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# Equilibrium effort in games with homogeneous production functions and homogeneous valuation

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## Abstract

In this study, I analyze games in which the functions mapping a vector of efforts into each player's share of the prize and its value exhibit an arbitrary degree of homogeneity. I present a simple way to compute the equilibrium strategy and sufficient conditions for a unique interior symmetric pure-strategy Nash equilibrium. The setup nests Malueg and Yates (2006), who exploit homogeneity for rent-seeking contests with exogenous prize valuation, and shows that homogeneity can be used to solve (i) a wider range of rent-seeking contests and (ii) other classes of games, like Cournot games with nonlinear inverse demand and possibly non homogeneous goods.

**KEYWORDS**

equilibrium existence, equilibrium uniqueness, homogeneous functions

**JEL CLASSIFICATION**

C70, D43, D72

## 1 | INTRODUCTION

In many real-life situations, like R&D competition, patent races, procurements, lobbying, and wars, agents make a costly investment and compete to obtain part or all of a prize (a rent).

In the rent-seeking contest introduced by Tullock (1980), players exert a non-negative effort to increase the probability of winning a prize, whose value is equal across contenders and exogenously given. The probability of winning the prize or equivalently each player's prize share is captured by the so-called contest success function (CSF henceforth).

One of the main features of a CSF is homogeneity of degree zero, which implies the realistic feature that the contest winner is determined according to relative efforts. One of the reasons of the wide use of this setting is its analytical tractability (Nitzan, 1994).

However, in other contests, which clearly still require a homogeneous CSF of degree zero, effort can be productive/destructive in the sense that it also positively/negatively affects the size of the rent and its value. Thus, when the value of the prize is endogenous, players do not simply care about increasing their probability of winning the prize, but also of the fact that their effort imposes externalities on the remaining contenders.

Other classes of games, like Cournot games, do not share the same features of rent-seeking contests, but clearly represent other valid examples of players competing to obtain part or all of a prize.[1]

In this paper, I focus on symmetric pure-strategy Nash equilibria in $n$-player games where the part of the prize that each player obtains in the game and its value can be a function of the efforts exerted by any subset of contenders.

I exploit the mathematical properties of homogeneous functions and (i) present a simple way of computing the equilibrium strategy and (ii) show sufficient conditions for the existence of a unique interior symmetric pure-strategy Nash equilibrium. In line, for example, with Szidarovszky and Okuguchi (1997) and Cornes and Hartley (2005), I refer to a function mapping a vector of efforts into a part of the prize as a production function, and to a player's revenue as a valuation.

For what concerns rent-seeking contests, the model includes more sophisticated and therefore more computationally complex models than Tullock's, for which, however, analytical tractability can be recovered.

Furthermore, the setting includes other classes of games such as a wide range of Cournot games. In particular, it is shown that through homogeneity the problems of existence of an equilibrium and its computation can be solved for a Cournot model with non linear inverse demand and possibly non homogeneous goods.

It is worth noting that Szidarovszky and Okuguchi (1997) have shown that a Tullock contest with concave production functions, convex cost function, and in which the value of the rent is normalized to one is strategically equivalent to a Cournot game where the market inverse demand exhibits a unitary elasticity and each player's quantity and cost functions also have specific functional forms.

By looking at a more general class of Cournot games, such connection with rent-seeking contests cannot be stated any more. However, in this paper it is shown that all of these games have something in common, being solvable relying on the same technique.

The paper is structured as follows: in Section 2, I review the related literature; in Section 3, I introduce the model; in Section 4, I study the implications of homogeneity and present practical applications; Section 5 concludes. All proofs are in the Appendix.

[1] Cournot games, for example, do not require a CSF.

## 2 | RELATED LITERATURE

The use of homogeneity to show the existence of a unique interior symmetric pure-strategy Nash equilibrium has been applied by Malueg and Yates (2006, MY henceforth) to rent-seeking contests. In particular, MY (2006) focus on contests in which the CSF is homogeneous of degree zero, the value of the rent is exogenous and equal across contenders, and the cost of effort is linear.

Thus, the current paper shows that the exploitation of homogeneity to tackle equilibrium existence and to provide a simple way of computing the equilibrium strategy: *i*) can be applied to a wider range of rent-seeking contests than those analyzed in MY (2006); and *ii*) can be enlarged to other classes of games.

As shown by Skaperdas (1996), homogeneity of degree zero, alongside with other four axioms, is one of the main properties of a CSF á la Tullock.[2] This property is also retained in different settings than Tullock's (Hirshleifer, 1989; Clark and Riis, 1998; Rai and Sarin, 2009; Bevia and Corchón, 2015). In particular, Clark and Riis (1998) drop the anonymity axiom, which implies that players have a fair treatment in the game and introduce a generalized Tullock CSF able to describe unfair contests. Their analysis is further generalized by Rai and Sarin (2009) by allowing players to make different types of investment. Hirshleifer (1989) proposes a CSF in which the difference in efforts affects the probability of success, whereas in Bevia and Corchón (2015) relative differences matter.

However, other contests, despite clearly requiring a homogeneous CSF of degree zero, also entail that effort is productive/destructive in the sense that it positively/negatively affects the value of the rent. In these settings, the function describing the endogenous value of the prize can exhibit a nonzero degree of homogeneity, thereby it seems reasonable to setup a general model able to include them and in which the exogenous prize valuation (as in MY, 2006) represents a special case.[3]

Several examples of productive/destructive effort can be found in the literature. Chung (1996) discusses a scenario in which the value of the rent is an increasing and concave function in the aggregate effort. As also acknowledged by Posner (1992), a proper example is litigation where a larger expenditure can be associated with a better-informed court that is more likely to take the right and socially desirable decision.

As in Chung (1996) the value of the rent can increase only in a concave way, Shaffer (2006) studies contests with different types of externalities and allows the value of the rent to increase/decrease with aggregate effort. His setting captures, for instance, wars, where effort is exerted to destroy part of the rival's facilities. Konrad (2000) studies rent-seeking contests among groups, where each group can not only invest resources to increase its chance to win the prize but also make sabotage effort to decrease the probability that a rival group obtains the prize.[4,5]

Without exploiting homogeneity, the literature also focused on solving for equilibria in contests with endogenous prize (Hirai, 2012; Hirai and Szidarovszky, 2013).

As pointed out, the paper goes beyond rent-seeking contests, and thereby it is related to other strands of literature. In particular, I refer to that strand that focuses on the existence of Nash equilibria in Cournot games. In this regard, the literature mainly tackled existence under different assumptions on the cost structure of firms and the shape of the inverse demand, while the homogeneous product assumption is retained.

[2]The other four axioms are probability, monotonicity, anonymity, and independence of irrelevant alternatives.

[3]Indeed, the expression of the equilibrium effort in MY (2006) is a special case of the current analysis.

[4]Lobbying is a classic example.

[5]Other examples of productive/destructive effort are Gershkov *et al.* (2009), Matros and Armanios (2009), Che and Humphreys (2014), and Chowdhury and Sheremeta (2011a,b, 2015).

For example, McManus (1962, 1964), Roberts and Sonnenschein (1976), and Szidarovszky and Yakowitz (1977) show existence assuming linear and/or convex costs. Moreover, Szidarovszky and Yakowitz (1977) allow for concave inverse demand. Novshek (1985) shows that when each firm's marginal revenue decreases with aggregate output, several common assumptions on the cost structure such as cost convexity can be relaxed.

Kolstad and Mathiesen (1986) discuss necessary and sufficient conditions for existence and uniqueness of an equilibrium relying on the properties of the Jacobian matrix of the profit functions, whereas Van Long and Soubeyran (2000) rely on a contraction mapping argument.

With regard to differentiated products, a relevant contribution is Hoernig (2003), who shows that, whenever the best replies are such that, if all firms but one symmetrically increase their production, the remaining firm responds in such a way that market price does not strictly rise, then a symmetric equilibrium exists. With additional assumptions, uniqueness can be established as well.

Thus, the paper shows that these two strands of literature can, in a sense, be unified as both the homogeneous and the differentiated product cases can be solved by exploiting the properties of homogeneous functions.

## 3 | THE MODEL

There are $n$ risk neutral players $i = 1, \ldots, n$. Each player exerts effort $e_i \in \mathbb{R}_+ = [0, \infty)$. Let $\mathbf{e} \in \mathbb{R}_+^n$ denote an $n$-dimensional vector of efforts. Players compete to win a share of a possibly divisible prize. Player $i$'s share is obtained according to the production function $\phi_i(e_i, \mathbf{e}_{-i})$: $\mathbb{R}_+^n \to \mathbb{R}_+$, where $\mathbf{e}_{-i} = (e_1, \ldots, e_{i-1}, e_{i+1}, \ldots e_n)$ is the vector of efforts exerted by all players, but player $i$. Player $i$'s payoff is:

$$V_i(\phi_i(e_i, \mathbf{e}_{-i}), \boldsymbol{\phi}_{-i}(\mathbf{e}_{-i}, e_i)) - C_i(e_i), \tag{1}$$

where $V_i$: $\mathbb{R}_+^n \to \mathbb{R}_+$ is player $i$'s valuation of his/her share $\phi_i$, which in turn is allowed to depend on the efforts exerted by any subset of players. Moreover, $V_i$ is also allowed to depend on the production of the remaining contenders $\boldsymbol{\phi}_{-i} = (\phi_1, \ldots, \phi_{i-1}, \phi_{i+1}, \ldots \phi_n)$. Finally, $C_i(e_i)$: $\mathbb{R}_+ \to \mathbb{R}_+$ is the cost of effort. In addition to differentiability and symmetry across players on $\phi$, $V$, and $C$, the following assumptions are made:

A1) $\phi(\lambda \mathbf{e}) = \lambda^\alpha \phi(\mathbf{e}), \lambda > 0, \alpha \in \mathbb{R}$;
A2) $V(\lambda \boldsymbol{\phi}) = \lambda^\delta V(\boldsymbol{\phi}), \lambda > 0, \delta \in \mathbb{R}$;
A3) For all $\mathbf{e}_{-i} = \tilde{\mathbf{e}} \in \mathbb{R}_+^{n-1}$, $V_i(\phi_i(e_i, \mathbf{e}_{-i} = \tilde{\mathbf{e}}), \boldsymbol{\phi}_{-i}(\mathbf{e}_{-i} = \tilde{\mathbf{e}}, e_i)) - C_i(e_i)$ is bounded above in $e_i \in \mathbb{R}_+$;
A4) $C$ is the power function $ce^s$, with $c > 0, s \geq 1$;
A5) $V_i(\phi_i(e_i, \mathbf{e}_{-i}), \boldsymbol{\phi}_{-i}(\mathbf{e}_{-i}, e_i))$ is increasing in $e_i$;[6]
A6) $s > \alpha\delta$.

A1) and A2) state that $\phi$ and $V$ are homogeneous of degree $\alpha$ and $\delta$, respectively. A3) rules out the possibility of a too large production or, equivalently, that players exert extremely high efforts. The only observation regarding the cost structure is that to preserve homogeneity of degree $s$, $C$ does not contain a fixed component. A5) is the equivalent of the monotonicity

[6] I will study situations where no effort could imply a positive valuation for player $i$ (i.e. $V_i(\phi_i(\mathbf{0}), \boldsymbol{\phi}_{-i}(\mathbf{0})) \geq 0$) and where a positive effort by player $i$ implies a positive valuation (i.e. $V_i(\phi_i(e_i, \mathbf{e}_{-i} = \tilde{\mathbf{e}}), \boldsymbol{\phi}_{-i}(\mathbf{e}_{-i} = \tilde{\mathbf{e}}, e_i)) > 0$, for $e_i > 0$ and $\forall\, \tilde{\mathbf{e}} \in \mathbb{R}_+^{n-1}$).

axiom (2′) in MY (2006). The reason why I need to take care of the value of $V$ (and $\phi$ later on in the paper) in a vector of ones is directly related to homogeneity and will be clarified in the following sections. Finally, A6) excludes the null vector as an equilibrium candidate, from which player $i$ has a profitable deviation by infinitesimally increasing his effort.[7]

### 3.1 | The Tullock contest

Although the present setting goes beyond rent-seeking contests, the Tullock model can be useful to clarify the meaning of the $\phi$ and $V$ functions. Player $i$'s payoff is:

$$u_i(e_i, \mathbf{e}_{-i}) = \varphi_i(\mathbf{e})T - e_i, \tag{2}$$

where:

$$\varphi_i(\mathbf{e}) = \begin{cases} \dfrac{e_i^r}{\sum_j e_j^r} & \text{if} \quad \mathbf{e} \neq \mathbf{0} \\ \dfrac{1}{n} & \text{if} \quad \mathbf{e} = \mathbf{0}, \end{cases}$$

with $r > 0$ is the CSF that captures player $i$'s probability of winning the contest or his/her prize share.[8],[9]

Provided that the null vector is not an equilibrium, a first approach is to interpret the CSF as a production function $\phi$, which therefore depends on the efforts of all contenders and it is homogeneous of degree zero. Thus, $V_i(\phi_i) = \phi_i(e_i, \mathbf{e}_{-i})T$, namely it is a linear and homogeneous of degree one function in the CSF $\phi_i$. Alternatively, one can set $\phi_i = e_i^r$, so that player $i$'s production only depends on his own effort. In this case $\phi$ is homogeneous of degree $r$ and $V_i(\phi_i, \boldsymbol{\phi}_{-i}) = \frac{\phi_i}{\sum_j \phi_j}T$ is homogeneous of degree zero.[10]

## 4 | ANALYSIS

### 4.1 | Homogeneous production functions and homogeneous valuation

In this section I present the implications of homogeneity of arbitrary degrees $\alpha, \delta$, and $s$ on $\phi$, $V$, and $C$, respectively. The first result is the following:

**Proposition 1.** *Under* A1)–A6)*, if a pure-strategy symmetric Nash equilibrium exists, then it is interior, unique, and the equilibrium effort is given by:*

[7]The setting can also deal with cases in which $\phi$ and/or $V$ are piecewise functions splitting in the origin and therefore, although not necessarily, discontinuous in such point. It follows that A1), A2), and A6), despite possibly fulfilled $\forall\ e_i \in [0, +\infty)$, are necessarily valid for $\forall\ e_i \in (0, +\infty)$. It also follows that the degrees of homogeneity of $\phi$ and $V$ can possibly change in the origin. However, in all of these cases, A6) excludes the null vector as an equilibrium candidate.

[8]Other assumptions can be made for $\varphi_i(\mathbf{0})$. In Serena (2017), for example, $\varphi_i(\mathbf{0}) = 0$.

[9]In this example, where the prize valuation is exogenous, the fact that the CSF must satisfy the probability axiom, alongside the fact that the cost of effort is sufficiently high, is important for the boundedness of $u$. More generally, however, boundedness can be satisfied for payoffs that do not involve a CSF and therefore in settings where the probability axiom is not required.

[10]In this case, the literature also refers to $\phi_i$ as an impact function. See, for example, Guigou *et al.* (2017). Also note that in both cases the product of the two degrees of homogeneity $\alpha\delta$ is equal to zero. I highlight this feature as such a product will explicitly appear in the expression of the equilibrium effort.

$$e^* = \left(\frac{cs}{\phi(\mathbf{1})^{\delta-1}\left(\frac{\partial V_i}{\partial \phi_i}(\mathbf{1})\frac{\partial \phi_i}{\partial e_i}(\mathbf{1}) + (n-1)\frac{\partial V_j}{\partial \phi_i}(\mathbf{1})\frac{\partial \phi_j}{\partial e_i}(\mathbf{1})\right)}\right)^{\frac{1}{\alpha\delta - s}}. \quad (3)$$

As pointed out, the analysis nests MY (2006), who focus on rent-seeking contests with a homogeneous CSF of degree zero and an exogenous and equal across contenders prize valuation. In MY (2006) the equilibrium effort level is $e^* = \frac{\partial \phi_i}{\partial e_i}(\mathbf{1})T$. This result is a special case of (3).

For example, relying on the two alternative formulations of the $\phi$ and $V$ functions through which the Tullock contest has been constructed in Subsection 3.1, it follows that $\alpha\delta = 0$. The cost of effort is linear, so that $s = 1$, and the parameter $c = 1$ as well. This leads to $e^* = \phi(\mathbf{1})^{\delta-1}\left(\frac{\partial V_i}{\partial \phi_i}(\mathbf{1})\frac{\partial \phi_i}{\partial e_i}(\mathbf{1}) + (n-1)\frac{\partial V_i}{\partial \phi_j}(\mathbf{1})\frac{\partial \phi_j}{\partial e_i}(\mathbf{1})\right)$.

Interpreting a production function as a CSF and since player $i$'s payoff is zero if another contender $j \neq i$ wins the prize, the term $(n-1)\frac{\partial V_i}{\partial \phi_j}(\mathbf{1})\frac{\partial \phi_j}{\partial e_i}(\mathbf{1}) = 0$. Being $V_i = \phi_i \quad T$, then its degree of homogeneity is $\delta = 1$, and $\phi(\mathbf{1})^{\delta-1} = 1$. Finally $\frac{\partial V_i}{\partial \phi_i}(\mathbf{1}) = T$.[11]

Note that (3) is very informative, in the sense that $e$ can always be isolated on the left-hand side, and the remaining terms, because of homogeneity, can be easily computed in a vector of ones.[12] The next proposition provides sufficient conditions for the existence of a unique interior symmetric pure-strategy Nash equilibrium.

**Proposition 2.** *Under* A1)–A6), *if:*

*i)* $\phi(\mathbf{1})^{\delta-1}\left(\frac{\partial V_i}{\partial \phi_i}(\mathbf{1})\frac{\partial \phi_i}{\partial e_i}(\mathbf{1}) + (n-1)\frac{\partial V_i}{\partial \phi_j}(\mathbf{1})\frac{\partial \phi_j}{\partial e_i}(\mathbf{1})\right) > 0$;

*ii)* $$\frac{\phi_i(\mathbf{1})^{\delta}V_i(\mathbf{1}) - \phi_i(\mathbf{1})^{\delta}V_i\left(\frac{\phi_i(0, \mathbf{e}_{-i} = \mathbf{1})}{\phi_{j\neq i}(\mathbf{e}_{-i} = \mathbf{1}, 0)}, \mathbf{1}\right)}{\phi(\mathbf{1})^{\delta-1}\left(\frac{\partial V_i}{\partial \phi_i}(\mathbf{1})\frac{\partial \phi_i}{\partial e_i}(\mathbf{1}) + (n-1)\frac{\partial V_i}{\partial \phi_j}(\mathbf{1})\frac{\partial \phi_j}{\partial e_i}(\mathbf{1})\right)} \geq \frac{1}{s};$$

*iii)* $\exists\ \bar{e}$ s.t. $$\frac{\partial}{\partial e_i}\left[\phi_{j\neq i}(e_i, \mathbf{1})^{\delta-1}\left(\frac{\partial V_i}{\partial \phi_i}\left(\frac{\phi_i(e_i, \mathbf{1})}{\phi_{j\neq i}(\mathbf{1}, e_i)}, \mathbf{1}\right)\frac{\partial \phi_i}{\partial e_i}(e_i, \mathbf{1}) \right.\right. ;$$
$$\left.\left. + (n-1)\frac{\partial V_i}{\partial \phi_j}\left(\frac{\phi_i(e_i, \mathbf{1})}{\phi_{j\neq i}(\mathbf{1}, e_i)}, \mathbf{1}\right)\frac{\partial \phi_j}{\partial e_i}(e_i, \mathbf{1})\right)\right]\begin{cases} >0, & e < \bar{e} \\ <0, & e > \bar{e}\end{cases}$$

*then a unique interior symmetric pure-strategy Nash equilibrium exists, in which players exert the effort level in (3).*

[11] Consider again the first approach of Section 3.1. As $\phi_i$ is player $i$'s CSF, then it is homogeneous of degree zero and $\frac{\partial \phi_i}{\partial e_i}(\mathbf{1}) = \frac{r(n-1)}{n^2}$. Moreover, being $\delta = 1$, $\phi(\mathbf{1})^{\delta-1} = 1$, $\frac{\partial V_i}{\partial \phi_i}(\mathbf{1}) = T$, $(n-1)\frac{\partial V_i}{\partial \phi_j}(\mathbf{1})\frac{\partial \phi_j}{\partial e_i}(\mathbf{1}) = 0$, and according to (3), the well-known expression for the equilibrium effort $e^* = \frac{r(n-1)}{n^2}T$ is obtained. The same result emerges by setting $\phi_i = e_i^r$, from which $\frac{\partial \phi_i}{\partial e_i}(\mathbf{1}) = r$. In this case $V_i = \frac{\phi_i}{\sum_j \phi_j}T$ is homogeneous of degree zero and $\frac{\partial V_i}{\partial \phi_i} = \frac{\sum_{j\neq i}\phi_j - \phi_i}{\left(\sum_j \phi_j\right)^2}T$. Because $\frac{\partial V_i}{\partial \phi_i}(\mathbf{1}) = \frac{n-1}{n^2}T$, then according to (3), $e^* = \frac{r(n-1)}{n^2}T$.

[12] A function $f: \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree $k$ in its argument $\mathbf{x} \equiv (x_1, x_2, ..., x_n)$ if $f(t\mathbf{x}) = t^k f(\mathbf{x})$, $\forall\ t > 0$. Thus, in a vector of positive and symmetric efforts $\mathbf{e}$, one can write $f(\mathbf{e}) = e^k f(\mathbf{1})$. Moreover, according to Euler's Theorem, the derivatives of order $z$ are homogeneous of degree $k - z$, so that they can be evaluated in a vector of ones as well.

Conditions i), ii), and iii) are a generalized version of those in Proposition 3 in MY (2006).[13] Condition i) ensures that the equilibrium effort is interior. Condition ii) ensures that player $i$, given that the remaining players choose the effort level in (3), is not worse off by himself exerting the effort level in (3) rather than exerting no effort.[14]

Finally, iii) is a condition on the second derivative of $V$ and states that although initially the returns on effort can be increasing, eventually they become decreasing.

## 4.2 | Some examples

I now provide two examples to show that homogeneity can be exploited to solve a wider range of rent-seeking contests than those in MY (2006) and that its use does not have to be restricted to rent-seeking contests.[15]

### 4.2.1 | Rent-Seeking contests

I solve a generalized Chung (1996) contest, where players exert effort not only to increase their chance of winning the prize but also to affect the value of the prize. Such value is assumed to be an increasing and concave function in the aggregate effort.[16]

**Example 1.** A Chung (1996) contest.

By symmetry, I focus on the problem faced by a representative player $i$, whose payoff is:

$$u_i(e_i, \mathbf{e}_{-i}) = \varphi_i(\mathbf{e}) A \left( \sum_{j=1}^{n} e_j \right)^{\eta} - e_i, \quad \eta \in (0, 1), \quad A > 0, \tag{4}$$

where $A$ and $\left(\sum_{j=1}^{n} e_j\right)^{\eta}$ are the exogenous and the endogenous parts of the prize valuation, respectively, and $\varphi_i(\mathbf{e})$ is the Tullock CSF as defined in Section 3.1.

[13] The arguments in $V_i$, $\frac{\phi_i(0, \mathbf{e}_{-i} = \mathbf{1})}{\phi_{j \neq i}(\mathbf{e}_{-i} = \mathbf{1}, 0)}$ and $\frac{\phi_i(e_i, \mathbf{1})}{\phi_{j \neq i}(\mathbf{1}, e_i)}$ are two $(n-1)$-dimensional vectors.

[14] The comparison with the payoff obtained by exerting a null effort is crucial. It is well known that there exist cases in which, although both the local first- and second-order conditions for a maximum hold in a symmetric vector of efforts, this is not sufficient for such a vector to be a Nash equilibrium. Baye *et al.* (1994) show that this is the case for a two-player Tullock contest with $r \geq 2$. If, say, player 1 exerts the positive effort level implied by the first-order conditions, the best reply for player 2 is to exert a null effort. Exerting the same effort than player 1 would also satisfy the local second-order conditions, but would assign to player 2 a negative payoff. However, if player 2 exerts no effort, player 1 in turn has a profitable deviation by exerting an arbitrarily small effort $\epsilon > 0$.

[15] Although the paper does not consider sequential contests, it is possible to find cases in which, at least for specific stages, homogeneity can be exploited. An example is the second stage of the efficient tournament problem by Gershkov *et al.* (2009), where players, after having agreed on a sharing rule of the endogenous prize, non-cooperatively choose their effort.

[16] It is possible to solve other endogenous prize contests, like Shaffer (2006), where, differently from Chung (1996), players are allowed to generate negative externalities by exerting destructive effort, and the value of the rent is not constrained to be a concave function in the aggregate effort. The details are available from the author upon request.

At this point, set $\phi_i = e_i$, so that $\alpha = 1$. Thus, $V_i \equiv \varphi_i(\mathbf{e})A\left(\sum_{j=1}^{n} e_j\right)^{\eta}$ and its degree of homogeneity is $\delta = \eta$. As the cost of effort is linear $s = 1$, which excludes the null vector as an equilibrium candidate.[17]

Moreover:

$$\phi_i(\mathbf{1})^{\eta-1} = 1; \tag{5}$$

$$\frac{\partial \phi_i}{\partial e_i}(\mathbf{1}) = 1; \tag{6}$$

$$(n-1)\frac{\partial V_i}{\partial \phi_j}(\mathbf{1})\frac{\partial \phi_j}{\partial e_i}(\mathbf{1}) = 0, \tag{7}$$

so that condition i) of Proposition 2 simply becomes $\frac{\partial V_i}{\partial \phi_i}(\mathbf{1}) > 0$. The first-order conditions write:

$$\underbrace{A\left(\frac{re_i^{r-1}\left(\sum_{j=1}^{n} e_j^r\right) - re_i^{2r-1}}{\left(\sum_{j=1}^{n} e_j^r\right)^2}\right)\left(\sum_{j=1}^{n} e_j\right)^{\eta} + \frac{e_i^r}{\sum_j e_j^r}\frac{A}{\frac{1}{\eta}\left(\sum_{j=1}^{n} e_j\right)^{1-\eta}}}_{\frac{\partial V_i}{\partial \phi_i}} = 1. \tag{8}$$

It follows that (3) becomes $e = \left(\frac{\partial V_i}{\partial \phi_i}(\mathbf{1})\right)^{\frac{1}{1-\eta}}$, where:

$$\frac{\partial V_i}{\partial \phi_i}(\mathbf{1}) = A\left(\frac{r(n-1)}{n^2}n^{\eta} + \frac{1}{\eta^{-1}n^{2-\eta}}\right) \tag{9}$$

$$= A\left(\frac{r(n-1)+\eta}{n^{2-\eta}}\right) > 0, \tag{10}$$

thereby condition i) of Proposition 2 holds and the solution is interior. Thus, one can avoid the majority of the computation and easily obtain the expression:

$$e = \left(\frac{A(r(n-1)+\eta)}{n^{2-\eta}}\right)^{\frac{1}{1-\eta}}. \tag{11}$$

It is straightforward to show that condition ii) of Proposition 2 becomes:

$$\frac{V(\mathbf{1})}{\frac{\partial V_i}{\partial \phi_i}(\mathbf{1})} = \frac{n^{\eta-1}}{\frac{(r(n-1)+\eta)}{n^{2-\eta}}} \geq 1. \tag{12}$$

The previous condition is satisfied for $r \leq \frac{n-\eta}{n-1}$.[18] As $\phi_i = e_i$, condition iii) of Proposition 2 needs to be checked for:

[17] In particular, it is straightforward to show that $\forall\ e_i \in \left(0, A^{\frac{1}{1-\eta}}\right)$, player $i$ can strictly improve his payoff given that the remaining contenders are exerting no effort.

[18] This is consistent with the fact that, for every nonnegative vector of efforts, if $\eta = 0$, then $\left(\sum_{j=1}^{n} e_j\right)^{\eta} = 1$ and the prize valuation only contains the exogenous component $A$. In this case, the payoff function is as in Tullock (1980), in which the known condition $r \leq \frac{n}{n-1}$ for the existence of an interior equilibrium applies.

$$A\left(\frac{r(n-1)e_i^{r-1}(e_i+n-1)^{\eta}}{(e_i+n-1)^2}+\frac{\eta e_i^r}{\left(e_i^r+n-1\right)(e_i+n-1)^{1-\eta}}\right). \tag{13}$$

The left-hand side of (13) strictly decreases in $e_i$, whereas the right-hand side initially increases and eventually decreases. Overall it turns out that, for $r \leq 1$, (13) decreases in $e_i$ and for $r > 1$ it follows the behavior of its right-hand side. Therefore, all the conditions of Proposition 2 are satisfied, and in the unique interior symmetric pure-strategy Nash equilibrium all players exert the effort level in (11).

### 4.2.2 | Cournot games

Szidarovszky and Okuguchi (1997) identified a link between rent-seeking contests and Cournot games by showing that a Tullock game with payoff $\frac{\phi_i(e_i)}{\sum_j \phi_j(e_j)} - e_i$, where, for all $i$, $\frac{\partial \phi_i}{\partial e_i} > 0$, $\frac{\partial^2 \phi_i}{\partial e_i^2} < 0$ and $\phi_i(0) = 0$ is strategically equivalent to a Cournot game with unitary elasticity inverse demand $p = \frac{1}{\sum_j \phi_j}$, quantity $\phi_i$, and convex cost function $g_i(\phi_i) = \phi^{-1}$.

In this case, player $i$'s revenue $\frac{1}{\sum_j \phi_j}\phi_i$: $\mathbb{R}_+^n \to [0, 1]$ mimics a homogeneous of degree zero CSF.

By looking at a generalized version of such Cournot game, this equivalence cannot be stated any more. However, I here show that this class of games can be included in the present analysis and solved by exploiting homogeneity.

Moreover, the existence of Cournot equilibria under homogeneous products (see, e.g., Kolstad and Mathiesen, 1986; McManus, 1962, 1964; Novshek, 1985; Roberts and Sonneschein, 1976; Szidarovszky and Yakowitz, 1977; Svizzero, 1997) and under differentiated products (see, e.g., Hoernig, 2003) developed in two separated strands of literature.

In what follows I show that existence of an equilibrium and its uniqueness can be tackled for both the homogeneous and the differentiated products case (with nonlinear inverse demand) relying on the same approach.

**Example 2.** A Cournot duopoly with nonlinear inverse demand.

Two firms $i = \{1, 2\}$ invest resources to produce their quantity which consistently with the notation of the paper are denoted $\phi_i$, with $\phi_i(e_i) = e_i^{\nu}$, $\nu \in \mathbb{R}_+$.[19] Firm $i$'s inverse demand is $p_i(\phi_i(e_i), \phi_j(e_j)) = A\frac{1}{(\phi_i(e_i)+\beta\phi_j(e_j))^{\sigma}}$, $i \neq j$, where $A$ is a positive parameter and $\beta \in [-1, 1]$ specifies the nature of the goods: when $\beta \in (0, 1]$ the goods are substitutes, when $\beta = 0$ the goods are independent, and when $\beta \in [-1, 0)$ the goods are complements.

Since a reasonable assumption is that no production from both firms implies no revenue, player 1's payoff writes:

$$\pi_1(e_1, e_2) = V_1(\phi_1(e_1), \phi_2(e_2)) - e_1, \tag{14}$$

with:

[19] When $\nu = 1$, there is a one-to-one relation between effort and quantity, so that a more classical Cournot game is described. In this example, more generally, I allow for economies or diseconomies of scale.

$$V_1(\phi_1(e_1), \phi_2(e_2)) = \begin{cases} A\dfrac{\phi_1(e_1)}{(\phi_1(e_1) + \beta\phi_2(e_2))^\sigma} & \text{if} \quad (\phi_1, \phi_2) \neq (0, 0) \\ 0 & \text{if} \quad (\phi_1, \phi_2) = (0, 0). \end{cases}$$

The degrees of homogeneity of $\phi$ and $V$ are $\nu$ and $1 - \sigma$, respectively.[20] First I make sure that the null vector is not a Nash equilibrium. Being $s = 1$, A6) requires that $\nu < \frac{1}{1-\sigma}$.

Moreover:

$$\phi_1(\mathbf{1})^{\delta-1} = 1, \tag{15}$$

$$\frac{\partial \phi_1}{\partial e_1}(\mathbf{1}) = \nu; \tag{16}$$

$$(n-1)\frac{\partial V_1}{\partial \phi_2}(\mathbf{1})\frac{\partial \phi_2}{\partial e_1}(\mathbf{1}) = 0, \tag{17}$$

and condition i) of Proposition 2 requires that:

$$\nu\frac{\partial V_1}{\partial \phi_1}(\mathbf{1}) = \nu\frac{A(1+\beta-\sigma)}{(1+\beta)^{1+\sigma}} > 0, \tag{18}$$

which is satisfied for $\sigma \in (0, 1+\beta)$.[21] According to (3), the expression:

$$e = \left(\frac{A\nu(1+\beta-\sigma)}{(1+\beta)^{\sigma+1}}\right)^{\frac{1}{1-\nu(1-\sigma)}} \tag{19}$$

is easily obtained. Condition ii) of Proposition 2 becomes:

$$\frac{V(\mathbf{1})}{\frac{\partial V_1}{\partial \phi_1}(\mathbf{1})\frac{\partial \phi_1}{\partial e_1}(\mathbf{1})} = \frac{(1+\beta)}{\nu(1+\beta-\sigma)} \geq 1, \tag{20}$$

which holds if $\nu \leq \frac{1+\beta}{1+\beta-\sigma}$. At this point, as an illustrative example, consider the homogeneous good case $(\beta, \sigma, \nu) = (1, \frac{1}{2}, 1)$.[22] In this case, (19) becomes:

$$e = \frac{9}{32}A^2. \tag{21}$$

For condition iii) of Proposition 2:

$$\frac{\partial V_1}{\partial \phi_1}(\phi_1, \phi_2 = 1) = A\frac{\beta + \phi_1(1-\sigma)}{(\beta + \phi_1)^{1+\sigma}}; \tag{22}$$

$$\frac{\partial \phi_1}{\partial e_1}(e_1, e_2 = 1) = \nu e_1^{\nu-1}. \tag{23}$$

Replacing $\phi_1 = e_1^\nu$ in (22), the expression for which condition iv) of Proposition 2 needs to be checked is:

[20] Notice that the image of $A\frac{\phi_1(e_1)}{(\phi_1(e_1)+\beta\phi_2(e_2))^\sigma}$ is not in general $[0, 1]$, which departs the game from rent-seeking contests.

[21] Thus, more compactly $\sigma \in (0, \min\{1, 1+\beta\})$.

[22] In this case, A5) is implied by $\sigma \in (0, 1)$.

$$Ave_1^{\nu-1}\frac{\beta + e_1^{\nu}(1-\sigma)}{\left(\beta + e_1^{\nu}\right)^{1+\sigma}}. \tag{24}$$

Evaluating (24) in the above triple yields:

$$A\frac{(2+e_1)}{2(1+e_1)^{\frac{3}{2}}}, \tag{25}$$

which is a strictly decreasing function in $e_1$. Therefore, all the assumptions of Proposition 2 hold, and in the unique interior symmetric pure-strategy Nash equilibrium both players exert the effort level in (21).

## 5 | CONCLUSIONS

I studied a general setting where $n$ players exert effort to obtain part or all of a prize, whose valuation can be either exogenously given or endogenously determined. Under homogeneity assumptions on the functions mapping the efforts into the part of the prize that each player obtains in the game and on its value, I proposed a simple way of evaluating the equilibrium effort and the conditions for the existence of a unique interior symmetric pure-strategy Nash equilibrium. The paper shows that the adoption of homogeneity is useful for a wider range of contests than those in Malueg and Yates (2006), and to solve other classes of games, like Cournot games with nonlinear inverse demand.

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## APPENDIX A:

*Proof of Proposition* 1. First, I make sure that the null vector $(0, 0, ..., 0)$ is not an equilibrium. By symmetry I focus on the problem faced by a representative player $i$. If all players but player $i$ exert no effort and player $i$ exerts a positive effort, his payoff is:

$$V_i(\phi_i(e_i, \mathbf{e}_{-i} = \mathbf{0}), \phi_{j \neq i}(\mathbf{e}_{-i} = \mathbf{0}, e_i)) - ce_i^s. \tag{A1}$$

By homogeneity of degree $\alpha$ on $\phi$, as specified in footnote 12, I can bring $e_i > 0$ outside all $\phi$'s and (A1) becomes:

$$V_i\left(e_i^\alpha \phi_i(1, \mathbf{e}_{-i} = \mathbf{0}), e_i^\alpha \phi_{j \neq i}(\mathbf{e}_{-i} = \mathbf{0}, 1)\right) - ce_i^s. \tag{A2}$$

By homogeneity of degree $\delta$ on $V$, (A2) can be further manipulated into:

$$e_i^{\alpha\delta} V_i(\phi_i(1, \mathbf{e}_{-i} = \mathbf{0}), \phi_{j \neq i}(\mathbf{e}_{-i} = \mathbf{0}, 1)) - ce_i^s. \tag{A3}$$

If instead player $i$ exerts no effort as well, then his payoff is:

$$V_i(\phi_i(\mathbf{0}), \phi_{j \neq i}(\mathbf{0})), \tag{A4}$$

where by symmetry $\phi_i(\mathbf{0}) = \phi_j(\mathbf{0}), \quad \forall\, i \neq j$. I introduce the following Lemma: □

**Lemma A1.** *Let* (A5) hold. Then $\alpha\delta \geq 0$.

*Proof.* $e_i^{\alpha\delta} V_i(\phi_i(1, \mathbf{e}_{-i} = \mathbf{0}), \phi_{j \neq i}(\mathbf{e}_{-i} = \mathbf{0}, 1)) - V_i(\phi_i(\mathbf{0}), \phi_{j \neq i}(\mathbf{0}))$ is the difference in $V_i$ when $i$ exerts some effort and $-i$ exerts no effort, and when all $i$'s exert no effort. From A5), the constant term $V_i(\phi_i(1, \mathbf{e}_{-i} = \mathbf{0}), \phi_{j \neq i}(\mathbf{e}_{-i} = \mathbf{0}, 1)) > V_i(\phi_i(\mathbf{0}), \phi_{j \neq i}(\mathbf{0})) \geq 0$. Moreover, again by A5), $\frac{\partial}{\partial e_i}\left[e_i^{\alpha\delta} V_i(\phi_i(1, \mathbf{e}_{-i} = \mathbf{0}), \phi_{j \neq i}(\mathbf{e}_{-i} = \mathbf{0}, 1))\right] = \alpha\delta \underbrace{e_i^{\alpha\delta - 1} V_i(\phi_i(1, \mathbf{e}_{-i} = \mathbf{0}), \phi_{j \neq i}(\mathbf{e}_{-i} = \mathbf{0}, 1))}_{>0} \geq 0$. This proves the claim. □

At this point, I need to distinguish two cases:

i) $\phi_i(\mathbf{0}) = 0$;
ii) $\phi_i(\mathbf{0}) > 0$.

In case i), player $i$ is better off by infinitesimally increasing his effort rather than exerting no effort if there exists an $\hat{e}_i > 0$ such that:

$$\frac{e_i^{\alpha\delta} V_i(\phi_i(1, \mathbf{e}_{-i} = \mathbf{0}), \phi_{j \neq i}(\mathbf{e}_{-i} = \mathbf{0}, 1)) - V_i(\mathbf{0})}{ce_i^s} > 1 \tag{A5}$$

holds $\forall \ e_i \in (0, \hat{e}_i)$.

If $V_i(\mathbf{0}) = 0$ and $s > \alpha\delta$, the left-hand side of (A5) is a continuous, positive valued, and strictly decreasing function in $e_i$ with $\lim_{e_i \to 0^+}() = +\infty$, thereby the aforementioned threshold exists.

If $V_i(\mathbf{0}) > 0$, being a possibly piecewise function splitting in the origin, I make a further distinction based on whether its degree of homogeneity does or does not change in this point.

I introduce the following lemma:

**Lemma A2.** *If a function* $f: \mathbb{R}_+^n \to \mathbb{R}$ *is homogeneous of degree* $k$ *with* $f(\mathbf{0}) > 0 \Rightarrow \ k = 0$.

*Proof.* By homogeneity of degree $k, f(t\mathbf{0}) = f(\mathbf{0}) = t^k f(\mathbf{0})$. Since $t > 0$ and $f(\mathbf{0}) > 0$, then previous equation is true if $k = 0$. "9" □

If despite being a piecewise function, the degree of homogeneity of $V_i$ is the same over $[0, +\infty)$, then from Lemma A2, $\delta = 0$. Thus, player $i$ is better off by infinitesimally increasing his effort rather than exerting no effort if there exists an $\hat{e}_i > 0$ such that:

$$\frac{V_i(\phi_i(1, \mathbf{e}_{-i} = \mathbf{0}), \phi_{j\neq i}(\mathbf{e}_{-i} = \mathbf{0}, 1)) - V_i(\mathbf{0})}{ce_i^s} > 1 \tag{A6}$$

holds $\forall\ e_i \in (0, \hat{e}_i)$. From A5), the numerator of (A6) is positive, thereby $s > 0$ ensures the existence of such threshold.

If instead the degree of homogeneity of $V_i$ does change in the origin, label it as $\bar{\delta}$. From Lemma A2, $\bar{\delta} = 0$.

In this case, $\lim_{e_i\to 0^+} \frac{e_i^{\alpha\delta} V_i(\phi_i(1, \mathbf{e}_{-i} = \mathbf{0}), \phi_{j\neq i}(\mathbf{e}_{-i} = \mathbf{0}, 1))}{ce_i^s} = \lim_{e_i\to 0^+} \frac{V_i(\phi_i(1, \mathbf{e}_{-i} = \mathbf{0}), \phi_{j\neq i}(\mathbf{e}_{-i} = \mathbf{0}, 1))}{ce_i^{s-\alpha\delta}}$ and $\lim_{e_i\to 0^+} \frac{V_i(\mathbf{0})}{ce_i^s}$ need to be compared. Suppose $\alpha\delta > 0$. Then $s > s - \alpha\delta$, so that the left-hand side of (A-5) tends to $-\infty$ as $e_i \to 0^+$, and player $i$ cannot profitably increase its payoff by exerting an infinitesimal amount of effort. If $\alpha\delta = 0$, from A5) and $s > 0$, the left-hand side of (A5) is a continuous, positive valued, and strictly decreasing function in $e_i$ with $\lim_{e_i\to 0^+}() = +\infty$, thereby player $i$ can increase his payoff by exerting an infinitesimal amount of effort. Therefore, if $V_i(\mathbf{0}) > 0$, then $\alpha\delta = 0$.

I now move to case ii) and being $\phi$ a possibly piecewise function splitting in the origin, I make a further distinction based on whether its degrees of homogeneity does or does not change in this point. If despite being a piecewise function, its degree of homogeneity is the same over $[0, +\infty)$, player $i$ is better off by exerting a sufficiently small effort rather than no effort if there exists an $\hat{e}_i > 0$ such that:

$$\frac{V_i(\phi_i(1, \mathbf{e}_{-i} = \mathbf{0}), \phi_{j\neq i}(\mathbf{e}_{-i} = \mathbf{0}, 1)) - V_i(\phi_i(\mathbf{0}), \phi_{j\neq i}(\mathbf{0}))}{ce_i^s} > 1 \tag{A7}$$

holds $\forall\ e_i \in (0, \hat{e}_i)$. From A5), the numerator of (A7) is positive, thereby $s > 0$ ensures the existence of such threshold.

If instead the degree of homogeneity of $\phi$ does change in the origin, label it as $\bar{\alpha}$. In this case $V_i(\phi_i(\mathbf{0}), \phi_{j\neq i}(\mathbf{0})) = e_i^{\bar{\alpha}\delta} V_i(\phi_i(\mathbf{0}), \phi_{j\neq i}(\mathbf{0}))$, and from Lemma A2, $\bar{\alpha} = 0$. Thus, (A7) becomes:

$$\frac{e_i^{\alpha\delta} V_i(\phi_i(1, \mathbf{e}_{-i} = \mathbf{0}), \phi_{j\neq i}(\mathbf{e}_{-i} = \mathbf{0}, 1)) - V_i(\phi_i(\mathbf{0}), \phi_{j\neq i}(\mathbf{0}))}{ce_i^s} > 1. \tag{A8}$$

With an equivalent analysis of the $V_i(\mathbf{0}) > 0$ case, $s > \alpha\delta = 0$ ensures the existence of an $\hat{e}_i > 0$ such that (A8) holds $\forall\ e_i \in (0, \hat{e}_i)$. Therefore, if $\phi_i(\mathbf{0}) > 0$, then $\alpha\delta = 0$.

Thus, if an equilibrium exists, it is interior and when the representative player $i$ solves:

$$\max_{e_i} V_i(\phi_i(\mathbf{e}), \phi_{j\neq i}(\mathbf{e})) - C_i(e_i),$$

the necessary first-order conditions require that:

$$\frac{\partial V_i}{\partial \phi_i}\frac{\partial \phi_i}{\partial e_i} + \sum_{j\neq i}\frac{\partial V_i}{\partial \phi_j}\frac{\partial \phi_j}{\partial e_i} - \frac{\partial C_i}{\partial e_i} = 0. \tag{A9}$$

Applying symmetry to (A9) yields:

$$\frac{\partial V_i}{\partial \phi_i}\frac{\partial \phi_i}{\partial e_i} + (n-1)\frac{\partial V_i}{\partial \phi_j}\frac{\partial \phi_j}{\partial e_i} - \frac{\partial C_i}{\partial e_i} = 0. \tag{A10}$$

By homogeneity of degree $\delta$ on $V$ and by symmetry on $\phi$:

$$\frac{\partial V_i}{\partial \phi_i}(\boldsymbol{\phi}) = \phi^{\delta-1}\frac{\partial V_i}{\partial \phi_i}(\mathbf{1}); \tag{A11}$$

$$\frac{\partial V_i}{\partial \phi_j}(\boldsymbol{\phi}) = \phi^{\delta-1}\frac{\partial V_i}{\partial \phi_j}(\mathbf{1}). \tag{A12}$$

By homogeneity of degree $\alpha$ on $\phi$, I can bring the common level of effort $e > 0$ outside the function, so that:

$$\frac{\partial \phi_i}{\partial e_i}(\mathbf{e}) = e^{\alpha-1}\frac{\partial \phi_i}{\partial e_i}(\mathbf{1}); \tag{A13}$$

$$\frac{\partial \phi_j}{\partial e_i}(\mathbf{e}) = e^{\alpha-1}\frac{\partial \phi_j}{\partial e_i}(\mathbf{1}); \tag{A14}$$

$$\phi_i(\mathbf{e}) = e^{\alpha}\phi_i(\mathbf{1}); \tag{A15}$$

$$\phi_j(\mathbf{e}) = e^{\alpha}\phi_j(\mathbf{1}). \tag{A16}$$

Furthermore, by homogeneity of degree $s$ on $C$:

$$\frac{\partial C_i}{\partial e_i}(e) = e^{s-1}\frac{\partial C_i}{\partial e_i}(1) = cse^{s-1}. \tag{A17}$$

Substituting (A11)–(A17) in (A10) and after some algebra yields:

$$e^{\alpha\delta-1}\phi(\mathbf{1})^{\delta-1}\left(\frac{\partial V_i}{\partial \phi_i}(\mathbf{1})\frac{\partial \phi_i}{\partial e_i}(\mathbf{1}) + (n-1)\frac{\partial V_i}{\partial \phi_j}(\mathbf{1})\frac{\partial \phi_j}{\partial e_i}(\mathbf{1})\right) = cse^{s-1}. \tag{A18}$$

Rearranging one obtains the unique solution:

$$e^* = \left(\frac{cs}{\phi(\mathbf{1})^{\delta-1}\left(\frac{\partial V_i}{\partial \phi_i}(\mathbf{1})\frac{\partial \phi_i}{\partial e_i}(\mathbf{1}) + (n-1)\frac{\partial V_i}{\partial \phi_j}(\mathbf{1})\frac{\partial \phi_j}{\partial e_i}(\mathbf{1})\right)}\right)^{\frac{1}{\alpha\delta-s}}. \tag{A19}$$

*Proof of Proposition* 2. The proof is a generalization of Proposition 2 in MY (2006) and follows the same logic.

Being $cs > 0$, (A19) is positive if $\phi(\mathbf{1})^{\delta-1}\left(\frac{\partial V_i}{\partial \phi_i}(\mathbf{1})\frac{\partial \phi_i}{\partial e_i}(\mathbf{1}) + (n-1)\frac{\partial V_i}{\partial \phi_j}(\mathbf{1})\frac{\partial \phi_j}{\partial e_i}(\mathbf{1})\right) > 0$, which is condition i). By symmetry, from now on I focus on the problem faced by a representative player $i$, who takes as given the symmetric effort level $e^*$ exerted by the remaining players and maximizes (1) w.r. to $e_i$. The first-order conditions write:

$$0 = \frac{\partial V_i}{\partial \phi_i}\left(\phi_i\left(e_i, \mathbf{e}^*_{-i}\right), \phi_{j\neq i}\left(\mathbf{e}^*_{-i}, e_i\right)\right)\frac{\partial \phi_i}{\partial e_i}\left(e_i, \mathbf{e}^*_{-i}\right) + (n-1)\frac{\partial V_i}{\partial \phi_j}\left(\phi_i\left(e_i, \mathbf{e}^*_{-i}\right), \phi_{j\neq i}\left(\mathbf{e}^*_{-i}, e_i\right)\right)\frac{\partial \phi_j}{\partial e_i}\left(e_i, \mathbf{e}^*_{-i}\right) - \frac{\partial C_i}{\partial e_i}, \quad \text{(A20)}$$

where $\mathbf{e}^*_{-i}$ denotes that all players, but player $i$, exert the effort $e^*$. By homogeneity of degree $\delta$ on $V$, its first derivative is homogeneous of degree $\delta - 1$. Moreover, I can bring $\phi_{j\neq i}(\mathbf{e}^*_{-i}, e_i) > 0$ outside $\frac{\partial V_i}{\partial \phi_i}$ and $\frac{\partial V_i}{\partial \phi_j}$ and A20 can be rewritten as:

$$0 = \phi_{j\neq i}\left(\mathbf{e}^*_{-i}, e_i\right)^{\delta-1}\left(\frac{\partial V_i}{\partial \phi_i}\left(\frac{\phi_i\left(e_i, \mathbf{e}^*_{-i}\right)}{\phi_{j\neq i}\left(\mathbf{e}^*_{-i}, e_i\right)}, \mathbf{1}\right)\frac{\partial \phi_i}{\partial e_i}\left(e_i, \mathbf{e}^*_{-i}\right) + (n-1)\frac{\partial V_i}{\partial \phi_j}\left(\frac{\phi_i\left(e_i, \mathbf{e}^*_{-i}\right)}{\phi_{j\neq i}\left(\mathbf{e}^*_{-i}, e_i\right)}, \mathbf{1}\right)\frac{\partial \phi_j}{\partial e_i}\left(\mathbf{e}^*_{-i}, e_i\right)\right) - \frac{\partial C_i}{\partial e_i}. \quad \text{(A21)}$$

By homogeneity of degree $\alpha$ on $\phi$, its first derivative is homogeneous of degree $\alpha - 1$. Moreover I can bring $e^* > 0$ outside $\phi_{j\neq i}(\mathbf{e}^*_{-i}, e_i)^{\delta-1}$, $\frac{\partial \phi_i}{\partial e_i}(e_i, \mathbf{e}^*_{-i})$ and $\frac{\partial \phi_j}{\partial e_i}(\mathbf{e}^*_{-i}, e_i)$. Thus, (A21) can be rewritten as:

$$0 = \left((e^*)^{\alpha}\phi_{j\neq i}\left(\mathbf{1}, \frac{e_i}{e^*}\right)\right)^{\delta-1}(e^*)^{\alpha-1}\left(\frac{\partial V_i}{\partial \phi_i}\left(\frac{\phi_i\left(e_i, \mathbf{e}^*_{-i}\right)}{\phi_{j\neq i}\left(\mathbf{e}^*_{-i}, e_i\right)}, \mathbf{1}\right)\frac{\partial \phi_i}{\partial e_i}\left(\frac{e_i}{e^*}, \mathbf{1}\right) + (n-1)\frac{\partial V_i}{\partial \phi_j}\left(\frac{\phi_i\left(e_i, \mathbf{e}^*_{-i}\right)}{\phi_{j\neq i}\left(\mathbf{e}^*_{-i}, e_i\right)}, \mathbf{1}\right)\frac{\partial \phi_j}{\partial e_i}\left(\mathbf{1}, \frac{e_i}{e^*}\right)\right) - \frac{\partial C_i}{\partial e_i}. \quad \text{(A22)}$$

Replacing (A17) in (A22) and after some simple algebra yields:

$$0 = (e^*)^{\alpha\delta - s}\phi_{j\neq i}\left(\mathbf{1}, \frac{e_i}{e^*}\right)^{\delta-1}\left(\frac{\partial V_i}{\partial \phi_i}\left(\frac{\phi_i\left(e_i, \mathbf{e}^*_{-i}\right)}{\phi_{j\neq i}\left(\mathbf{e}^*_{-i}, e_i\right)}, \mathbf{1}\right)\frac{\partial \phi_i}{\partial e_i}\left(\frac{e_i}{e^*}, \mathbf{1}\right) + (n-1)\frac{\partial V_i}{\partial \phi_j}\left(\frac{\phi_i\left(e_i, \mathbf{e}^*_{-i}\right)}{\phi_{j\neq i}\left(\mathbf{e}^*_{-i}, e_i\right)}, \mathbf{1}\right)\frac{\partial \phi_j}{\partial e_i}\left(\mathbf{1}, \frac{e_i}{e^*}\right)\right) - cs. \quad \text{(A23)}$$

Replacing (A19) in (A23), one obtains:

$$0 = \frac{\phi_{j\neq i}\left(\mathbf{1}, \frac{e_i}{e^*}\right)^{\delta-1}\left(\frac{\partial V_i}{\partial \phi_i}\left(\frac{\phi_i\left(e_i, \mathbf{e}^*_{-i}\right)}{\phi_{j\neq i}\left(\mathbf{e}^*_{-i}, e_i\right)}, \mathbf{1}\right)\frac{\partial \phi_i}{\partial e_i}\left(\frac{e_i}{e^*}, \mathbf{1}\right) + (n-1)\frac{\partial V_i}{\partial \phi_j}\left(\frac{\phi_i\left(e_i, \mathbf{e}^*_{-i}\right)}{\phi_{j\neq i}\left(\mathbf{e}^*_{-i}, e_i\right)}, \mathbf{1}\right)\frac{\partial \phi_j}{\partial e_i}\left(\mathbf{1}, \frac{e_i}{e^*}\right)\right)}{\phi(\mathbf{1})^{\delta-1}\left(\frac{\partial V_i}{\partial \phi_i}(\mathbf{1})\frac{\partial \phi_i}{\partial e_i}(\mathbf{1}) + (n-1)\frac{\partial V_i}{\partial \phi_j}(\mathbf{1})\frac{\partial \phi_j}{\partial e_i}(\mathbf{1})\right)} - 1 \quad \text{(A24)}$$

$$= \frac{\phi_{j\neq i}\left(\mathbf{1}, \frac{e_i}{e^*}\right)^{\delta-1}\left(\frac{\partial V_i}{\partial \phi_i}\left(\frac{\phi_i\left(\frac{e_i}{e^*},\mathbf{1}\right)}{\phi_{j\neq i}\left(\mathbf{1},\frac{e_i}{e^*}\right)},\mathbf{1}\right)\frac{\partial \phi_i}{\partial e_i}\left(\frac{e_i}{e^*},\mathbf{1}\right)+(n-1)\frac{\partial V_i}{\partial \phi_j}\left(\frac{\phi_i\left(\frac{e_i}{e^*},\mathbf{1}\right)}{\phi_{j\neq i}\left(\mathbf{1},\frac{e_i}{e^*}\right)},\mathbf{1}\right)\frac{\partial \phi_j}{\partial e_i}\left(\mathbf{1},\frac{e_i}{e^*}\right)\right)}{\phi(\mathbf{1})^{\delta-1}\left(\frac{\partial V_i}{\partial \phi_i}(\mathbf{1})\frac{\partial \phi_i}{\partial e_i}(\mathbf{1})+(n-1)\frac{\partial V_i}{\partial \phi_j}(\mathbf{1})\frac{\partial \phi_j}{\partial e_i}(\mathbf{1})\right)} - 1. \tag{A25}$$

By symmetry on $\phi$, $e_i = e^*$ is clearly a solution to (A25). From iii), at most two solutions are available for (A25), and A3) excludes very high effort levels. Thus, an alternative solution is either $e = 0$ or $e = \bar{\bar{e}} > 0$. I first compare the payoff in case of a null effort. Formally, if the payoff by exerting $e_i = e^*$ is no lower than the one by exerting a null effort, then:

$$V_i(\phi_i(\mathbf{e}^*), \phi_{-i}(\mathbf{e}^*)) - C_i(e^*) \geq V_i\left(\phi_i\left(e_i = 0, \mathbf{e}^*_{-i}\right), \phi_{-i}\left(e_i = 0, \mathbf{e}^*_{-i}\right)\right). \tag{A26}$$

If $\phi_i(\mathbf{e}^*) > 0$ and $\phi_{-i}\left(e_i = 0, \mathbf{e}^*_{-i}\right) > 0$, applying homogeneity on $V$ and $\phi$ to the LHS and the RHS of (A26), respectively, yields:

$$(e^*)^{\alpha\delta}\phi_i(\mathbf{1})^{\delta}V_i(\mathbf{1}) - (e^*)^{s}c; \tag{A27}$$

$$(e^*)^{\alpha\delta}\phi_{-i}(0, \mathbf{e}_{-i} = \mathbf{1})^{\delta}V_i\left(\frac{\phi_i(0, \mathbf{e}_{-i} = \mathbf{1})}{\phi_{j\neq i}(\mathbf{e}_{-i} = \mathbf{1}, 0)}, \mathbf{1}\right). \tag{A28}$$

Thus, (A27) becomes:

$$(e^*)^{\alpha\delta - s}\left(\phi_i(\mathbf{1})^{\delta}V_i\left(\frac{\phi_i(0, \mathbf{e}_{-i} = \mathbf{1})}{\phi_{j\neq i}(\mathbf{e}_{-i} = \mathbf{1}, 0)}, \mathbf{1}\right)\right) \geq c. \tag{A29}$$

Plugging (A19) in (A29) and rearranging terms yield condition ii). This means that if $e^*$ is the unique solution to (A25), then it also provides the global maximum.

Now, if another solution exists and being the numerator of (A25) a single-peaked function in $\bar{e}$, then according to iii), two possibilities are available:

a) $\frac{e^*}{e^*} < \bar{e} < \frac{\bar{\bar{e}}}{e^*}$;

b) $\frac{\bar{\bar{e}}}{e^*} < \bar{e} < \frac{e^*}{e^*}$.

Let:

$$\Omega_1(t) \equiv \phi_{j\neq i}\left(\mathbf{1}, \frac{t}{e^*}\right)^{\delta-1}\frac{\partial V_i}{\partial \phi_i}\left(\frac{\phi_i\left(\frac{t}{e^*},\mathbf{1}\right)}{\phi_{j\neq i}\left(\mathbf{1},\frac{t}{e^*}\right)},\mathbf{1}\right)\frac{\partial \phi_i}{\partial e_i}\left(\frac{t}{e^*},\mathbf{1}\right); \tag{A30}$$

$$\Omega_2(t) \equiv (n-1)\phi_{j\neq i}\left(\mathbf{1}, \frac{t}{e^*}\right)^{\delta-1}\frac{\partial V_i}{\partial \phi_i}\left(\frac{\phi_i\left(\frac{t}{e^*},\mathbf{1}\right)}{\phi_{j\neq i}\left(\mathbf{1},\frac{t}{e^*}\right)},\mathbf{1}\right)\frac{\partial \phi_j}{\partial e_i}\left(\frac{t}{e^*},\mathbf{1}\right); \tag{A31}$$

$$\Gamma_1 \equiv \phi_i(\mathbf{1})^{\delta-1}\frac{\partial V_i}{\partial \phi_i}(\mathbf{1})\frac{\partial \phi_i}{\partial e_i}(\mathbf{1}); \tag{A32}$$

$$\Gamma_2 \equiv (n-1)\phi_i(\mathbf{1})^{\delta-1}\frac{\partial V_i}{\partial \phi_j}(\mathbf{1})\frac{\partial \phi_j}{\partial e_i}(\mathbf{1}). \tag{A33}$$

At this point, it is possible to rewrite the payoff difference by choosing any positive effort level $e_+$ and no effort as:

$$\Delta_{e_+,0} \equiv \frac{\int_0^e (\Omega_1(t) - \Gamma_1)dt + \int_0^e (\Omega_2(t) - \Gamma_2)dt}{\frac{\partial V_i}{\partial \phi_i}(\mathbf{1})\Gamma_1 + \frac{\partial V_i}{\partial \phi_j}(\mathbf{1})\Gamma_2} \tag{A34}$$

$$\equiv \frac{\int_0^e (\Omega_1(t) + \Omega_2(t) - (\Gamma_1 + \Gamma_2))dt}{\frac{\partial V_i}{\partial \phi_i}(\mathbf{1})\Gamma_1 + \frac{\partial V_i}{\partial \phi_j}(\mathbf{1})\Gamma_2}. \tag{A35}$$

*case a)*

Since $\frac{e^*}{e^*} < \bar{e}$, according to condition iii), for each $t \in [0, e^*)$, $\Gamma_1 + \Gamma_2 > \Omega_1(t) + \Omega_2(t)$. Thus, for $e = e^*$, the integrand in (A35) is negative almost everywhere, thereby $\Delta_{e^*,0} < 0$. This implies that the payoff by exerting no effort is larger than the one by exerting the effort level in (A19). This contradicts condition ii).

*case b)*

Since $\frac{\bar{\bar{e}}}{e^*} < \bar{e}$, according to condition iii), for each $t \in [0, \bar{\bar{e}})$, $\Omega_1(t) + \Omega_2(t) < \Omega_1(\bar{\bar{e}}) + \Omega_2(\bar{\bar{e}})$, so that for $e = \bar{\bar{e}}$, the integrand in (A35) is such that $\Omega_1(t) + \Omega_2(t) - (\Gamma_1 + \Gamma_2) < \Omega_1(\bar{\bar{e}}) + \Omega_2(\bar{\bar{e}}) - (\Gamma_1 + \Gamma_2) = 0$, where the equality follows from the fact that $\bar{\bar{e}}$ is assumed to be a solution to (A25). It follows that $\Delta_{\bar{\bar{e}},0} < 0$, thereby $\bar{\bar{e}}$ is not the best response to all the remaining players choosing $e = e^*$. □