pi}{3m_P^2} V(\phi). \tag{59}$$

in equation (58) has also been neglected the $\ddot{\phi}$ term because this has to be small in order to ensure that $\dot{\phi}$ is also small. Then, provided a potential for the scalar field it can be defined the first two slow-roll parameters[26] as:

$$\epsilon_V(\phi) = \frac{m_P^2}{16\pi}\left(\frac{V'}{V}\right)^2, \tag{60}$$

$$\eta_V(\phi) = \frac{m_P^2}{8\pi}\left(\frac{V''}{V}\right). \tag{61}$$

[26]More precise is to say that these are the Potential Slow-Roll (PSR) parameters.

This two parameters allow us to know the validity of the SRA. If SRA holds then:

$$\epsilon_V \ll 1, \quad |\eta_V| \ll 1. \tag{62}$$

These definitions could seem arbitrary, however, there is a direct connection between the slow roll conditions (62) and the definition of inflation ($\ddot{a} > 0$), as will be shown next.
Taking equation (59) and deriving with respect to time one gets:

$$2H\dot{H} = \frac{8\pi}{3m_P^2}\frac{d\phi}{dt}\frac{d}{d\phi}\left(V(\phi)\right) \quad \Rightarrow \quad \dot{H} = \frac{4\pi\dot{\phi}V'(\phi)}{3m_P^2 H}. \tag{63}$$

Now, derivating the Hubble parameter with respect to the time one has,

$$H = \frac{\dot{a}}{a} \quad \Rightarrow \quad \dot{H} = \frac{\ddot{a}}{a} - H^2. \tag{64}$$

Considering the condition that inflation takes place ($\ddot{a} > 0$) and equation (64), then one obtains:

$$\frac{\ddot{a}}{a} > 0 \quad \Rightarrow \quad \dot{H} + H^2 > 0 \quad \Rightarrow \quad -\frac{\dot{H}}{H^2} < 1. \tag{65}$$

Combining equations (58),(59),(63) and (65) and after some operations one obtains:

$$-\frac{\dot{H}}{H^2} = \frac{m_P^2}{16\pi}\left(\frac{V'}{V}\right) < 1. \tag{66}$$

Thus, this proved the connection between the slow roll parameters and inflation. Inflation lasts until $\epsilon_V \sim 1$, and afterwards, the SRA is not valid any more. Besides, it is important to clarify that SRA implies inflation, while the converse is not strictly true.

The two slow-roll parameters introduced until now are known as Potential Slow Roll (PSR) parameters and have as fundamental quantity the potential of the inflaton. They have to be small in order to neglect the kinetic term of the equation of motion, however, it can be shown[27] that the smallness of the PSR parameters is a necessary consistency condition, but not a sufficient condition to ensure that kinetic terms can be neglected. For this, the inflaton has to approach the asymptotic attractor solution[28]:

$$\dot{\phi} = \frac{-V'}{3H}. \tag{67}$$

In general, the assumption of an attractor solution at the end of inflation can be tested for a wide range of initial conditions for the different inflationary potential models, so it is not a very strong restriction[29].

2.5 A measure of the amount of inflation

The amount of inflationary expansion is usually specified by the logarithm of the scale factor at a particular moment, for instance, at the beginning of inflation divided by the scale factor at the end of inflation, this is the number of *e-foldings*, $\mathcal{N}$. In other words, $\mathcal{N}$ measures how much the scale factor increases; concretely, one *e-folding* is the amount of time for $a(t)$ to grow

[27]For more details look at [15].

[28] The justification of why an asymptotic attractor solution is needed is explained in [15].

[29]In this work, it is going to work for simplicity only with the first order PSR parameters. Higher order PSR parameters can be obtained as a series expansion based on the definitions of the first PSR parameters. For more details see [15].

by a factor e. SRA requires H to be nearly constant during this regime, that is why the scale factor has an exponential dependence with the Hubble parameter.
Mathematically, it is expressed by:

$$\mathcal{N} = ln\left(\frac{a(t_{end})}{a(t_{intial})}\right) = \int_{t_i}^{t_e} H(t)dt. \tag{68}$$

Using the SRA, equation (68) can be written in terms of the potential and its derivative. Dividing equation (58) by equation (59), one obtains:

$$\frac{H}{3\dot{\phi}} = -\frac{8\pi V(\phi)}{3m_P^2 V'(\phi)}, \quad \mathcal{N} = \int_{\phi_i}^{\phi_e} H\frac{dt}{d\phi}d\phi. \tag{69}$$

Combining equations (69) one finds that in the SRA the number of e-$foldings$ is:

$$\mathcal{N} = -\frac{8\pi}{m_P^2}\int_{\phi_i}^{\phi_e} \frac{V}{V'}d\phi. \tag{70}$$

This last equation permits to calculate the amount of inflation without having to solve the equations of motion.

In the literature one can find a formula for the number of e-$foldings$[30] of inflation in terms of the inflationary potential and some features of the entropy generation process (reheating) at the end of inflation:

$$\mathcal{N}_* \approx 71.21 - \log\left(\frac{k_*}{a_0 H_0}\right) + \frac{1}{4}\log\left(\frac{8\pi V_{hor}}{m_P^4}\right) + \frac{1}{4}\log\left(\frac{V_{hor}}{\rho_{end}}\right) + \frac{1-3w_{int}}{12(1+w_{int})}\log\left(\frac{\rho_{th}}{\rho_{end}}\right). \tag{71}$$

where $a_0 H_0$ is the present horizon scale, ρ_{end} is the energy density at the end of inflation, ρ_{th} is the scale energy density at which the universe has thermalized, V_{hor} is the value of the inflationary potential when the present horizon scale left the horizon during inflation and w_{int} characterizes the equation of state between the end of inflation and the energy scale ρ_{th}.

Equation (71) depends on several parameters, beyond the scope of this work. However, it is interesting to make a magnitude analysis of the terms of (71). The first two terms of (71) are independent of the inflationary potential, and taking[31] $k_* = 0.05\ Mpc^{-1}$ the second term is about 5. Moreover, if the thermalization is supposed to occur quickly, or if the reheating period is assumed to be radiation-dominated then the magnitude of the last term is ≤ 1. For a wide range of inflationary models, the fourth term is $O(1)$ and the third term ~ -10. With these values the number of e-$foldings$ takes the common range of $50 < \mathcal{N}_* < 60$. Nevertheless, $\mathcal{N}_*$ can vary over this range depending on the inflationary model chosen.

In addition, it can be proved[32] that the number of e-$foldings$ must be within the range of [50-60] in order to solve the cosmological problems as the horizon, flatness and monopole problems.

3 Perturbations in the Inflationary period

3.1 Qualitative description of the perturbations

All the introduction done to the inflation paradigm, has not given an answer to the fundamental question of how the large-structure of the universe can be created. The answer is that the

[30] For more details look [22].
[31] This value is taken according to [22].
[32] For a more detailed explanation see [4].

fluctuations of the scalar field and the metric, respectively, at very early times of the universe; are responsible for the current large-structure observed in our universe.

There are two kinds of perturbations according to their nature: density perturbations (or scalar perturbations), which are quantum fluctuations of the scalar field and are eventually the responsible of the formation of matter clusters and galaxies; and gravitational perturbations (or tensor perturbations), these are due to variations of the metric and although, they do not contribute to the the galaxy creation; however, their effect is expected to be observed analysing accurately the anisotropies of the CMB spectrum. The names scalar and tensor perturbations come from the transformation law they follow.

As seen in section 2.1, the characteristic scale of the universe during inflation is the Hubble length, H^{-1}, which marks a boundary for the possible causal processes. Perturbations are usually described as fluctuations described through a power spectrum (via Fourier analysis) assigning a comoving wavenumber k for each mode. During inflation these fluctuations grow exponentially as the universe expands, so that they can grow so much that they become greater than H^{-1} and they extend beyond the Hubble radius. The comoving mode k_* at which the perturbations cross the Hubble radius for the first time is:

$$k_* = a_* H_*, \tag{72}$$

where a_* and H_* are the scale factor and Hubble parameter, respectively, at the exiting moment.

When the fluctuations are outside the Hubble radius, they become disconnected from the cause which produces them, and their amplitude is frozen in[33]. Once the inflation has finished, the Hubble length grows faster than the scale factor and all the perturbations can re-enter the horizon during radiation- or matter-dominated eras. These perturbations, which have re-entered the horizon, will later be responsible for generating the large structure of the universe.

Moreover, the exactly exponential expansion of the De Sitter spacetime has the property to generate an scale-invariant spectrum[34]. During inflation, the only important physical length is the Hubble length, which remains nearly constant during inflation. With these two reasons one expects an approximately scale-invariant spectrum.

An accurate treatment of the cosmological perturbations produced during inflation is beyond the scope of this work[35], in the next pages will be briefly summarized the key results of the cosmological perturbations theory.

3.2 Quantum density fluctuations

Because of the observed anisotropies on the spectrum are so small, of the order of 10^{-5}, it is sufficient to study the cosmological perturbations at first order, giving these linearized perturbations, an accurate description of the spectrum. One can write a perturbation of the scalar field as:

$$\phi(\vec{\mathbf{r}}, t) = \overline{\phi}(t) + \delta\phi(\vec{\mathbf{r}}, t). \tag{73}$$

Combining equations (73) and (46), and after operating one obtains:

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} - \frac{\nabla^2 \delta\phi}{a^2} + V''\delta\phi = 0. \tag{74}$$

[33]For more details on this statement look at [21].
[34]For a complete proof see [4].
[35]For more details look [9] or [21].

Then, one has to expand the perturbation in comoving Fourier modes:

$$\delta\phi(\vec{\mathbf{r}},t) = \int \frac{d^3\vec{\mathbf{k}}}{(2\pi)^{3/2}} e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}}\delta\phi_{\mathbf{k}}(t). \tag{75}$$

However, these two conditions give a classical perturbation theory. That is why one has to go further, quantize the perturbations, introduce a Gauge invariant potential[36] and after some calculations one obtains:

$$(a\delta\phi_{\mathbf{k}})'' + \left(k^2 - \frac{z''}{z}\right)(a\delta\phi_{\mathbf{k}}) = 0, \tag{76}$$

where $z = a\dot{\phi}/H$ and the prime means derivatives with respect to the conformal time(τ), defined as $\tau = \int dt/a(t)$. This is the evolution equation for the scalar perturbations and in next sections will be showed how it can be solved.
Key quantities to describe the characteristics of the perturbations are the comoving curvature perturbation and the power spectrum of perturbations. They are defined as:

$$\mathcal{R} = -H\frac{\delta\phi}{\dot{\phi}}, \tag{77}$$

$$\langle 0|\,\delta\phi^*_{\mathbf{k_1}}\delta\phi_{\mathbf{k_2}}\,|0\rangle = \delta^{(3)}(\mathbf{k_1} - \mathbf{k_2})\frac{2\pi^2}{k^3}\mathcal{P}_{\delta\phi}(k), \tag{78}$$

where the state $|0\rangle$ represents the ground state of the system.

3.3 Metric fluctuations

In a similar way, one can study linear perturbations of the metric with the form:

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu}, \tag{79}$$

where $|h_{\mu\nu}| \ll 1$ and $\overline{g}_{\mu\nu}$ is a FRW metric.

It can be shown[37] that the tensor perturbation $h_{\mu\nu}$ has only 2 degrees of freedom, which are the two polarizations states (+,×) predicted for the gravitational waves.
After applying the formalism of perturbation theory[38] one gets the evolution equation:

$$(ah_k^{+,\times})'' + \left(k^2 - \frac{a''}{a}\right)(ah_k^{+,\times}) = 0, \tag{80}$$

where the prime denotes derivatives with respect to the conformal time. Equation (80) is very similar to equation (76), therefore, their solutions have also to be very similar.

3.4 Description of the primordial spectrum

There are mainly three approaches for solving equations (76) and (80). One manner is solving these set of differential equations numerically, then, one has to take into account that for a fixed comoving wavenumber, the evolution of the perturbations has to be separated in different stages[39]. Secondly, using a development of the HSR parameters and using its dependence with the primordial perturbations[40]. And, thirdly, one can also use the SRA and expand the power spectra of density and tensor perturbations in a phenomenological way as:

$$\mathcal{P}_{\mathcal{R}}(k) = A_s\left(\frac{k}{k_*}\right)^{n_s - 1 + \frac{1}{2}\frac{dn_s}{d\log k}\log(k/k_*) + \ldots}, \tag{81}$$

[36] For more details look at [12] and [21].
[37] For more details of the properties of the linear perturbations of the metric look [1] and [2].
[38] For more details look at [9].
[39] For more details see [22] and [23].
[40] HSR parameters means Hubble Slow Roll parameters and they are very similar to the PSR parameters, but they have the Hubble parameter as a fundamental quantity. For more details see [23].

$$\mathcal{P}_t(k) = A_t \left(\frac{k}{k_*}\right)^{n_t + \frac{1}{2}\frac{dn_t}{d\log k}\log(k/k_*)+\ldots}; \tag{82}$$

where A_s and A_t are the scalar and tensor amplitudes; n_s and n_t are the scalar and tensor spectral indices; whereas, the terms with logarithms are called the running of the scalar or the tensor index, respectively.

It can be proved[41] using the SRA that the terms of the scalar and tensor power spectra are related with the PSR parameters by:

$$A_s \approx \frac{8V}{3m_P^4\epsilon_V}, \tag{83}$$

$$A_t \approx \frac{128V}{3m_P^4}, \tag{84}$$

$$n_s - 1 \approx 2\eta_V - 6\epsilon_V, \tag{85}$$

$$n_t \approx -2\epsilon_V, \tag{86}$$

where the ϵ_V and η_V are given by equations (60) and (61), and the symbol $\approx$ indicates that SRA has been used. Besides, in the SRA the two power spectra can be related by the consistency relation[42]:

$$r = \frac{\mathcal{P}_{\mathcal{R}}(k_*)}{\mathcal{P}_t(k_*)} \approx 16\epsilon_V \approx -8n_t, \tag{87}$$

the quantity r is called the tensor-to-scalar ratio.

4 Inflation Models

In the last two decades, a great amount of inflation models have emerged. Some of them based on some particle theory or GUT, although some of them have no relation with particle theories and only are taken in a phenomenological way, without worrying about the pre-initial stages of inflation which would determine the inflationary potential .

In this work, distinct models of inflation are going to be solved, first analytically using the SRA and also numerically using the equation of motion including the kinetic terms, out of the SRA. In the framework of the SRA, all the fundamental parameters of the inflation model are going to be calculated, paying attention especially to the values of n_s and r obtained for each model, which can directly be compared with the experimental values obtained by the Planck mission.

The most simple models are the **chaotic inflation models**[43] . Their main features are:

1. They have a single scalar field.
2. A potential $V(\phi)$ has a minimum and in some regions satisfies the SRA conditions.
3. Initials conditions depend on the parameters of the potential $V(\phi)$.

There is a long list of inflation potential of interest, before giving the results for some of the chaotic inflation models, which have been calculated, it is going to give a detailed example of the calculations done for a specific model.

[41] For more details look at [9] and [12].

[42] For a detailed demonstration of the expression see [21] and [4].

[43] The name of chaotic models was coined by Linde because in this model the scalar field any value at different places of the early universe. For a more extended explanation on chaotic inflation models see [10].

4.1 Exponential Inflation

This model has a potential of the form:

$$V(\phi) = \Lambda^4 e^{-\lambda \frac{\sqrt{8\pi}}{m_P}\phi}, \tag{88}$$

where Λ and λ are constants describing the amplitude and the strength of the exponent of the potential, respectively. Applying the SRA and substituting $V(\phi)$ in equations (58) and (59); the equations of motion are:

$$3H\dot{\phi} = \Lambda^4 \lambda \frac{\sqrt{8\pi}}{m_P} e^{-\lambda \frac{\sqrt{8\pi}}{m_P}\phi}, \tag{89}$$

$$H^2 = \frac{8\pi}{3m_P^2} \Lambda^4 e^{-\lambda \frac{\sqrt{8\pi}}{m_P}\phi}. \tag{90}$$

These two equations can be solved analytically for the scalar field by integration, giving the solution:

$$\phi(t) = \frac{m_P}{\sqrt{2\pi}\lambda} \log\left[e^{\lambda \frac{\sqrt{2\pi}}{m_P}\phi_i} + t\sqrt{\frac{2\pi\lambda^4\Lambda^4}{3m_P^2}} \right], \tag{91}$$

where ϕ_i is the value of the inflaton at the beginning of the inflationary period.
Now, one can also find an expression for the scale factor using the definition of the Hubble parameter; and combining equations (90) and (91):

$$a(t) = a_i \left(1 + t^{2/\lambda^2} \sqrt{\frac{2\pi\lambda^4\Lambda^4}{3m_P^2}} e^{\lambda \frac{\sqrt{2\pi}}{m_P}\phi_i} \right), \tag{92}$$

where a_i is the value of the scale factor at the beginning of inflation. With equations (91) and (92) one can describe the behaviour of the scalar field and the scale factor in the slow roll regime. Moreover, one can also calculate the PSR parameters for this potential:

$$\epsilon_V = \frac{\lambda^2}{2} \qquad \text{and} \qquad \eta_V = \lambda^2. \tag{93}$$

Another important characteristic of the model is the number of *e-foldings* which can be calculated from equation (70) and using the potential given by (88):

$$\mathcal{N} = -\frac{8\pi}{m_P^2} \int_{\phi_i}^{\phi_e} \frac{\Lambda^4 e^{-\lambda \frac{\sqrt{8\pi}}{m_p}\phi}}{-\Lambda^4 \lambda \frac{\sqrt{8\pi}}{m_p} e^{-\lambda \frac{\sqrt{8\pi}}{m_p}\phi}} d\phi = \sqrt{\frac{8\pi}{m_P^2 \lambda^2}} (\phi_e - \phi_i), \tag{94}$$

where ϕ_e is the value of the inflaton at the end of the inflation. The end of inflation can be assumed to be approximately when $\epsilon_V = 1$; applying this condition to equation (93), that at the end of inflation (when $\phi = \phi_e$) $\lambda^2 = 2$. A common procedure would be writing with the condition of the end of inflation the value of ϕ_i as a function of $\mathcal{N}$, nevertheless, in the case of this model it is not useful, because in the PSR parameters do not appear ϕ_e, and because, as it will be shown, this model has no natural way of ending its expansion.

For this potential, the value of the scalar and tensor indices; and for the tensor-to-scalar ratio are:

$$n_s = 1 + 2\lambda^2 - 6\left(\frac{\lambda^2}{2}\right) = 1 - \lambda^2, \tag{95}$$

$$n_t = -\lambda^2, \tag{96}$$

$$r = 8\lambda^2. \tag{97}$$

If now one uses the value of λ for the end of inflation, the results for the observables quantities are: $n_s = -1$ and $r = 16$, which are excluded by the Planck mission data[44]. These results are obtained because this model does not stop inflating the universe at any time. For this model inflation only ends if some external mechanism is introduced in order to stop it.

The previous analysis of the inflationary model has been performed in the framework of the SRA. However, it is interesting to solve the equation of motion for the inflaton with the kinetic terms. Then, one has a set of two coupled differential non-lineal equations:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \tag{98}$$

$$H^2 = \frac{8\pi}{3m_P^2}\left[V(\phi) + \frac{1}{2}(\dot{\phi})^2\right]. \tag{99}$$

These set of equations can be solved numerically. Before doing that, it is useful to introduce a set of dimensionless variables:

$$Y = \frac{\phi(t)}{\phi_i}, \qquad X = \frac{H(t)}{H_i}, \qquad W = \frac{V(\phi)}{H_i}, \quad \tau = H_i t. \tag{100}$$

It is important not to confuse τ with the conformal time introduced before; in this equations τ is simply a dimensionless variable for the time integration. Taking these new set of dimensionless variables the equations of motion can be written as:

$$Y'' + 3XY' + \frac{dW}{dY} = 0, \tag{101}$$

$$X^2 = \frac{1}{W_i}\left[\frac{1}{2}(Y')^2 + W\right], \tag{102}$$

where the prime means derivatives with respect to τ. With these dimensionless variables, the initials conditions for these set of equations are immediately found as:

$$Y(\tau = 0) = 1, \qquad \text{and} \qquad Y'(\tau = 0) = -\left.\frac{1}{3}\frac{dW}{dY}\right|_i \quad . \tag{103}$$

Therefore, one initial condition will be the same for all the models, while the initial condition on the derivative will depend on the inflation model chosen. Besides, the set of equations (100-103) are completely general. Specifying the power law potential in the previous equations one has, after some calculations[45]

$$W = W_i\, e^{\kappa(Y-1)}, \tag{104}$$

where W_i and κ are constants that can be chosen arbitrarily. They are free parameters which will determine the strength of the potential. The equations of motion (101) and (102) will be solved numerically using a fifth order Runge-Kutta-Fehlberg method with a fixed integration step.

The results of the numerical integration, can be seen in Figure 5, numerical curves reproduce SRA predicted behaviour[46] for $\phi(t)$ and $a(t)$ for all the range of integration, this is because inflation remains in the slow roll regime for the scalar field all the integration time. From this behaviour one can see that this is an incomplete model which needs an external mechanism in order to end inflation. As it will be seen later, this model is rejected by the experimental data.

[44]For more details on the main results of the Planck mission look at [24].

[45]In this calculations W_i and κ are constants which simplify the notation and contain a group of constant like ϕ_i, λ or Λ.

[46]To compare the numerical results and the SRA curves we have rewritten the SRA curves in terms of the dimensionless variables.

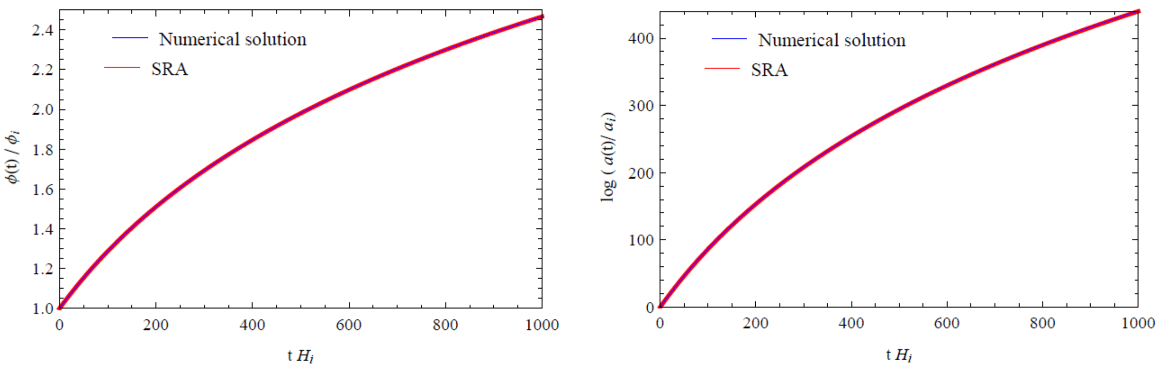

Fig. 5 – Numerical solution for the scalar field and the scale factor and also the analytical curves within SRA. In these graphics have been used values for $W_i = 0.005$ and $\kappa = 2.0$. The behaviour of the scalar field confirms that for this inflation model never decay into the reheating era because the inflaton does not decrease its value by its own.

4.2 Starobinsky model

The Staborinsky model, which is one of the first models proposed for inflation, can be analysed within an alternative theory of gravity, the $f(R)$ theory, which consider higher order curvature terms in the Hilbert-Einstein action[47]. Within the $f(R)$ theory are defined two frameworks: the Jordan frame and the Einstein frame. The Einstein frame is equivalent to the one in GR theory, and uses the Hilbert-Einstein action given by (37); whereas the Jordan frame is related to the previous one through a conformal transformation.

Starobinsky proposed a model for inflation which is based on an action equal to the Einstein-Hilbert action with an extra curvature quadratic term:

$$S_S = \int \sqrt{-g}\frac{m_P^2}{16\pi}\left(R + \frac{R^2}{6M^2}\right) d^4x, \tag{105}$$

where M is the mass of the scalar field and R is the Ricci Scalar.

From equation (105) one can extract the Lagrangian density. Making a conformal transformation and after some technicalities[48]one obtains:

$$\mathcal{L} = \left[\frac{1}{2}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \Lambda^4\left(1 - e^{-\sqrt{\frac{\pi}{3}}\frac{4\phi}{m_p}}\right)^2\right], \tag{106}$$

where Λ is a constant. From this Lagrangian density one can straightforwardly identify the potential for the scalar field:

$$V(\phi) = \Lambda^4\left(1 - e^{-\sqrt{\frac{\pi}{3}}\frac{4\phi}{m_p}}\right)^2. \tag{107}$$

This is the Starobinsky potential in four dimensions[49]. Now, the same calculations done in section 4.1, for the previous model can be applied here. Using the SRA one can solve analytically the equations of motion and obtain a result for the scalar field and the scale factor:

$$\phi(t) = \frac{m_p}{4}\sqrt{\frac{\pi}{3}}\log\left(e^{\sqrt{\frac{\pi}{3}}\frac{4\phi_i}{m_p}} - \frac{8}{3}\sqrt{\frac{2\pi}{3}}\frac{\Lambda^2 t}{m_p}\right). \tag{108}$$

[47]For more details of the f(R) theory see [18] , [19] and [20]
[48]If one is interested in the details of the transformation can read [16].
[49]There exists a generalization of the Starobinsky at higher dimensions, see [16] and [17].

$$a(t) = a_i \left(1 - \frac{8\Lambda^2}{3m_p}\sqrt{\frac{2\pi}{3}}e^{-\sqrt{\frac{\pi}{3}}\frac{4\phi}{m_p}}\right)^{3/4} e^{\sqrt{\frac{2\pi}{3}}\frac{2\Lambda^2 t}{m_p}}. \tag{109}$$

where ϕ_i and a_i are initial values for the scalar field and the scale factor.

The results for the PSR parameters are:

$$\epsilon_V = \frac{4}{3\left(e^{\sqrt{\frac{\pi}{3}}\frac{4\phi}{m_p}} - 1\right)^2}, \tag{110}$$

$$\eta_V = \frac{4\left(e^{\sqrt{\frac{\pi}{3}}\frac{4\phi}{m_p}} - 2\right)}{3\left(e^{\sqrt{\frac{\pi}{3}}\frac{4\phi}{m_p}} - 1\right)^2}. \tag{111}$$

Then, computing the number of $e\text{-}foldings$ using equations (70) and (107) one obtains:

$$\mathcal{N} = \frac{3}{4}\left(e^{\sqrt{\frac{\pi}{3}}\frac{4\phi_i}{m_p}} - e^{\sqrt{\frac{\pi}{3}}\frac{4\phi_{end}}{m_p}}\right) + \frac{\sqrt{3\pi}}{m_p}(\phi_{end} - \phi_i) \tag{112}$$

This last equation for $\mathcal{N}$ can then be simplified using the fact that during inflation the scalar field decreases its value and, hence, $\phi_{end} \ll \phi_i$, moreover, if ϕ_i has a large value, then, the exponential term containing ϕ_i will be much larger than the others. With these approximations one obtains:

$$\mathcal{N} = \frac{3}{4}e^{\sqrt{\frac{\pi}{3}}\frac{4\phi_i}{m_p}}. \tag{113}$$

Using equations (113), one can write the PSR parameters in terms of the $\mathcal{N}$, and therefore, one can also write the spectral index and the tensor-to-scalar ratio (equations (85) and (87), respectively) as a function of $\mathcal{N}$:

$$n_s = 1 - \frac{2}{\mathcal{N}}, \tag{114}$$

$$r = \frac{12}{\mathcal{N}^2}. \tag{115}$$

Taking the range of $\mathcal{N}$ between [50-60], one obtains the values shown in Table 3. It can be observed that the values of n_s and r calculated within the SRA, approach significantly to the experimental data provided by Planck[50].

$\mathcal{N}$	n_s	r
50	0.9600	0.0048
55	0.9636	0.0039
600	0.9667	0.0033

Table 3 – Values for the scalar index and the tensor-to-scalar ratio for the range of $50 < \mathcal{N} < 60$.

One can also compute the Starobinsky model numerically using equations (101-103). With this, the potential for this model can be written in terms of the dimensionless variables given by (100) as:

$$W = W_i\left(\frac{1-\kappa^Y}{1-\kappa}\right)^2, \tag{116}$$

[50]Some key results of Planck measurements will be shown in section 5.

where W_i and κ are dimensionless variables which completely determine the intensity of the scalar field during inflation. Their values will be chosen here arbitrarily, considering that the model has to accomplish at least the minimum number of *e-foldings* necessary to solve the cosmological problems.

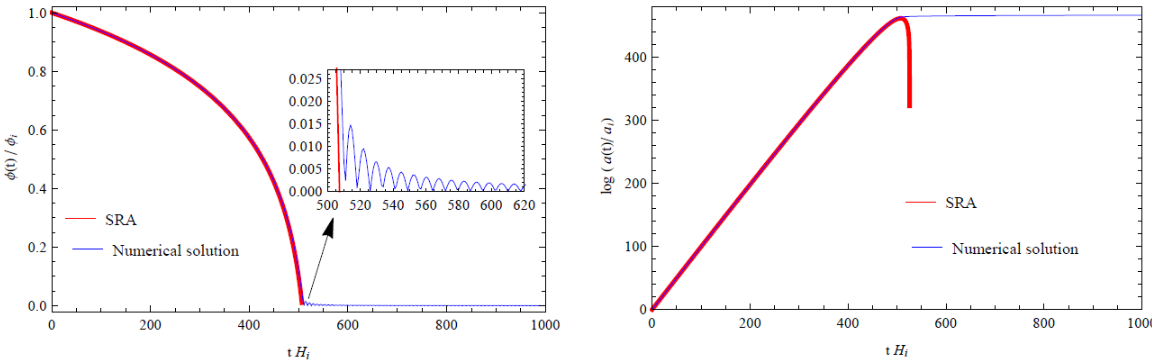

Fig. 6 – Numerical solution for the scalar field and the scale factor for $\kappa = 0.0035$ and $W_i = 0.007$, in both plots numerical results are compared with the analytical curves from SRA.

The results for the scalar field are shown in the first plot of Figure 6. First, the scalar field has a large value and then begins to slowly roll down, as long as it is rolling down the decay goes faster, as it happens in a classical system, and, finally, it has a period of very small oscillations, this is the reheating period, with these oscillations the scalar field transform its energy in thermal energy. Furthermore, the SRA curve fits really well the numerical result until the Slow Roll period finishes, then SRA is not able to reproduce the really small oscillations of the potential which are the seed of the future anisotropies of the universe.
In Figure 6, it is shown also the behaviour of the scale factor and the SRA result. At the beginning the SRA curve reproduces the exponential expansion of the universe, however, afterwards instead of stopping the increase, it decreases strongly, that is why, at the time of the decreasing SRA is not valid because the PSR (ϵ_V and η_V) have become greater than 1. There is a logarithmic term in the scale factor analytical solution for $a(t)$ within SRA, which becomes negative when SRA finishes, that is why a great descent is observed. Besides the graphic shows that the scale factor gives about 450 *e-foldings*. However, the observationally relevant part of the inflation are the last 60 *e-foldings* where the regime is clearly non-exponential. This is important to understand because observational data correspond just to the last part of the inflationary period, where the scalar field does not present great changes in its values.

As discussed before, the oscillatory behaviour at the end of inflation has a great importance; that is why, it is going to be tested what is the behaviour of the oscillations if the step of integration changes. The results can be seen in Figure 7, where it is observed that the oscillatory behaviour changes a few, specially the depth of the oscillations while the spikes remain the same. This fact is important because the energy is released when ϕ reaches a spike and then decrease, hence, that solution means that the energy recovered by the scalar field will be the same in both models. Looking at both plots in Figure 7 one can see that the main behaviour of the solution of the non-linear system remains practically unchanged when one modifies[51] the $\Delta\tau$.

[51] A deeper analysis of the numerical error of the solution can be done studying the convergence, consistency and stability of the solution. However, the lack of space in this work impede us to treat such an interesting test.

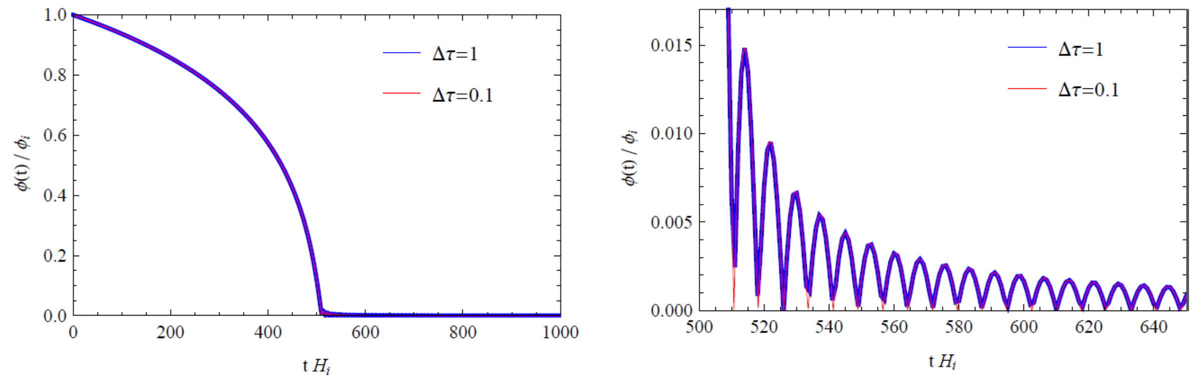

Fig. 7 – Graphic showing the numerical solution for different steps of integration: in blue $\Delta\tau = 1$ and in red $\Delta\tau = 0.1$.

One can follow the same procedure with other inflation models and obtain results for $\phi(t)$ and $a(t)$. However, the more relevant quantities of an inflation model are the spectral index and the tensor-to-scalar ratio as they are observational magnitudes that can be tested by experimental measurements. That is why we have computed for distinct models the values of n_s and r for $50 < \mathcal{N} < 60$ within the SRA.

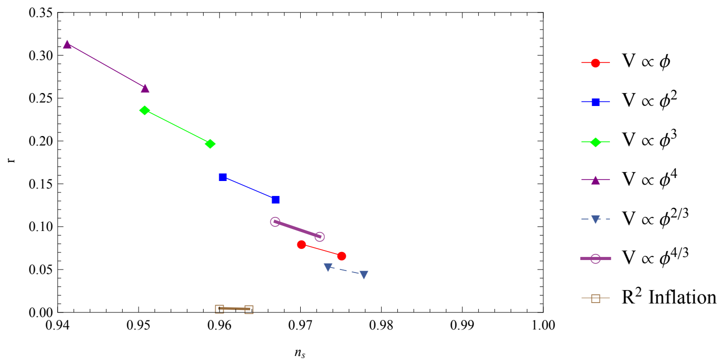

Fig. 8 – In this Figure one can observe the results for the different models within the SRA for the values of n_s and r for the range of $50 < \mathcal{N} < 60$.

The results are shown in Figure 9, as it will be seen this configuration fits properly the experimental data provided by Planck.

5 Experimental evidences of Inflation

Different experiments have been designed in order to reduce the number of models allowed for the inflationary period. These experiments consist basically in studying accurately the anisotropies of the CMRB. The most recent one has been the Planck mission. Planck is the name of a satellite launched by the European Space Agency in 2009, whose objective was not only generating constraints on inflation, but also providing detailed data of the CMBR[52].

The Planck mission made highly accurate measurements of the temperature anisotropies on the spectrum of the CMBR. These data can be used to constrain inflation models, particularly, the most important results launched on 2015 for the scalar spectral index and tensor-to-scalar

[52] For a detailed explanation of the products and scientific results of Planck mission look at [24].

ratio are:

$$n_s = 0.9603 \pm 0.0073, \tag{117}$$

$$r < 0.11 \quad (95\% CL), \tag{118}$$

where CL means Confidence Level. These results are really important because they are a strong constraint on the possible models describing inflation.

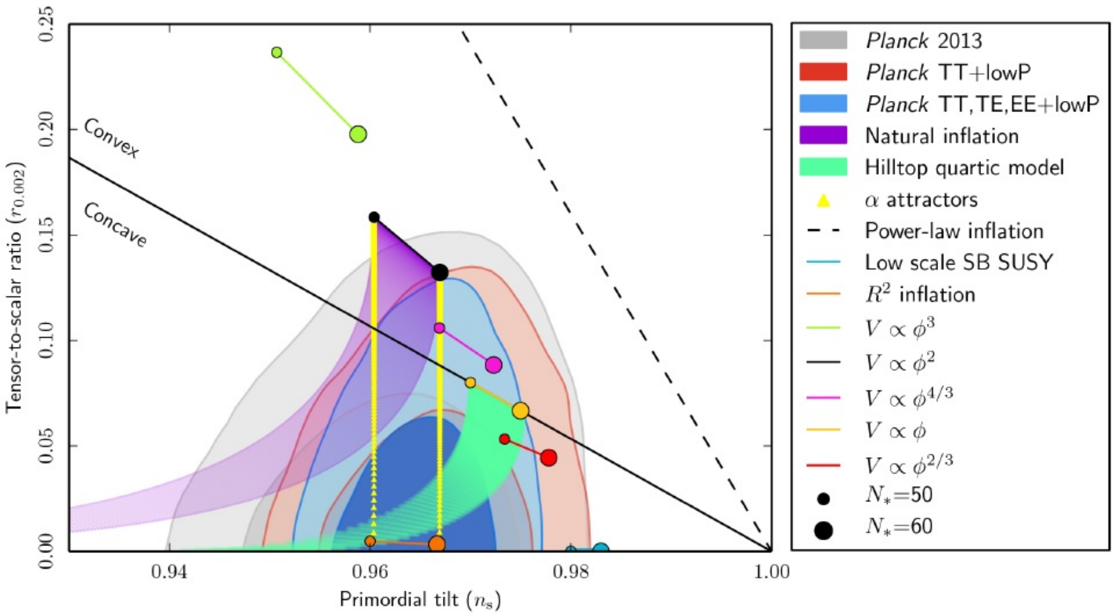

Fig. 9 – Constraints on the inflation models provided by the Planck mission. In this graphic it is also shown the data from the polarizations modes of radiation.

In Figure 9, the Planck results are shown with the results for different inflationary models in the first order SRA. Moreover, in the Planck article has been used[53] $k_* = 0.002 Mpc^{-1}$ as a pivot scale when the perturbations are supposed to leave the horizon. This value is chosen according to the criteria of others articles[54]. However, while the tensor-to-scalar ratio depends slightly on k_*, the spectral index is independent of the pivot scale, and these results can be taken quite general. On the other hand, the contour plots are likelihood distributions for distinct modes of the radiation, for example, Planck TT means a combination of different data from the temperature likelihood[55] at multipoles $l > 30$.

These results show that polynomial inflation models with $n > 2$ are disfavoured and out of the CL, whereas the inflation models with low values for n_s and r, like the Starobinsky model, are compatible and strengthen by Planck results. Furthermore, it is important to note that Figure 9 is quite similar to Figure 8, confirming that SRA is a great tool to constrain the acceptable inflation models.

[53] In this work we have used basically two values for the pivot scale (k_*) following the criteria of [22]. That is why in equation (72) appears $k_* = 0.05 Mpc^{-1}$ and now $k_* = 0.002 Mpc^{-1}$.

[54] For more details look at [22] and [23].

[55] If one is interested in the meaning of these labels, look at [22] and [23].

6 Conclusions

The well accepted big bang theory, which describes the beginning of the universe, has a great number of successes, however, this theory presents a list of problems like the flatness, the horizon or the monopole problem which require a modified scenario, inflation, which is entirely compatible with the big bang theory. It is like an accessory which improves it.

Inflation is a period usually assumed to take place between 10^{-35} and $10^{-32}s$ after the big bang, during which the universe suffered a great expansion. This huge expansion is due to the anomalous behaviour of the scale factor that has $\ddot{a} > 0$. This accelerated expansion solves directly some of the cosmological problems, which the big bang theory could not solve.

The standard model of particle physics may not hold at inflationary era and it is currently not possible to derive from first principles an equation of state for that epoch. However, particle physics suggests that during the process of expansion and cooling down of the universe, phase transitions could have occurred due to symmetry breakings. All this process is modelled by an scalar field and its potential, which releases its latent heat during the cooling down of the universe.

The freedom of choosing the potential of the scalar field, has caused that during the last 20 years a great amount of models for inflation have emerged. Not only theoretical models have been developed, but also a large number of experiments have been designed in order to constrain the possible inflation models. The most recent one has been Planck, which was a satellite launched on 2009 and destined to make accurate measurements of the CMBR.

The results of Planck have discarded some models like the polynomial with $n > 2$, whereas some others like the Starobinsky model have been reinforced. This model was developed by Starobinsky at 1980 with the idea to include quantum corrections to the model adding extra curvature terms to the action of GR. This was one of the first models for inflation and during the years has not been rejected by any experimental data from the different experiments. In general, one can conclude that every single inflation model, apart from being theoretically well explained, it has also to provide values of n_s and r according to their experimental range of values.

On the other hand, inflation is not a perfect theory, it also presents its own shortcomings. The question of what causes inflation remains unanswered. There is no physical theory, which gives a well accepted answer. Besides, in most models the scalar field and its potential are chosen arbitrarily to fit the experimental data, so that they do not have a strong physical basis, which explains its origin.

Inflation provides a solution to the cosmological problems, however, it is not the only theory proposed to solve them, for example, Steinhardt and Turok proposed in 2001 an alternative to inflation called ekpyrosis. This alternative proposal is based on string theory and consists in the collision of branes in higher dimension universe. Both Ekpyrosis and inflation are a topics of intense discussion in theoretical physics. Nevertheless, the lack of experimental evidences supporting Ekpyrosis currently makes inflation the most natural option and the most accepted scenario for the very early universe.

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