ARTICLE IN PRESS

FSS:7983

Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems ••• (••••) •••–•••

FUZZY sets and systems

www.elsevier.com/locate/fss

The aggregation of transitive fuzzy relations revisited

T. Calvo Sánchez [a,b,*], P. Fuster-Parra [b,c], O. Valero [b,c]

[a] Department of Computer Science, Universidad de Alcalá, 28871 Alcalá de Henares, Madrid, Spain
[b] Departament de Ciències Matemàtiques i Informàtica, Universitat de les Illes Balears, 07122 Palma de Mallorca, Illes Balears, Spain
[c] Institut d' Investigació Sanitària Illes Balears (IdISBa), Hospital Universitari Son Espases, 07120 Palma de Mallorca, Illes Balears, Spain

Received 25 April 2020; received in revised form 30 October 2020; accepted 19 November 2020

Abstract

In this paper, we consider the problem of aggregating a collection of transitive fuzzy binary relations in such a way that the aggregation process preserves transitivity. Specifically, we focus our efforts on the characterization of those functions that aggregate a collection of fuzzy binary relations which are transitive with respect to a collection of t-norms preserving the transitivity. We characterize them in terms of triangular triplets. Further, the relationship between triangular triplets, the monotonicity of the aggregation function and an appropriate dominance notion is explicitly stated. Special attention is paid to a few classes of transitive fuzzy binary relations that are relevant in the literature. Concretely, we describe, in terms of triangular triplets, those functions that aggregate a collection of fuzzy pre-orders, fuzzy partial orders, relaxed indistinguishability relations, indistinguishability relations and equalities. A surprising relationship between functions that aggregate transitive fuzzy relations into a T_M-transitive fuzzy relation and those that aggregate relaxed indistinguishability relations is shown. A few consequences of the new results are also provided for those cases in which all the t-norms of the given collection are the same. Some celebrated results are retrieved as a particular case from the exposed theory.

Keywords: Aggregation; Transitivity; Fuzzy binary relation; Triangular triplet; T-norm; Fuzzy pre-order; Indistinguishability relation; Equality

1. Introduction

Nowadays, the concept of fuzzy binary relation plays a central role in many problems that arise naturally in applied sciences like Artificial Intelligence and Decision-Making (see, for instance, [5,10,17,18,25]). According to [1,5,10,18,22], many times in the aforesaid fields a collection of fuzzy binary relations must be fused, by means of a function, in order to get a new fuzzy binary relation that, on the one hand, takes into account the information that can be derived from all members of such a collection and, on the other hand, allows us to make working decisions.

Let us recall that, on account of [26], a fuzzy binary relation R on a non-empty set X is a function $R : X \times X \to [0, 1]$. Thus, following [5], given a collection of fuzzy binary relations $\{R_i\}_{i=1}^n$ on X, the aggregated fuzzy binary

* Corresponding author at: Department of Computer Science, Universidad de Alcalá, 28871 Alcalá de Henares, Madrid, Spain.
E-mail addresses: tomasa.calvo@uah.es (T. Calvo Sánchez), pilar.fuster@uib.es (P. Fuster-Parra), o.valero@uib.es (O. Valero).

https://doi.org/10.1016/j.fss.2020.11.012
0165-0114/

relation through a function $F : [0,1]^n \to [0,1]$ is a fuzzy binary relation $R : X \times X \to [0,1]$ given by $R(x, y) = F(R_1(x, y), \ldots, R_n(x, y))$ for all $x, y \in X$.

In order to carry out the above aggregation process, the choice of the function F that must be used is a very important fact. According to [19], one way to achieve such an objective is to impose a constraint to the aggregation process. Thus, the selection is made in such a way that the function F preserves any property that each fuzzy binary relation of the collection $\{R_i\}_{i=1}^n$ to be merged fulfills, i.e., R satisfies the aforementioned property when each R_i does. Many authors have focused their efforts on the study of those functions F that allow us to preserve properties such as reflexivity, irreflexivity, symmetry, asymmetry, antisymmetry, connectedness and transitivity.

Among the aforementioned properties, transitivity has attracted the attention of many works. Recall that, given a t-norm $T : [0,1] \times [0,1] \to [0,1]$, a fuzzy binary relation R on a non-empty set X is transitive with respect to T (T-transitive for short) provided that

$$T(R(x,z), R(z,y)) \leq R(x,y)$$

for all $x, y, z \in X$ ([5]). We assume that the reader is familiar with the basics of triangular norms (we refer the reader to [11] for a full treatment of the topic).

Sufficient conditions for preserving the transitivity were explored in [5] and in [2] when the function F under consideration belongs to the family of the so-called aggregation functions in the sense of [1,8], i.e., F satisfies $F(0_n) = 0$, $F(1_n) = 1$ and is monotonic ($F(a) \leq F(b)$ when $a \preceq b$) where $0_n, 1_n \in [0,1]^n$ with $0_n = (0, \ldots, 0)$, $1_n = (1, \ldots, 1)$ and $a \preceq b \Leftrightarrow a_i \leq b_i$ for all $i = 1, \ldots, n$. However, a characterization of aggregation functions preserving transitivity was provided in [21]. Such a characterization was stated in terms of the so-called domination property.

From now on, in order to recall it, we will say that a function $F : [0,1]^n \to [0,1]$ aggregates T-transitive fuzzy relations on a non-empty set X when $F(R_1, \ldots, R_n)$ is a T-transitive fuzzy relation on X provided that $\{R_i\}_{i=1}^n$ is a collection of T-transitive fuzzy binary relations on X. Concretely, the following characterization was obtained in [21]:

Theorem 1. *Let $n \in \mathbb{N}$ and let $F : [0,1]^n \to [0,1]$ be an aggregation function. Then the following assertions are equivalent:*

1) F aggregates T-transitive fuzzy relations.
2) F dominates the t-norm T ($T(F(a), F(b)) \leq F(T(a_1, b_1), \ldots, T(a_n, b_n))$ for all $a, b \in [0,1]^n$).

In [3,4], an extension of Theorem 1 was obtained in the case in which no assumptions about the considered function F are assumed and, thus, more general functions than the aggregation ones are considered. To this end, according to [3,4], a function $F : [0,1]^n \to [0,1]$ dominates a t-norm T whenever

$$T(F(a), F(b)) \leq F(T(a_1, b_1), \ldots, T(a_n, b_n))$$

for all $a, b \in [0,1]^n$. In this new approach, without further assumptions about the function F, only partial results could be obtained. Specifically, the next relationships were proved in [3,4]:

Theorem 2. *Let $n \in \mathbb{N}$ and let $F : [0,1]^n \to [0,1]$ be a function. If F aggregates T-transitive fuzzy relations, then F dominates the t-norm T.*

The converse of the implication exposed in the previous result can be obtained when the function F is assumed to be monotonic.

Theorem 3. *Let $n \in \mathbb{N}$ and let $F : [0,1]^n \to [0,1]$ be a monotonic function. If F dominates the t-norm T, then F aggregates T-transitive fuzzy relations.*

Considering Theorems 2 and 3, the next characterization was stated in [3] when monotonicity is assumed:

Theorem 4. *Let $n \in \mathbb{N}$ and let $F : [0,1]^n \to [0,1]$ be a monotonic function. Then the following assertions are equivalent:*

1) F aggregates T-transitive fuzzy relations.
2) F dominates the t-norm T.

Notice that Theorem 4 improves Theorem 1 by removing the assumptions $F(0_n) = 0$ and $F(1_n) = 1$.

It must be stressed that, contrary to the transitivity, those functions that preserve reflexivity, irreflexivity, symmetry, asymmetry, antisymmetry and connectedness were characterized in [3,4].

A very important class of fuzzy binary relations are those that are at the same time reflexive, symmetric and transitive. This kind of fuzzy relations was first considered in [26], where the transitivity was considered with respect to the minimum t-norm T_M, and they were called similarities relations. Later on, the transitivity was considered with respect to any t-norm T and, thus, the notion of T-indistinguishability relation or T-equivalence was introduced in [23]. Since then, a comprehensive study of indistinguishability relations has been carried out, and we refer to the reader to [20] as a reference monograph on this topic.

The description of the functions that aggregate indistinguishability relations into a new one was obtained in [14].

According [19], let us recall that a function $F : [0, 1]^n \to [0, 1]$ aggregates T-indistinguishability operators on a non-empty set X when $F(R_1, \dots, R_n)$ is a T-indistinguishability relation on X and $\{R_i\}_{i=1}^n$ is a collection of T-indistinguishability operators on X. A new characterization of such functions, which were not assumed a priori as aggregation functions, was withdrawn by means of the notion of triangular triplet with respect to a t-norm. On account of [14], the alluded new notion can be formulated in the following way.

Given a t-norm T, a triplet $(a, b, c) \in [0, 1]^n$ is said to be an n-dimensional T-triangular triplet whenever

$$T(a_i, b_i) \le c_i \quad T(a_i, c_i) \le b_i \quad \text{and} \quad T(b_i, c_i) \le a_i$$

for all $i = 1, \dots, n$.

Considering the previous notion the next result supplies the announced characterization.

Theorem 5. *Let $n \in \mathbb{N}$ and let T be a t-norm. If $F : [0, 1]^n \to [0, 1]$ is a function, then the following assertions are equivalent:*

1) F aggregates T-indistinguishability relations.
2) F satisfies the following conditions:
2.1) F transforms n-dimensional T-triangular triplets into 1-dimensional T-triangular triplets.
2.2) $F(1_n) = 1$.

Notice that assertion 2.1) in the statement of the previous result means that $(F(a), F(b), F(c))$ is a 1-dimensional T-triangular triplet provided that $(a, b, c) \in [0, 1]^n$ is an n-dimensional T-triangular triplet.

In [9] (see also [20]), the need to consider more general ways to aggregate a collection of fuzzy binary relations was exposed. Concretely, the notion of a function that aggregates fuzzy binary relations with respect to a collection of t-norms was introduced (see also, [19]). In particular, such a notion can be formulated as follows:

A collection $\{R_i\}_{i=1}^n$ of fuzzy relations on a non-empty set X is said to be transitive with respect to a collection $\mathcal{T} = \{T_i\}_{i=1}^n$ of t-norms (or $\mathcal{T}$-transitive for short) provided that R_i is T_i-transitive for each $i = 1, \dots, n$. Thus, given a t-norm T, a function $F : [0, 1]^n \to [0, 1]$ aggregates $\mathcal{T}$-transitive fuzzy relations into a T-transitive fuzzy binary relation provided that $F(R_1, \dots, R_n)$ is a T-transitive fuzzy relation on a non-empty set X when $\{R_i\}_{i=1}^n$ is a collection of $\mathcal{T}$-transitive fuzzy relations on X. Clearly, in the particular case in which all members of the collection $\mathcal{T}$ are exactly T, we retrieve from the previous notion that $F : [0, 1]^n \to [0, 1]$ aggregates T-transitive fuzzy relations.

In the references [9,19], relationships are established between those functions that aggregate $\mathcal{T}$-transitive fuzzy relations into a T-transitive fuzzy binary relation and those that aggregate a collection of distances into a new distance when the t-norms belonging to $\mathcal{T}$ and the t-norm T are continuous and Archimedean. However, this approach will not be discussed here and we will propose it as future work in Section 4.

Motivated by the facts above, this paper focuses our attention on exploring the description of those functions that aggregate $\mathcal{T}$-transitive fuzzy relations into a T-transitive fuzzy binary in terms of triangular triplets. Nevertheless, in this case, we will need to introduce a new notion of triangular triplet, which we have called n-dimensional asymmetric $\mathcal{T}$-triangular triplet. The relationship between triangular triplets, the monotonicity of the aggregation function and an appropriate dominance notion is explicitly stated. Special attention is paid to a few classes of transitive fuzzy binary relations that are relevant in the literature. In particular, we describe, inspired by Theorem 5, in terms of $\mathcal{T}$-triangular triplets the functions, that aggregate a collection of fuzzy pre-orders, fuzzy partial orders, relaxed indistinguishability relations, indistinguishability relations and equalities. A surprising link between functions that aggregate $\mathcal{T}$-transitive fuzzy relations into a T_M-transitive fuzzy relation and those that aggregate relaxed indistinguishability relations is obtained. A few consequences of the new results are also provided for those cases in which all the t-norms belonging to the collection $\mathcal{T}$ are exactly the same and equal to T. Thus Theorems 2, 3, 4 and 5 are retrieved as a particular case from the exposed theory. Finally, conclusions and future work are exposed.

2. Aggregation of transitive fuzzy binary relations: a characterization

Next, we focus on the study of those properties that a function must satisfy in order to merge a collection of $\mathcal{T}$-transitive fuzzy relations into a single one. First, we answer to the posed question giving partial descriptions of such properties extending Theorems 2, 3 and 4 to this new context. Secondly, we go deeper into such a description and we provide a characterization of those functions under consideration in the spirit of Theorem 5.

With this aim we need to introduce the following notion.

Definition 6. A function $F : [0, 1]^n \to [0, 1]$ dominates a t-norm T with respect to a collection $\mathcal{T} = \{T_i\}_{i=1}^n$ of t-norms whenever F satisfies, for all $a, b \in [0, 1]^n$, the condition below:

$$T(F(a), F(b)) \leq F(T_1(a_1, b_1), \dots, T_n(a_n, b_n)).$$

Of course when $T_i = T$ for all $i = 1, \dots, n$ we get from the preceding definition that F dominates the t-norm T.

The next result yields a first approach to the aforementioned description.

Proposition 7. *Let $n \in \mathbb{N}$ and let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of t-norms. If $F : [0, 1]^n \to [0, 1]$ is a function that aggregates $\mathcal{T}$-transitive fuzzy relations into a T-transitive fuzzy relation, then F dominates the t-norm T with respect to $\mathcal{T}$.*

Proof. Let $a, b \in [0, 1]^n$. Consider a non-empty set $X = \{x, y, z\}$ where x, y, z are different. Define the fuzzy relations $(R_i)_{i=1}^n$ on X by $R_i(x, x) = R_i(y, y) = R_i(z, z) = 1$ and $R_i(y, x) = R_i(x, y) = T_i(a_i, b_i)$, $R_i(z, x) = R_i(x, z) = a_i$ and $R_i(y, z) = R_i(z, y) = b_i$ for all $i = 1, \dots, n$. Then it is easily seen that each R_i is a T_i-transitive fuzzy relation on X. Since F aggregates $\mathcal{T}$-transitive fuzzy relations into a T-transitive fuzzy relation we have that $F(R_1, \dots, R_n)$ is a T-transitive fuzzy relation on X. It follows that

$$T(F(R_1, \dots, R_n)(x, z), F(R_1, \dots, R_n)(z, y)) \leq F(R_1, \dots, R_n)(x, y).$$

Moreover,

$$\begin{aligned}
&T(F(a), F(b)) = T(F(R_1(x, z), \dots, R_n(x, z)), F(R_1(z, y), \dots, R_n(z, y))) = \\
&T(F(R_1, \dots, R_n)(x, z), F(R_1, \dots, R_n)(z, y)) \leq \\
&F(R_1, \dots, R_n)(x, y) = F(R_1(x, y), \dots, R_n(x, y)) = \\
&F(T_1(a_1, b_1), \dots, T_n(a_n, b_n)).
\end{aligned}$$

This ends the proof. □

Notice that the preceding result allows us to discard those functions that are not useful to merge transitive fuzzy relations preserving the transitivity. The next example illustrates this fact.

Example 8. Consider a collection $\mathcal{T} = \{T_i\}_{i=1}^n$ of t-norms such that $T_i = T_M$ for all $i = 1, \dots, n$ and $T = T_M$. Define the function $F : [0, 1]^3 \to [0, 1]$ by

$$F(a) = F(a_1, a_2, a_3) = \begin{cases} a_1 \cdot a_2 + a_3 & a_1, a_2, a_3 \in [0, 0.6] \\ 1 & \text{otherwise.} \end{cases}$$

Consider $a = (0.2, 0.4, 0.2)$ and $b = (0.4, 0.2, 0.4)$. Then we have that

$$0.24 = F(0.2, 0.2, 0.2) < T_{Min}(F(0.2, 0.4, 0.2), F(0.4, 0.2, 0.4)) = 0.28.$$

Therefore, by Proposition 7, we conclude that F is not a function that aggregates T_M-transitive fuzzy relations.

When the collection of t-norms $\mathcal{T}$ satisfies that $T_i = T$ for all $i = 1, \ldots, n$, then Proposition 7 retrieves as a particular case Theorem 2.

Corollary 9. *Let T be a t-norm and let $F : [0, 1]^n \to [0, 1]$ be a function that aggregates T-transitive fuzzy relations. Then F dominates the t-norm T.*

The next result, whose easy proof we omit, can be obtained immediately from Proposition 7 when we assume that the aggregation function is monotonic.

Corollary 10. *Let $n \in \mathbb{N}$ and let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of t-norms. If $F : [0, 1]^n \to [0, 1]$ is a monotonic function that transforms $\mathcal{T}$-transitive fuzzy relations into a T-transitive fuzzy relation, then F satisfies*

$$T(F(a), F(b)) \leq F(\min\{a_1, b_1\}, \ldots, \min\{a_n, b_n\}) \leq F(a + b)$$

for all $a, b \in [0, 1]^n$ such that $a + b \in [0, 1]^n$.

Next we provide a converse of Proposition 7.

Proposition 11. *Let $n \in \mathbb{N}$ and let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of t-norms. If $F : [0, 1]^n \to [0, 1]$ is a monotonic function which dominates a t-norm T with respect to $\mathcal{T}$, then F aggregates $\mathcal{T}$-transitive fuzzy relations into a T-transitive fuzzy relation.*

Proof. Consider a collection $\{R_i\}_{i=1}^n$ of $\mathcal{T}$-transitive fuzzy relations on a non-empty set X. Then

$$R_i(x, y) \geq T_i(R_i(x, z), R_i(z, y))$$

for all $i = 1, \ldots, n$ and for all $x, y, z \in X$. We have to show that $F(R_1, \ldots, R_n)$ is a T-transitive fuzzy relation on X. Indeed,

$$\begin{aligned} T(F(R_1, \ldots, R_n)(x, z), F(R_1, \ldots, R_n)(z, y)) &= \\ T(F(R_1(x, z), \ldots, R_n(x, z)), F(R_1(z, y), \ldots, R_n(z, y))) &\leq \\ F(T_1(R_1(x, z), R_1(z, y)), \ldots, T_n(R_n(x, z), R_n(z, y))). \end{aligned}$$

Since F is monotonic we obtain that

$$F(T_1(R_1(x, z), R_1(z, y)), \ldots, T_n(R_n(x, z), R_n(z, y))) \leq F(R_1(x, y), \ldots, R_n(x, y)).$$

Thus we deduce that

$$T(F(R_1, \ldots, R_n)(x, z), F(R_1, \ldots, R_n)(z, y)) \leq F(R_1, \ldots, R_n)(x, y)$$

and, hence that $F(R_1, \ldots, R_n)$ is a T-transitive fuzzy relation on X. □

In the light of Proposition 11, as a natural question one can wonder whether the converse of Proposition 11 is always true. However, the next example shows that there are functions that aggregate transitive fuzzy relations which are not monotonic.

Example 12. Define the function $F : [0,1]^2 \to [0,1]$ by

$$F(a) = \begin{cases} \frac{1}{4} & \text{if } a = (\frac{1}{2}, \frac{1}{2}) \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

for all $a \in [0,1]^2$. Consider the collection $\mathcal{T} = \{T_i\}_{i=1}^2$ of t-norms with $T_i = T_p$ for $i = 1, 2$. It is easy to check that F aggregates $\mathcal{T}$-transitive fuzzy relations into a T_D-transitive fuzzy relation, where T_D stands for the drastic t-norm. Clearly F satisfies that

$$T_D(F(a), F(b)) \leq F(T_P(a_1, b_1), T_P(a_2, b_2))$$

for all $a \in [0,1]^2$. Nevertheless, F is not monotonic, since $F(\frac{1}{2}, \frac{1}{2}) \leq F(0,0)$.

When the collection of t-norms $\mathcal{T}$ satisfies that $T_i = T$ for all $i = 1, \ldots, n$, then Proposition 11 retrieves as a particular case Theorem 3.

Corollary 13. *Let T be a t-norm. If a function $F : [0,1]^n \to [0,1]$ is monotonic and dominates the t-norm T, then F aggregates T-transitive fuzzy relations.*

Considering Propositions 7 and 11 provide only a partial description about the functions in which we are interested, we state the relationship between the properties assumed in the statement of the aforesaid results and the transformation of triangle triplets in the spirit of Theorem 5. We make this with the aim of going one step further towards the promised characterization. To this end, we need to introduce new triangle triplet notions that extend the concept given in [14].

Definition 14. Given a collection of t-norms $\mathcal{T} = \{T_i\}_{i=1}^n$, a triplet $(a, b, c) \in [0,1]^n$ is said to be:

1) an n-dimensional asymmetric $\mathcal{T}$-triangular triplet whenever $T_i(a_i, b_i) \leq c_i$ for all $i = 1, \ldots, n$.
2) an n-dimensional $\mathcal{T}$-triangular triplet $T_i(a_i, b_i) \leq c_i$, $T_i(a_i, c_i) \leq b_i$ and $T_i(b_i, c_i) \leq a_i$ for all $i = 1, \ldots, n$.

As illustrative example of the both introduced notions, we have that $(a, b, c) \in [0,1]^n$ with $a_i = R_i(x, z)$, $b_i = R_i(z, y)$ and $c_i = R_i(x, y)$ is always an n-dimensional $\mathcal{T}$-triangular triplet provided that $\{R_i\}_{i=1}^n$ is a transitive collection of fuzzy binary relations with respect to a collection $\mathcal{T} = \{T_i\}_{i=1}^n$ of t-norms. Clearly every n-dimensional $\mathcal{T}$-triangular triplet is always an n-dimensional asymmetric $\mathcal{T}$-triangular triplet. The converse is not true in general. Indeed, $(a, b, 1_n) \in [0,1]^n$ is always an n-dimensional asymmetric $\mathcal{T}$-triangular triplet for every collection $\mathcal{T}$. However, it is an n-dimensional $\mathcal{T}$-triangular triplet if and only if $a = b$ ($a_i = b_i$ for all $i = 1, \ldots, n$).

The reason for which we have included "asymmetric" in the name of the new type of triangular triplets introduced in assertion 1) in Definition 14 will be clarified in Section 3.

Note that when all the t-norms belonging to $\mathcal{T} = \{T_i\}_{i=1}^n$ satisfy that $T_i = T$, for any t-norm T, for all $i = 1, \ldots, n$ then assertion 2) in the previous definition retrieves the notion of n-dimensional T-triangular triplet.

The link between all notions exposed so far is given in the next result.

Proposition 15. *Let $n \in \mathbb{N}$ and let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of t-norms. If T is a t-norm and $F : [0,1]^n \to [0,1]$ is a function, then among the below assertions, $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$:*

1) F is monotonic and dominates the t-norm T with respect to $\mathcal{T}$.
2) F aggregates $\mathcal{T}$-transitive fuzzy relations into a T-transitive fuzzy relation.
3) F transforms n-dimensional asymmetric $\mathcal{T}$-triangular triplets into 1-dimensional asymmetric T-triangular triplets.
4) F transforms n-dimensional $\mathcal{T}$-triangular triplets into 1-dimensional asymmetric T-triangular triplets.
5) F dominates the t-norm T with respect to $\mathcal{T}$.

Proof. 1) $\Rightarrow$ 2). By Proposition 11 we have that F aggregates $\mathcal{T}$-transitive fuzzy relations into a T-transitive fuzzy relation.

2) $\Rightarrow$ 3). Suppose that $(a, b, c) \in [0, 1]^n$ is an asymmetric $\mathcal{T}$-triangular triplet. Fix $X = \{x, y\}$ with x, y different. Consider the collection of fuzzy binary relations $\{R_i\}_{i=1}^n$ on X given by $R_i(x, y) = a_i$, $R_i(y, x) = b_i$, $R_i(x, x) = R_i(y, y) = c_i$, for all $i = 1, \ldots, n$. Since $(a, b, c) \in [0, 1]^n$ is an asymmetric $\mathcal{T}$-triangular triplet we deduce easily that $\{R_i\}_{i=1}^n$ is a collection of $\mathcal{T}$-transitive fuzzy binary relations on X. Let us prove that $(F(a), F(b), F(c))$ is an asymmetric T-triangular triplet. The fact that F aggregates $\mathcal{T}$-transitive fuzzy relations into a T-transitive fuzzy relation, guarantees that $F(R_1, \ldots, R_n)$ is a T-transitive fuzzy binary relation on X. Thus

$$T(F(R_1, \ldots, R_n)(x, y), F(R_1, \ldots, R_n)(y, x)) \leq F(R_1, \ldots, R_n)(x, x).$$

It follows that

$$\begin{aligned} &T(F(a), F(b)) &= \\ &T(F(R_1, \ldots, R_n)(x, y), F(R_1, \ldots, R_n)(y, x)) &\leq \\ &F(R_1, \ldots, R_m)(x, x) = F(c). \end{aligned}$$

Whence we conclude that $(F(a), F(b), F(c))$ is an asymmetric T-triangular triplet.

3) $\Rightarrow$ 4). It is straightforward.

4) $\Rightarrow$ 5). Let $a, b \in [0, 1]^n$. Clearly, (a, b, c) is a $\mathcal{T}$-triangle triplet with $c_i = T_i(a_i, b_i)$ for all $i = 1, \ldots, n$. Since F transforms n-dimensional $\mathcal{T}$-triangular triplets into 1-dimensional T-triangular triplets we have that $(F(a), F(b), F(c))$ is a 1-dimensional T-triplet. Therefore

$$T(F(a), F(b)) \leq F(c) = F(T_1(a_1, b_1), \ldots, T_n(a_n, b_n)),$$

as claimed. □

From Proposition 15 we immediately obtain the following result.

Corollary 16. *Let T be a t-norm, $n \in \mathbb{N}$ and let $F : [0, 1]^n \to [0, 1]$ be a function. Then, among the below assertions,* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5)*:*

1) F is monotonic and dominates the t-norm T.
2) F aggregates T-transitive fuzzy relations.
3) F transforms asymmetric n-dimensional T-triangular triplets into 1-dimensional asymmetric T-triangular triplets.
4) F transforms n-dimensional T-triangular triplets into 1-dimensional T-triangular triplets.
5) F dominates the t-norm T.

Of course Proposition 15 gives a characterization of those functions that aggregate $\mathcal{T}$-transitive fuzzy relations into a T-transitive fuzzy relation and they are, in addition, monotonic. Observe that Corollary 16 retrieves the celebrated Theorem 4 under the hypothesis of monotonicity and, in addition, improves it.

Example 12 shows that there are functions satisfying the condition in the assertion 5) in Proposition 15 which are not monotonic. In the light of this handicap, we clarify which are the functions that aggregate $\mathcal{T}$-transitive fuzzy relations into a T-transitive fuzzy relation in the result below.

Theorem 17. *Let $n \in \mathbb{N}$ and let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of t-norms. If T is a t-norm and $F : [0, 1]^n \to [0, 1]$ is a function, then the following assertions are equivalent:*

1) F aggregates $\mathcal{T}$-transitive fuzzy relations into a T-transitive fuzzy relation.
2) F transforms n-dimensional asymmetric $\mathcal{T}$-triangular triplets into 1-dimensional asymmetric T-triangular triplets.

Proof. By Proposition 15 we have that 1) $\Rightarrow$ 2). Next we show 2) $\Rightarrow$ 1). With this aim we assume that $\{R_i\}_{i=1}^n$ is a collection of $\mathcal{T}$-transitive fuzzy binary relations on a non-empty set X. Then, for any $x, y, z \in X$, we have that

$$R_i(u, v) \geq T_i(R_i(u, w), R_i(w, v))$$

for all $i = 1, \ldots, n$ and for all $u, v, w \in \{x, y, z\}$. Next we show that the binary fuzzy relation $F(R_1, \ldots, R_n)$ is T-transitive. To this end, we only show that

$$T(F(R_1, \ldots, R_n)(x, y), F(R_1, \ldots, R_n)(y, z)) \leq F(R_1, \ldots, R_n)(x, z).$$

The remainder inequalities can be proved following similar arguments. Set $a, b, c \in [0, 1]^n$ such that $a_i = R_i(x, y), b_i = R_i(y, z), c_i = R_i(x, z)$ for all $i = 1, \ldots, n$. It follows that (a, b, c) forms an n-dimensional asymmetric $\mathcal{T}$-triangle triplet. The fact that F transforms n-dimensional $\mathcal{T}$-triangular triplets into 1-dimensional T-triangular triplets provides that $(F(a), F(b), F(c))$ forms a 1-dimensional T-triangular triplet. Whence we deduce that $T(F(a), F(b)) \leq F(c)$ and, therefore, that

$$T(F(R_1, \ldots, R_n)(x, y), F(R_1, \ldots, R_n)(y, z)) = \\ T(F(a), F(b)) \leq F(c) = F(R_1, \ldots, R_n)(x, z).$$

So the binary fuzzy relation $F(R_1, \ldots, R_n)$ is T-transitive. Consequently, we have that F aggregates $\mathcal{T}$-transitive fuzzy relations into a T-transitive fuzzy relation. □

In view of the preceding result we can obtain the next characterization which describes those functions that aggregate T-transitive fuzzy relations and, in addition, improves the partial description obtained in Corollaries 9 and 13.

Corollary 18. *Let T be a t-norm and let $n \in \mathbb{N}$. If $F : [0, 1]^n \to [0, 1]$ is a function, then the following assertions are equivalent:*

1) *F aggregates T-transitive fuzzy relations.*
2) *F transforms n-dimensional asymmetric T-triangular triplets into 1-dimensional T-triangular triplets.*

We devote the end of this section to the case in which the t-norm T is exactly the T_M. Concretely, we obtain the following interesting results.

Proposition 19. *Let $n \in \mathbb{N}$ and let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of t-norms. If $F : [0, 1]^n \to [0, 1]$ is a function that aggregates $\mathcal{T}$-transitive fuzzy relations into a T_M-transitive fuzzy relation, then F is monotonic.*

Proof. Let $a, b \in [0, 1]^n$ such that $a \preceq b$. Consider a set $X = \{x, y\}$ with x, y different. Define the collection of fuzzy binary relations $\{R_i\}_{i=1}^n$ on X by $R_i(x, y) = R_i(y, x) = a_i$ and $R_i(x, x) = R_i(y, y) = b_i$ for all $i = 1, \ldots, n$. Then the collection $\{R_i\}_{i=1}^n$ is $\mathcal{T}$-transitive. We obtain that $F(R_1, \ldots, R_n)$ is a T_M-transitive fuzzy binary relation on X. So we deduce that

$$F(a) = T_M(F(a), F(a)) = \\ T_M(F(R_1(x, y), \ldots, R_n(x, y))), F(R_1(y, x), \ldots, R_n(y, x))) \leq \\ F(R_1(y, y), \ldots, R_n(y, y)) = F(b).$$

Whence we deduce that $F(a) \leq F(b)$. From the preceding fact we conclude that F is monotonic. □

From the above result we can state the next one.

Corollary 20. *Let $n \in \mathbb{N}$ and let $F : [0, 1]^n \to [0, 1]$ be a function. If F aggregates T_M-transitive fuzzy relations, then it is monotonic.*

The reciprocal of Proposition 19 and Corollary 20 does not hold such as the following example shows.

Example 21. Consider $x, y, z \in [0, 1]$ such that x, y, z are different and set $X = \{x, y, z\}$. Define the collection of fuzzy binary relations $(R_i)_{i=1}^3$ on X as follows: $R_1(x, y) = R_1(x, z) = 0.2$, $R_1(z, y) = 0.4$, $R_2(x, y) = R_2(z, y) = 0.2$,

$R_2(x,z) = 0.4$, and $R_3(x,y) = R_3(x,z) = 0.2$, $R_3(z,y) = 0.4$ and, for all $i = 1,2,3$, $R_i(x,y) = R_i(y,x)$, $R_i(x,z) = R_i(z,x)$, $R_i(z,y) = R_i(y,z)$ and $R_i(x,x) = R_i(y,y) = R_i(z,z) = 0.4$.

It is not hard to check that $\{R_i\}_{i=1}^3$ is a collection of fuzzy binary relations which is transitive with respect to any collection $\mathcal{T} = \{T_i\}_{i=1}^3$ of t-norms because it is transitive with respect to the collection $\mathcal{T}_M = \{T_i\}_{i=1}^3$, where $T_i \in \mathcal{T}_M \Leftrightarrow T_i = T_M$. Consider the monotonic function F introduced in Example 8. Then it does not aggregate $\mathcal{T}$-transitive fuzzy relations into a T_M-transitive fuzzy relation, since

$$F(R_1(x,y), R_2(x,y), R_3(x,y)) = 0.24 < T_M(F(R_1(x,z), R_2(x,z), R_3(x,z)), F(R_1(z,y), R_2(z,y), R_3(z,y))) = T_M(0.28, 0.48) = 0.28.$$

As a consequence of Propositions 15 and 19 we deduce the characterization below.

Theorem 22. *Let $n \in \mathbb{N}$ and let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of t-norms. If $F : [0,1]^n \to [0,1]$ is a function, then the following assertions are equivalent:*

1) F is monotonic and dominates the t-norm T_M with respect to $\mathcal{T}$.
2) F aggregates $\mathcal{T}$-transitive fuzzy relations into a T_M-transitive fuzzy relation.
3) F transforms n-dimensional asymmetric $\mathcal{T}$-triangular triplets into 1-dimensional asymmetric T_M-triangular triplets.
4) F transforms n-dimensional $\mathcal{T}$-triangular triplets into 1-dimensional asymmetric T_M-triangular triplets.

Proof. The unique thing to prove is the monotonicity of F provided that it transforms n-dimensional $\mathcal{T}$-triangular triplets into 1-dimensional asymmetric T_M-triangular triplets. To show this, consider $a, b \in [0,1]^n$ with $a \preceq b$. Clearly (a,a,b) is a $\mathcal{T}$-triangular triplet. Then $(F(a), F(a), F(b))$ is a T_M-triangular triplet. Thus $F(a) = T_M(F(a), F(a)) \leq F(b)$. So F is monotonic. □

From Theorem 22 the next result can be deduced.

Corollary 23. *Let $n \in \mathbb{N}$. If $F : [0,1]^n \to [0,1]$ is a function, then the following assertions are equivalent:*

1) F is monotonic and dominates the t-norm T_M.
2) F aggregates T_M-transitive fuzzy relations.
3) F transforms n-dimensional asymmetric T_M-triangular triplets into 1-dimensional asymmetric T_M-triangular triplets.
4) F transforms n-dimensional T_M-triangular triplets into 1-dimensional asymmetric T_M-triangular triplets.

3. Aggregation of transitive fuzzy binary relations: special cases

In this section we describe the functions that aggregate a few special classes of transitive fuzzy binary relations that are relevant in the literature. With this aim, we recall that a fuzzy binary relation R on a non-empty set X is said to be:

1. a T-fuzzy pre-order if $R(x,x) = 1$ for all $x \in X$ and it is T-transitive [7];
2. a T-fuzzy partial order if it is a T-fuzzy pre-order such that $R(x,y) = R(y,x) = 1 \Rightarrow x = y$ [12];
3. a T-relaxed indistinguishability relation if it is symmetric and T-transitive [6];
4. a SSI-T-relaxed indistinguishability relation if $R(x,y) \leq R(x,x)$ for all $x, y \in X$ [6] (where SSI stands for small self-indistinguishability);
5. a T-indistinguishability relation if it is a T-relaxed indistinguishability relation such that $R(x,x) = 1$ for all $x \in X$ [20,24];
6. a T-equality if it is a T-indistinguishability relation such that $R(x,y) = 1 \Rightarrow x = y$ [11].

It must be pointed out that a T-fuzzy pre-order is called T-pre-order in [24] and partial T-pre-order or T-quasiorder in [13]. A T-fuzzy partial order is called T-order in and [7]. Moreover, a fuzzy binary relation R on a non-empty set X

satisfying $R(x, y) \leq R(x, x)$ for all $x, y \in X$ is known as weakly reflexive (see [13]). So SSI-T-relaxed indistinguishability relations are T-relaxed indistinguishability relations that are weakly reflexive. Besides, T-indistinguishability relations are called T-equivalences in [11]. Furthermore, T-equalities are called T-indistinguishability relations that separate points in [20].

The introduced notions can be extended to the case in which a collection of fuzzy binary relations is under consideration in the following way. Given a collection $\mathcal{T} = \{T_i\}_{i=1}^n$ of t-norms, a collection $\{R_i\}_{i=1}^n$ of fuzzy relations on a non-empty set X is said to be:

1. a collection of $\mathcal{T}$-fuzzy pre-orders if it is $\mathcal{T}$-transitive and each R_i satisfies that $R_i(x, x) = 1$ for all $x \in X$;
2. a collection of $\mathcal{T}$-fuzzy partial orders if it is a collection of $\mathcal{T}$-fuzzy pre-orders such that $R_i(x, y) = R_i(y, x) = 1 \Rightarrow x = y$ for all $i = 1, \ldots, n$;
3. a collection of $\mathcal{T}$-relaxed indistinguishability relations if it is $\mathcal{T}$-transitive and R_i is symmetric for all $i = 1, \ldots, n$;
4. a collection of SSI-$\mathcal{T}$-relaxed indistinguishability relations if it is a collection of $\mathcal{T}$-relaxed indistinguishability relations such that each R_i fulfills that $R_i(x, y) \leq R_i(x, x)$ for all $x, y \in X$;
5. a collection of $\mathcal{T}$-indistinguishability relations if it is a collection of $\mathcal{T}$-relaxed indistinguishability relations such that $R_i(x, x) = 1$ for all $x \in X$ and for all $i = 1, \ldots, n$;
6. a collection of $\mathcal{T}$-equalities if it is a collection of $\mathcal{T}$-indistinguishability relations such that $R_i(x, y) = 1 \Rightarrow x = y$ for all $i = 1, \ldots, n$.

In the light of the preceding notions (different classes of fuzzy binary relations), and considering the exposed theory, we discuss those properties that a function must satisfy in order to aggregate the previous classes of transitive fuzzy binary relations. The next notions will play a crucial role in order to achieve our aim.

Definition 24. From now on, given a collection of t-norms $\mathcal{T} = \{T_i\}_{i=1}^n$, a collection $\{R_i\}_{i=1}^n$ of $\mathcal{T}$-transitive fuzzy relations on a non-empty set X is said to belong to the class C provided that R_i belongs to the class C for each $i = 1, \ldots, n$.

Definition 25. Given a t-norm T, a function $F : [0, 1]^n \to [0, 1]$ aggregates $\mathcal{T}$-transitive fuzzy relations belonging to the class C into a T-transitive fuzzy binary relation in the class C provided that $F(R_1, \ldots, R_n)$ is a T-transitive fuzzy relation on a non-empty set X belonging to the class C when $\{R_i\}_{i=1}^n$ is a collection of $\mathcal{T}$-transitive fuzzy relations on X which belongs to the class C.

The required notion of aggregation for each of the classes of fuzzy binary relations listed above can be stated formally following Definition 25.

First we discuss the class of fuzzy pre-orders.

Theorem 26. *Let $n \in \mathbb{N}$ and let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of t-norms. If T is a t-norm and $F : [0, 1]^n \to [0, 1]$ is a function, then the following assertions are equivalent:*

1) F aggregates $\mathcal{T}$-fuzzy pre-orders into a T-fuzzy pre-order.
2) F holds the following conditions:
  2.1) $F(1_n) = 1$.
  2.2) F transforms n-dimensional asymmetric $\mathcal{T}$-triangular triplets into 1-dimensional asymmetric T-triangular triplets.

Proof. 1) $\Rightarrow$ 2). First we show that $F(1_n) = 1$. To this end, consider a non-empty set X and define the collection of fuzzy binary relations $\{R_i\}_{i=1}^n$ on X by $R_i(x, y) = 1$ for all $x, y \in X$ and for all $i = 1, \ldots, n$. Then $(R_i)_{i=1}^n$ is a collection of $\mathcal{T}$-fuzzy pre-orders on X. So $F(R_1, \ldots, R_n)$ is a T-fuzzy pre-order on X. Hence $F(1_n) = F(R_1(x, x), \ldots, R_n(x, x)) = F(R_1, \ldots, R_n)(x, x) = 1$.

It remains to prove that $(F(a), F(b), F(c))$ is an asymmetric T-triangular triplet whenever $(a, b, c) \in [0, 1]^n$ is an asymmetric $\mathcal{T}$-triangular triplet. With this aim, fix $X = \{x, y, z\}$ with x, y, z different. Consider the collection of fuzzy binary relations $\{R_i\}_{i=1}^n$ on X given by $a_i = R_i(x,y), b_i = R_i(y,z), c_i = R_i(x,z)$ and $R_i(y,x) = R_i(z,y) = R_i(z,x) = R_i(x,x) = R_i(y,y) = R_i(z,z) = 1$ for all $i = 1, \ldots, n$. Since $(a,b,c) \in [0,1]^n$ is an asymmetric $\mathcal{T}$-triangular triplet it is not difficult to prove that $\{R_i\}_{i=1}^n$ is a collection of $\mathcal{T}$-fuzzy pre-orders. The fuzzy binary relation $F(R_1, \ldots, R_n)$ is a T-fuzzy pre-order on X, since F aggregates $\mathcal{T}$-fuzzy pre-orders into a T-fuzzy pre-order. Clearly $T(F(a), F(b)) \leq F(c)$ because $F(R_1, \ldots, R_n)$ is a transitive fuzzy binary relation on X and, thus,

$$\begin{aligned} &T(F(a), F(b)) &= \\ &T(F(R_1,\ldots,R_n)(x,y), F(R_1,\ldots,R_n)(y,z)) &\leq \\ &F(R_1,\ldots,R_m)(x,z) = F(c). & \end{aligned}$$

2) ⇒ 1). Consider a collection $\{R_i\}_{i=1}^n$ of $\mathcal{T}$-fuzzy pre-orders on a non-empty set X. We have to show that $F(R_1, \ldots, R_n)$ is a T-fuzzy pre-order on X. On the one hand, we have that $\{R_i\}_{i=1}^n$ is $\mathcal{T}$-transitive. By Theorem 17 we have that the fuzzy binary relation $F(R_1, \ldots, R_n)$ is T-transitive. On the other hand, $F(R_1, \ldots, R_n)(x,x) = F(R_1(x,x), \ldots, R_n(x,x)) = F(1_n) = 1$ for all $x \in X$. Consequently, we deduce that $F(R_1, \ldots, R_n)$ is a T-fuzzy pre-order on X. □

In view of the preceding result we have the next one.

Corollary 27. *Let $n \in \mathbb{N}$. If T is a t-norm and $F : [0,1]^n \to [0,1]$ is a function, then the following assertions are equivalent:*

1) F aggregates T-fuzzy pre-orders.
2) F holds the following conditions:
 2.1) $F(1_n) = 1$.
 2.2) F transforms n-dimensional asymmetric T-triangular triplets into 1-dimensional asymmetric T-triangular triplets.

In the case of fuzzy partial orders we have the following characterization.

Theorem 28. *Let $n \in \mathbb{N}$ and let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of t-norms. If T is a t-norm and $F : [0,1]^n \to [0,1]$ is a function, then the following assertions are equivalent:*

1) F aggregates $\mathcal{T}$-fuzzy partial orders into a T-fuzzy partial order.
2) F holds the following conditions:
 2.1) $F(1_n) = 1$;
 2.2) Let $a, b \in [0,1]^n$. If $F(a) = F(b) = 1$, then there exists $i \in \{1 \ldots, n\}$ such that $a_i = b_i = 1$.
 2.3) F transforms n-dimensional asymmetric $\mathcal{T}$-triangular triplets into 1-dimensional asymmetric T-triangular triplets.

Proof. 1) ⇒ 2). In order to prove that $F(1_n) = 1$, consider a non-empty set X and define the collection of fuzzy binary relations $\{R_i\}_{i=1}^n$ on X by

$$R_i(x,y) = \begin{cases} \frac{1}{2} & \text{if } x \neq y \\ 1 & \text{if } x = y \end{cases}$$

for all $i = 1, \ldots, n$. Then $\{R_i\}_{i=1}^n$ is a collection of $\mathcal{T}$-fuzzy partial orders on X. So $F(R_1, \ldots, R_n)$ is a T-fuzzy partial order on X. Hence $F(1_n) = F(R_1(x,x), \ldots, R_n(x,x)) = F(R_1, \ldots, R_n)(x,x) = 1$.

Next assume that $a, b \in [0,1]^n$ and that $F(a) = F(b) = 1$. For purpose of contradiction suppose that $a_i, b_i < 1$ for all $i = 1, \ldots, n$. Consider a non-empty set $X = \{x, y\}$ with x, y different. Define the collection of fuzzy binary relations $\{R_i\}_{i=1}^n$ on X given by $a_i = R_i(x,y), b_i = R_i(y,x)$ and $R_i(x,x) = R_i(y,y) = 1$ for all $i = 1, \ldots, n$. Then $\{R_i\}_{i=1}^n$ is a collection of $\mathcal{T}$-fuzzy partial orders on X. It follows that $F(R_1, \ldots, R_n)$ is a T-fuzzy partial order on X. Moreover, we have that $F(R_1, \ldots, R_n)(x,y) = F(a) = 1 = F(b) = F(R_1, \ldots, R_n)(y,x)$. Whence we deduce that $x = y$, which is a contradiction.

It remains to show that F transforms n-dimensional asymmetric $\mathcal{T}$-triangular triplets into 1-dimensional asymmetric T-triangular triplets. Assume that $(a, b, c) \in [0, 1]^n$ is an asymmetric $\mathcal{T}$-triangular triplet. Fix $X = \{x, y, z\}$ with x, y, z different. Consider the collection of fuzzy binary relations $\{R_i\}_{i=1}^n$ on X given by $a_i = R_i(x, y), b_i = R_i(y, z), c_i = R_i(x, z)$, $R_i(y, x) = R_i(z, y) = R_i(z, x) = \frac{1}{2}$ and $R_i(x, x) = R_i(y, y) = R_i(z, z) = 1$ for all $i = 1, \dots, n$. A straightforward computation shows that $\{R_i\}_{i=1}^n$ is a collection of $\mathcal{T}$-fuzzy partial orders. Then the fuzzy binary relation $F(R_1, \dots, R_n)$ is a T-fuzzy partial order on X, since F aggregates $\mathcal{T}$-fuzzy pre-orders into a T-fuzzy pre-order. Clearly $T(F(a), F(b)) \leq F(c)$ because $F(R_1, \dots, R_n)$ is a transitive fuzzy binary relation on X and, thus,

$$\begin{aligned} &T(F(a), F(b)) && = \\ &T(F(R_1, \dots, R_n)(x, y), F(R_1, \dots, R_n)(y, z)) && \leq \\ &F(R_1, \dots, R_m)(x, z) = F(c). \end{aligned}$$

2) $\Rightarrow$ 1). Consider a collection $\{R_i\}_{i=1}^n$ of $\mathcal{T}$-fuzzy partial orders on a non-empty set X. By Theorem 26 we have that $F(R_1, \dots, R_n)$ is a T-fuzzy pre-order on X. Next assume that, for any $x, y \in X$, $F(R_1, \dots, R_n)(x, y) = F(R_1, \dots, R_n)(y, x) = 1$. Then there exists $i \in \{1, \dots, n\}$ such that $R_i(x, y) = R_i(y, x) = 1$. Since R_i is a T_i-fuzzy partial order on X we have that $x = y$. So $F(R_1, \dots, R_n)$ is a T-fuzzy partial order on X. □

The preceding result retrieves the next one as a particular case.

Corollary 29. *Let $n \in \mathbb{N}$. If T is a t-norm and $F : [0, 1]^n \to [0, 1]$ is a function, then the following assertions are equivalent:*

1) F aggregates T-fuzzy partial orders.
2) F holds the following conditions:
  2.1) $F(1_n) = 1$.
  2.2) Let $a, b \in [0, 1]^n$. If $F(a) = F(b) = 1$, then there exists $i \in \{1 \dots, n\}$ such that $a_i = b_i = 1$.
  2.3) F transforms n-dimensional asymmetric T-triangular triplets into 1-dimensional asymmetric T-triangular triplets.

At this point, we should stress that, on the one hand, the fact that the notion of n-dimensional asymmetric $\mathcal{T}$-triangular triplet plays a central role in the characterization of those functions that aggregate $\mathcal{T}$-fuzzy pre-orders (partial orders) into a T-fuzzy pre-order (partial order) and, on the other hand, the fact that these fuzzy binary relations are not symmetric make the choice of the name clearer.

Next we treat the class of relaxed indistinguishability relations.

Theorem 30. *Let $n \in \mathbb{N}$ and let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of t-norms. If T is a t-norm and $F : [0, 1]^n \to [0, 1]$ is a function, then the following assertions are equivalent:*

1) F aggregates $\mathcal{T}$-relaxed indistinguishability relations into a T-relaxed indistinguishability relation.
2) F transforms n-dimensional $\mathcal{T}$-triangular triplets into 1-dimensional T-triangular triplets.

Proof. 1) $\Rightarrow$ 2). Assume that $(a, b, c) \in [0, 1]^n$ is a $\mathcal{T}$-triangular triplet. Consider a set $X = \{x, y, z\}$ with x, y, z different. Consider the collection of fuzzy binary relations $\{R_i\}_{i=1}^n$ on X given by $a_i = R_i(x, y) = R_i(y, x), b_i = R_i(y, z) = R_i(z, y), c_i = R_i(x, z) = R_i(z, x)$ and $R_i(x, x) = R_i(y, y) = R_i(z, z) = 1$ for all $i = 1, \dots, n$. Then $\{R_i\}_{i=1}^n$ is a collection of $\mathcal{T}$-relaxed indistinguishability relations on X. So $F(R_1, \dots, R_n)$ is a T-relaxed indistinguishability relation on X. Thus we deduce from the T-transitivity of $F(R_1, \dots, R_n)$ that $T(F(a), F(b)) \leq F(c)$, $T(F(a), F(c)) \leq F(b)$, and $T(F(a), F(b)) \leq F(c)$. Whence we conclude that $(F(a), F(b), F(c))$ is a 1-dimensional T-triangular triplet.

2) $\Rightarrow$ 1). Assume that $\{R_i\}_{i=1}^n$ is a collection of $\mathcal{T}$-relaxed indistinguishability relations on a non-empty set X. For each $x, y, z \in X$ set $a, b, c \in [0, 1]^n$ such that $a_i = R_i(x, y), b_i = R_i(y, z), c_i = R_i(x, z)$ for all $i = 1, \dots, n$. The fact that $R_i(x, y) = R_i(y, x)$ for all $x, y \in X$ gives that $F(R_1, \dots, R_n)(x, y) = F(R_1, \dots, R_n)(y, x)$. Hence

the fuzzy binary relation $F(R_1, \ldots, R_n)$ is symmetric. Moreover we have that (a, b, c) forms an n-dimensional $\mathcal{T}$-triangle triplet, since $\{R_i\}_{i=1}^n$ is a collection of $\mathcal{T}$-relaxed indistinguishability relations. The fact that F transforms n-dimensional $\mathcal{T}$-triangular triplets into 1-dimensional T-triangular triplets provides that $(F(a), F(b), F(c))$ forms a 1-dimensional T-triangular triplet. Whence we deduce that $T(F(a), F(b)) \leq F(c)$, $T(F(b), F(c)) \leq F(a)$ and that $T(F(a), F(c)) \leq F(b)$. So we have that

$$T(F(R_1, \ldots, R_n)(u, v), F(R_1, \ldots, R_n)(v, w)) \leq F(R_1, \ldots, R_n)(u, w)$$

for all $u, v, w \in \{x, y, z\}$. Hence the binary fuzzy relation $F(R_1, \ldots, R_n)$ is T-transitive.

Consequently, $F(R_1, \ldots, R_n)$ is a T-relaxed indistinguishability relation on X. Thus F aggregates $\mathcal{T}$-relaxed indistinguishability relations into a T-relaxed indistinguishability relation. □

As a corollary we get the result below.

Corollary 31. *Let $n \in \mathbb{N}$. If T is a t-norm and $F : [0, 1]^n \to [0, 1]$ is a function, then the following assertions are equivalent:*

1) F aggregates T-relaxed indistinguishability relations.
2) F transforms n-dimensional T-triangular triplets into 1-dimensional T-triangular triplets.

When we focus our attention on the class of SSI-relaxed indistinguishability relations, the following result can be stated.

Theorem 32. *Let $n \in \mathbb{N}$ and let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of t-norms. If T is a t-norm and $F : [0, 1]^n \to [0, 1]$ is a function, then the following assertions are equivalent:*

1) F aggregates SSI-$\mathcal{T}$-relaxed indistinguishability relations into a SSI-T-relaxed indistinguishability relation.
2) F holds the following conditions:
  2.1) F is monotonic.
  2.2) F transforms n-dimensional $\mathcal{T}$-triangular triplets into 1-dimensional T-triangular triplets.

Proof. 1) ⇒ 2). Assume that $(a, b, c) \in [0, 1]^n$ is a $\mathcal{T}$-triangular triplet. Consider a set $X = \{x, y, z\}$ with x, y, z different and the collection of fuzzy binary relations $\{R_i\}_{i=1}^n$ on X introduced in the proof of Theorem 30. Then $\{R_i\}_{i=1}^n$ is a collection of SSI-$\mathcal{T}$-relaxed indistinguishability relations on X. So $F(R_1, \ldots, R_n)$ is a SSI-T-relaxed indistinguishability relation on X. Hence, it is T-transitive. So $(F(a), F(b), F(c))$ is a 1-dimensional T-triangular triplet.

Next we show that F is monotonic. To this end, let $a, b \in [0, 1]^n$ such that $a \preceq b$. Consider a set $X = \{x, y\}$ with x, y different. Define the collection of fuzzy binary relations $\{R_i\}_{i=1}^n$ on X given by $R_i(x, y) = R_i(y, x) = a_i$ and $R_i(x, x) = R_i(y, y) = b_i$ for all $i = 1, \ldots, n$. Then $\{R_i\}_{i=1}^n$ is a collection of SSI-$\mathcal{T}$-relaxed indistinguishability relations on X. Then $F(R_1, \ldots, R_n)$ is SSI-T-relaxed indistinguishability relations on X. Thus we have that $F(a) = F(R_1, \ldots, R_n)(x, y) \leq F(R_1, \ldots, R_n)(x, x) = F(b)$. So F is monotonic.

2) ⇒ 1). Consider a collection $\{R_i\}_{i=1}^n$ of SSI-$\mathcal{T}$-relaxed indistinguishability relations. By Theorem 30 we have that $F(R_1, \ldots, R_n)$ is a $\mathcal{T}$-relaxed indistinguishability relation on X. Since $R_i(x, y) \leq R_i(x, x)$ for all $x, y \in X$ and for all $i = 1, \ldots, n$ and, in addition, F is monotonic we have that

$$F(R_1, \ldots, R_n)(x, y) \leq F(R_1, \ldots, R_n)(x, x)$$

for all $x, y \in X$ and, thus, that $F(R_1, \ldots, R_n)$ is a SSI-$\mathcal{T}$-relaxed indistinguishability relation on X. So we have shown that F aggregates SSI-$\mathcal{T}$-relaxed indistinguishability relations into a SSI-T-relaxed indistinguishability relation. □

The next result is an immediate consequence of the preceding one.

Corollary 33. *Let $n \in \mathbb{N}$. If T is a t-norm and $F : [0,1]^n \to [0,1]$ is a function, then the following assertions are equivalent:*

1) F aggregates SSI-T-relaxed indistinguishability relations.
2.1) F is monotonic.
2.2) F transforms n-dimensional T-triangular triplets into 1-dimensional T-triangular triplets.

The functions that aggregate indistinguishability relations can be characterized as follows.

Theorem 34. *Let $n \in \mathbb{N}$ and let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of t-norms. If T is a t-norm and $F : [0,1]^n \to [0,1]$ is a function, then the following assertions are equivalent:*

1) F aggregates $\mathcal{T}$-indistinguishability relations into a T-indistinguishability relation.
2) F holds the following conditions:
2.1) $F(1_n) = 1$.
2.2) F transforms n-dimensional $\mathcal{T}$-triangular triplets into 1-dimensional T-triangular triplets.

Proof. 1) $\Rightarrow$ 2). In order to show that F transforms n-dimensional $\mathcal{T}$-triangular triplets into 1-dimensional T-triangular triplets we can apply the same arguments as those given in 1) $\Rightarrow$ 2) in the proof of Theorem 32. It remains to prove that $F(1_n) = 1$. To achieve it, consider a non-empty set X and the collection of fuzzy binary relations $\{R_i\}_{i=1}^n$ on X introduced in the proof of Theorem 28. Then $(R_i)_{i=1}^n$ is a collection of $\mathcal{T}$-indistinguishability relations on X. Hence $F(R_1, \ldots, R_n)$ is T-indistinguishability relations on X. Whence we obtain that

$$F(1_n) = F(R_1(x,x), \ldots, R_n(x,x)) = F(R_1, \ldots, R_n)(x,x) = 1.$$

2) $\Rightarrow$ 1). Given a collection $\{R_i\}_{i=1}^n$ of $\mathcal{T}$-indistinguishability relations on a non-empty set X we have, by Theorem 30, that $F(R_1, \ldots, R_n)$ is a T-relaxed indistinguishability relation on X. We just need to prove that $F(R_1, \ldots, R_n)(x,x) = 1$ and, thus, that $F(R_1, \ldots, R_n)$ is a T-indistinguishability operator on X. Since $F(1_n) = 1$ we conclude that $F(R_1, \ldots, R_n)(x,x) = 1$ because $R_i(x,x) = 1$ for all $x \in X$ and for all $i = 1, \ldots, n$ and, hence, $F(R_1, \ldots, R_n)(x,x) = F(R_1(x,x), \ldots, R_n(x,x)) = F(1_n) = 1$. $\square$

Observe that Theorem 5 is an immediate consequence of the preceding one.

Corollary 35. *Let $n \in \mathbb{N}$. If T is a t-norm and $F : [0,1]^n \to [0,1]$ is a function, then the following assertions are equivalent:*

1) F aggregates T-indistinguishability relations.
2.1) $F(1_n) = 1$;
2.2) F transforms n-dimensional T-triangular triplets into 1-dimensional T-triangular triplets.

In the class of equalities the characterization is provided by the result below.

Theorem 36. *Let $n \in \mathbb{N}$ and let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of t-norms. If T is a t-norm and $F : [0,1]^n \to [0,1]$ is a function, then the following assertions are equivalent:*

1) F aggregates $\mathcal{T}$-equalities into a T-equality.
2) F holds the following conditions:
2.1) $F(1_n) = 1$.
2.2) Let $a \in [0,1]^n$. If $F(a) = 1$, then there exists $i \in \{1, \ldots, n\}$ such that $a_i = 1$.
2.3) If $a, b, c \in [0,1[^n$ such that (a,b,c) is an n-dimensional $\mathcal{T}$-triangular triplet, then $(F(a), F(b), F(c))$ is a 1-dimensional T-triangular triplet.

Proof. 1) ⇒ 2). In order to show that $F(1_n) = 1$, consider any collection $\{R_i\}_{i=1}^n$ of $\mathcal{T}$-equalities on a non-empty set X. Then $R_i(x, x) = 1$ for all $x \in X$ and for all $i = 1, \ldots, n$. Since F aggregates $\mathcal{T}$-equalities into a T-equality we have that $F(R_1, \ldots, R_n)$ is a T-equality on X. Whence we deduce that $F(R_1, \ldots, R_n)(x, x) = 1$. Then $F(1_n) = F(R_1(x, x), \ldots, R_n(x, x)) = F(R_1, \ldots, R_n)(x, x) = 1$.

Assume that there exists $a \in [0, 1]^n$ such that $F(a) = 1$. Suppose that $a_j \in [0, 1[$ for all $j = 1, \ldots, n$. Consider the non-empty set $X = \{x, y\}$ with x, y different. Define on X the collection $\{R_i\}_{i=1}^n$ of $\mathcal{T}$-equalities as follows: $R_i(x, y) = R_i(y, x) = a_i$ and $R_i(x, x) = R_i(y, y) = 1$ for all $i = 1, \ldots, n$. Then $F(R_1, \ldots, R_n)$ is a T-equality on X. Thus $F(R_1, \ldots, R_n)(x, y) = F(a) = 1$. It follows that $x = y$ which is a contradiction. Then there exists $i \in \{1, \ldots, n\}$ such that $a_i = 1$.

Let $(a, b, c) \in [0, 1[^n$ be a $\mathcal{T}$-triangular triplet. Consider a set $X = \{x, y, z\}$ with x, y, z different. Consider the collection of fuzzy binary relations $(R_i)_{i=1}^n$ on X introduced in the proof of Theorem 30. It is clear that $\{R_i\}_{i=1}^n$ is a collection of $\mathcal{T}$-equalities on X. So $F(R_1, \ldots, R_n)$ is a T-equality on X. By T-transitivity we deduce that $T(F(a), F(b)) \leq F(c)$, $T(F(a), F(c)) \leq F(b)$ and $T(F(a), F(b)) \leq F(c)$. Whence we conclude that $(F(a), F(b), F(c))$ is a 1-dimensional T-triangular triplet.

2) ⇒ 1). Consider a collection $\{R_i\}_{i=1}^n$ of $\mathcal{T}$-equalities on a non-empty set X. Clearly $F(R_1, \ldots, R_n)$ is a symmetric fuzzy binary relation on X. Take $x, y, z \in X$ and set $a_i = R_i(x, y)$, $b_i = R_i(y, z)$, $c_i = R_i(x, z)$ for all $i = 1, \ldots, n$. We distinguish two possible cases:

1. Case 1. There exists $i \in \{1, \ldots, n\}$ such that any element a_i, b_i, c_i is equal to 1. Suppose that $a_i = 1$. Then $x = y$ and $(a, b, c) \in [0, 1]^n$ matches up with the vector $(1_n, b, b) \in [0, 1]^n$. Clearly, the fact that $F(1_n) = 1$ gives that
$$T(F(R_1, \ldots, R_n)(s, t), F(R_1, \ldots, R_n)(t, v)) \leq F(R_1, \ldots, R_n)(s, v)$$
for all $s, t, v \in \{x, y, z\}$. Similar conclusions can be derived when we assume that either any $b_i = 1$ or any $c_i = 1$.
2. Case 2. $(a, b, c) \in [0, 1[^n$. Since R_i is T_i-transitive we have that $(a, b, c) \in [0, 1[^n$ is an n-dimensional $\mathcal{T}$-triangle triplet. Then $(F(a), F(b), F(c))$ is a 1-dimensional T-triangular triplet. Whence we have that
$$T(F(R_1, \ldots, R_n)(s, t), F(R_1, \ldots, R_n)(t, v)) \leq F(R_1, \ldots, R_n)(s, v)$$
for all $s, t, v \in \{x, y, z\}$.

We have shown that the fuzzy binary relation $F(R_1, \ldots, R_n)$ is T-transitive on X.

We only need to prove that $x = y$ provided that $F(R_1, \ldots, R_n)(x, y) = 1$. Assume that $F(R_1, \ldots, R_n)(x, y) = 1$ for any $x, y \in X$. So we have that $F(R_1(x, y), \ldots, R_n(x, y)) = 1$. Whence we deduce the existence of $i \in \{1, \ldots, n\}$ such that $R_i(x, y) = 1$. Since R_i is a T-equality on X we conclude that $x = y$.

Therefore we have proven that $F(R_1, \ldots, R_n)$ is a T-equality on X. □

The next result is an immediate consequence of the preceding one.

Corollary 37. *Let $n \in \mathbb{N}$. If T is a t-norm and $F : [0, 1]^n \to [0, 1]$ is a function, then the following assertions are equivalent:*

1) F aggregates T-equalities.
2) F holds the following conditions:
  2.1) $F(1_n) = 1$.
  2.2) Let $a \in [0, 1]^n$. If $F(a) = 1$, then there exists $i \in \{1, \ldots, n\}$ such that $a_i = 1$.
  2.3) If $a, b, c \in [0, 1[^n$ such that (a, b, c) is an n-dimensional T-triangular triplet, then $(F(a), F(b), F(c))$ is a 1-dimensional T-triangular triplet.

We end the paper paying special attention to the case in which the minimum t-norm is involved. In this direction we have the following surprising results.

Proposition 38. *Let $n \in \mathbb{N}$ and let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of t-norms. If $F : [0,1]^n \to [0,1]$ is a function that aggregates $\mathcal{T}$-relaxed indistinguishability relations into a T_M-relaxed indistinguishability relation, then F is monotonic.*

Proof. A reasoning similar to that given in the proof of Proposition 19 provides the conclusion. We only remark that the $\mathcal{T}$-transitive collection of fuzzy binary relations $\{R_i\}_{i=1}^n$ introduced in the proof of the aforementioned proposition is actually a collection of $\mathcal{T}$-relaxed indistinguishability relations on X. □

Corollary 39. *Let $n \in \mathbb{N}$ and let $F : [0,1]^n \to [0,1]$ be a function. If F aggregates T_M-relaxed indistinguishability relations, then it is monotonic.*

Again, the converse of Proposition 38 and Corollary 39 does not hold such as Example 21 shows.

Combining Theorems 22, 30 and 32 and Proposition 38, we get immediately the next characterization when the t-norm T_M is under consideration.

Theorem 40. *Let $n \in \mathbb{N}$ and let $\mathcal{T} = \{T_i\}_{i=1}^n$ be a collection of t-norms. If $F : [0,1]^n \to [0,1]$ is a function, then the next assertions are equivalent:*

1) F is monotonic and F dominates the t-norm T_M with respect to $\mathcal{T}$.
2) F aggregates $\mathcal{T}$-transitive fuzzy relations into a T_M-transitive fuzzy relation.
3) F transforms n-dimensional asymmetric $\mathcal{T}$-triangular triplets into 1-dimensional asymmetric T_M-triangular triplets.
4) F transforms n-dimensional $\mathcal{T}$-triangular triplets into 1-dimensional T_M-triangular triplets.
5) F aggregates $\mathcal{T}$-relaxed indistinguishability relations into a T_M-relaxed indistinguishability relations.
6) F aggregates SSI-$\mathcal{T}$-relaxed indistinguishability relations into a SSI-T_M-relaxed indistinguishability relation.

Proof. By Theorem 22 we have that 1) ⇔ 2) ⇔ 3) ⇔ 4). Theorem 30 gives 4) ⇔ 5). Proposition 38 gives the monotonicity of F and, thus, Theorem 32 gives 5) ⇔ 6). □

The next result can be derived directly from the preceding one. In order to state it observe that T_M-relaxed indistinguishability relations are exactly SSI-T_M-relaxed indistinguishability relations.

Corollary 41. *Let $n \in \mathbb{N}$. If $F : [0,1]^n \to [0,1]$ is a function, then the next assertions are equivalent:*

1) F is monotonic and F dominates the t-norm T_M.
2) F aggregates T_M-transitive fuzzy relations.
3) F transforms n-dimensional asymmetric T_M-triangular triplets into 1-dimensional asymmetric T_M-triangular triplets.
4) F transforms n-dimensional T_M-triangular triplets into 1-dimensional T_M-triangular triplets.
5) F aggregates T_M-relaxed indistinguishability relations into a T_M-relaxed indistinguishability relations.

4. Conclusions and further work

In this paper we have revisited the problem of aggregating a collection of fuzzy binary relations when all of them belong to a concrete class and, in addition, the aggregation process preserves such a class. In particular, we have focused on the characterization of those functions that aggregate a collection of transitive fuzzy binary relations with respect to a collection of t-norms in such a way that the transitivity is preserved. The aforementioned characterization has been provided in terms of triangular triplets. Moreover, we have studied when these functions satisfy a dominance property and monotonicity. We have also described in terms of triangular triplets the functions that aggregate a collection of fuzzy pre-orders, fuzzy partial orders, relaxed indistinguishability relations, indistinguishability relations and equalities. As a consequence, we have also obtained a surprising relationship between functions that aggregate transitive fuzzy relations into a T_M-transitive fuzzy relation and those that aggregate relaxed indistinguishability relations.

Finally, we have provided new results for those cases in which all the t-norms belonging to the collection are exactly the same in such a way that celebrated results are retrieved, as a particular case, from the exposed theory.

In [9,19], relationships are established between those functions that aggregate collections of indistinguishability relations with respect to a collection of continuous and Archimedean t-norms and those that aggregate a collection of extended pseudo-metrics. In this direction, as future work, we plan to study the relationships that exist between the characterizations obtained in the present paper and the corresponding characterizations of those functions that aggregate a collection of generalized metrics in the spirit of [15,16].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

This research was funded by FEDER/Ministerio de Ciencia, Innovación y Universidades-Agencia Estatal de Investigación/_Proyecto PGC2018-095709-B-C21, and by Spanish Ministry of Economy and Competitiveness under contract DPI2017-86372-C3-3-R (AEI, FEDER, UE). This work is also partially supported by Programa Operatiu FEDER 2014-2020 de les Illes Balears, by project PROCOE/4/2017 (Direcció General d'Innovació i Recerca, Govern de les Illes Balears) and by projects ROBINS and BUGWRIGHT2. These two latest projects have received funding from the European Union's Horizon 2020 research and innovation programme under grant agreements No 779776 and No. 871260, respectively. This publication reflects only the authors views and the European Union is not liable for any use that may be made of the information contained therein.

References

[1] G. Beliakov, A. Pradera, T. Calvo, Aggregation Functions: A Guide for Practitioners, Studies in Fuzziness and Soft Computing, Springer, Cham, 2007.
[2] J. Drewniak, U. Dudziak, Aggregation preserving classes of fuzzy relations, Kybernetika 41 (2005) 265–284.
[3] J. Drewniak, U. Dudziak, Preservation of properties of fuzzy relations during aggregation processes, Kybernetika 43 (2007) 115–132.
[4] U. Dudziak, Preservation of t-norm and t-conorm based properties of fuzzy relations during aggregation process, in: Proc. of 8th. Conference of the European Society for Fuzzy Logic and Technology, EUSFLAT'2013, 2013, pp. 376–383.
[5] J. Fodor, M. Roubens, Fuzzy Preference Modelling and Multicriteria Decision Support, Kluwer, Dordrecht, 1994.
[6] P. Fuster-Parra, J. Martín, J.-J. Miñana, O. Valero, A study on the relationship between relaxed metrics and indistinguishability operators, Soft Comput. 23 (2019) 6785–6795.
[7] G. Gerla, Representation theorems for fuzzy orders and quasi-metrics, Soft Comput. 8 (2004) 571–658.
[8] M. Grabisch, J.-L. Marichal, R. Mesiar, E. Pap, Aggregation Functions, Encyclopedia of Mathematics and its Applications, vol. 127, Cambridge University Press, New York, 2009.
[9] J. Jacas, J. Recasens, Aggregation of T-transitive relations, Int. J. Intell. Syst. 18 (2003) 1193–1214.
[10] G.J. Klir, B. Yuan, Fuzzy Sets and Fuzzy Logic: Theory and Applications, Prentice Hall, New Jersey, 1995.
[11] E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer, Dordrecht, 2000.
[12] H. Lai, D. Zhang, Fuzzy pre-order and fuzzy topology, Fuzzy Sets Syst. 157 (2006) 1865–1885.
[13] S. Montes, I. Montes, T. Iglesias, Fuzzy relations: past, present and future, in: J. Kacprzyk, W. Pedrycz (Eds.), Handbook of Computational Intelligence, Springer, Dordrecht, 2015, pp. 171–181.
[14] G. Mayor, J. Recasens, Preserving T-transitivity, in: À. Nebot, et al. (Eds.), Artificial Intelligence Research and Development, vol. 288, IOS Press, Amsterdam, 2016, pp. 79–87.
[15] G. Mayor, O. Valero, Metric aggregation functions revisited, Eur. J. Comb. 80 (2019) 390–400.
[16] J.-J. Miñana, O. Valero, Characterizing quasi-metric aggregation functions, Int. J. Gen. Syst. 48 (8) (2019) 890–909.
[17] K. Peeva, Y. Kyosev, Fuzzy Relational Calculus: Theory, Applications and Software, World Scientific, Singapore, 2004.
[18] F.E. Petry, P. Bosc, Fuzzy Databases: Principles and Applications, Kluwer, Dordrecht, 1996.
[19] A. Pradera, E. Trillas, E. Castiñeira, On the aggregation of some classes of fuzzy relations, in: B. Bouchon-Meunier, et al. (Eds.), Technologies for Constructing Intelligent Systems 2, in: Studies in Fuzziness and Soft Computing, vol. 90, Springer, Heidelberg, 2002, pp. 125–147.
[20] J. Recasens, Indistinguishability Relations: Modelling Fuzzy Equalities and Fuzzy Equivalence Relations, Springer, Heidelberg, 2010.
[21] S. Saminger, R. Mesiar, U. Bodenhofer, Domination of aggregation operators and preservation of transitivity, Int. J. Uncertain. Fuzziness Knowl.-Based Syst. 10 (2002) 11–35.
[22] R. Słowiński, Fuzzy Sets in Decision Analysis, Operations Research and Statistics, Kluwer, Dordrecht, 1998.
[23] E. Trillas, Assaig sobre les relacions d'indistingibilitat, in: Proc. of Primer Congrés Català de Lògica Matemàtica, 1982, pp. 51–59.

[24] L. Valverde, On the structure of F-indistinguishability relations, Fuzzy Sets Syst. 17 (1985) 313–328.
[25] X. Wang, D. Ruan, E.E. Kerre, Mathematics of Fuzziness-Basic Issues, Springer, Heidelberg, 2009.
[26] L. Zadeh, Similarity relations and fuzzy orderings, Inf. Sci. 1 (1971) 177–200.