Why are discrete implications necessary? An analysis through the discretization process

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***Abstract*—Discrete implications have been studied for almost two decades as those operators needed to perform inference processes when dealing with qualitative information from a finite chain. However, it is known that by means of some adequate transformations, fuzzy logic operators defined in $[0,1]$ can generate the corresponding discrete operators. Thus, an immediate question arises: do we need to study discrete implications or is it enough to study implications defined in $[0,1]$ and then to discretize them? The answer must rely on the preservation of the additional properties of fuzzy implication functions through these discretization methods. In this paper, for two specific methods based on the ceiling and floor functions it is proved that most of the additional properties are not preserved in general, showing that the preservation of the additional properties depends directly on the properties of the underlying operators considered in the discretization. Thus, sufficient, and for some properties necessary, conditions to guarantee the preservation are presented.**

***Index Terms*—discrete implications, discretization, finite chains**

I. Introduction

When we have to deal with qualitative information, the use of finite scales is a good framework to model this type of data. For example, considering the set of linguistic labels that an expert has to assess the behavior of a certain system:

$$S = \{\text{Very Slow}, \text{Slow}, \text{Medium}, \text{Fast}, \text{Very Fast}\}.$$

In this context, in order to generically represent this type of information, the finite chains $L_n = \{0, 1, \ldots, n\}$ or $\Gamma_n = \{i/n \mid i \in L_n\}$ are usually considered to model these linguistic labels. The operations defined on these two sets are often referred to as discrete operators, and their main objective is to handle qualitative information directly, avoiding the numerical conversion between the finite chain and the unit interval. Some well-known families of discrete operators are discrete negations [1], discrete t-norms ([1], [2], [3]) and discrete implications ([4], [5]), which are the discrete counterpart of fuzzy negations [6], t-norms [2] and fuzzy implication functions ([7], [8]), respectively, in the $[0,1]$ framework. In addition, for more information about other families of operators and their applications, we recommend the reader to visit [9] and [10] for discrete uninorms (a generalization of discrete t-norms), [11], [12], [13] and [14] for discrete fuzzy numbers and its applications in decision making, and finally, [15] and [16] for image processing applications of discrete implications and discrete t-norms in the framework of discrete mathematical morphology.

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Among all these discrete operators, we highlight discrete implications whose aim is to generalize in the discrete setting the conditional statement *if p then q*. Despite the undoubted importance that the concept of discrete implication presents, a question that arises is precisely in its reason to be studied separately. If the numerical conversion process preserves the properties that are satisfied in one domain into the other, it should not be necessary to consider discrete implications: it would be enough to convert a fuzzy implication function defined in $[0,1]$ to a discrete implication in L_n or Γ_n. As far as we know, the literature has only considered the study of the conversion between t-norms and discrete t-norms in [2], and between copulas and discrete copulas in [17]–[19]. In fact, it is only studied when the restriction of a t-norm and a copula over Γ_n is a discrete t-norm and a discrete copula, respectively. Apart from these references, we have not found any other studies that explores this question in depth.

From the above discussion, the main objective of this paper is to study whether the conversion between fuzzy implication functions and discrete implications, which we call the discretization process, preserves some well-known properties of fuzzy implication functions, pretending to argue that it is indeed necessary to consider discrete implications separately. To do so, the paper is structured as follows. First, Section II establishes the common notation to be used throughout the paper, and introduces the main concepts to work with. Section III shows that it is possible to convert a fuzzy implication function to a discrete implication and vice versa by means of the floor and ceiling functions. This fact could lead to the idea that, since the chosen discretization process is appropriate for the task of transforming fuzzy implication functions to discrete implications, a separate study of discrete implications would not be entirely necessary. Then, in Section IV we study the preservation of additional properties through this discretization process. For the discretization methods presented in Section III, it is shown that they are not entirely satisfactory since some properties are not translated directly from their version in $[0,1]$ to their discrete counterpart. The paper ends with some conclusions and some samples of future work.

II. Preliminaries

In order to this paper be self-contained, we recall in this section some concepts about fuzzy implication functions ([7], [8]), t-norms [2], negations [6] and their corresponding concept in the discrete framework ([1], [4]).

First of all, we give the definition of fuzzy negation.

Definition II.1. *[6] A* fuzzy negation *is a function $N : [0,1] \to [0,1]$ such that satisfies the following axioms:*

(N1) N is decreasing, i.e., for all $x, y \in [0,1]$ such that $x \leqslant y$, then $N(x) \geqslant N(y)$.
(N2) $N(0) = 1$ and $N(1) = 0$.

Another widely studied operator is the triangular norm, which is a well-known family of fuzzy conjunctions.

Definition II.2. *[2],[20] A* triangular norm *(briefly* t-norm*) is a binary operator $T : [0,1]^2 \to [0,1]$ such that, for all $x, y, z \in [0,1]$, satisfies the following axioms:*

(T1) T is commutative, i.e., $T(x,y) = T(y,x)$.
(T2) T is associative, i.e., $T(T(x,y),z) = T(x,T(y,z))$.
(T3) T is monotone increasing, i.e., $T(x,y) \leqslant T(x,z)$ whenever $y \leqslant z$.
(T4) $T(x,1) = x$.

We now recall the definition of a fuzzy implication function.

Definition II.3. *[7] A* fuzzy implication function *is a binary operator $I : [0,1]^2 \to [0,1]$ such that, for all $x, y, z \in [0,1]$, satisfies the following axioms:*

(I1) I is decreasing in the first argument, i.e., $I(x,y) \geqslant I(z,y)$ whenever $x \leqslant z$.
(I2) I is increasing in the second argument, i.e., $I(x,y) \leqslant I(x,z)$ whenever $y \leqslant z$.
(I3) $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$.

Note that from this definition, it follows that $I(0,x) = 1$ and $I(x,1) = 1$ for all $x \in [0,1]$, whereas the segments $I(x,0)$ and $I(1,x)$ are not derived from the definition. Now, we present some additional interesting properties for fuzzy implication functions:

- The *exchange principle*,
$$I(x, I(y,z)) = I(y, I(x,z)), \quad x,y,z \in [0,1]. \qquad \textbf{(EP)}$$
- The *left neutrality principle*,
$$I(1,y) = y, \quad y \in [0,1]. \qquad \textbf{(NP)}$$
- The *identity principle*,
$$I(x,x) = 1, \quad x \in [0,1]. \qquad \textbf{(IP)}$$
- The *ordering principle*,
$$I(x,y) = 1 \Leftrightarrow x \leqslant y, \quad x,y \in [0,1]. \qquad \textbf{(OP)}$$
- The *consequent boundary*,
$$I(x,y) \geqslant y, \quad x,y \in [0,1]. \qquad \textbf{(CB)}$$
- The *contrapositive symmetry* with respect to a negation N,
$$I(x,y) = I(N(y), N(x)), \quad x,y \in [0,1]. \qquad \textbf{(CP}(N)\textbf{)}$$
- The *law of importation* with respect to a t-norm T,
$$I(T(x,y),z) = I(x, I(y,z)), \quad x,y,z \in [0,1]. \qquad \textbf{(LI}(T)\textbf{)}$$

In addition to the previous properties, continuity is another property that will be taken into account in the following sections. Given a fuzzy implication function I, we say that it is continuous when it is with the usual concept of continuity for real-valued functions.

To avoid confusion between operators defined in $[0,1]$ and those defined in the discrete framework, we will add the subscript n to indicate that the operator is discrete, and we will denote discrete negations by $N_n : L_n \to L_n$, discrete t-norms by $T_n : L_n^2 \to L_n$ and discrete implications by $I_n : L_n^2 \to L_n$, which are defined simply by changing the domain of definition and keeping the axioms of its $[0,1]$ counterpart. However, due to its discrete nature, continuity is not available in the discrete case; therefore, an attempt to translate the continuity to the discrete framework is made with the idea of smoothness, which appeared for the first time in [21] and has been extensively studied in [1] for discrete t-norms, and in [4], [5], [22], [23] for discrete implications. Let us give the definition of smoothness.

Definition II.4. *[1] Let $f : L_n \to L_n$ be a unary operator. It is said that f is k-*smooth *(or simply* smooth *when $k = 1$) if*

$$0 \leqslant |f(x+1) - f(x)| \leqslant k, \tag{1}$$

for all $x \in L_n \setminus \{n\}$.

According to this definition, the idea of continuity corresponds in the discrete framework to the 1-smooth case. Now, given a binary operator, we give the definition of smoothness.

Definition II.5. *[1] Let $F : L_n^2 \to L_n$ be a binary operator. It is said that F is k-*smooth *(or simply* smooth *when $k = 1$) if it is k-smooth in each argument; that is,*

$$|F(x+1,y) - F(x,y)| \leqslant k, \tag{2}$$

for all $x \in L_n \setminus \{n\}$ and $y \in L_n$, and

$$|F(x,y+1) - F(x,y)| \leqslant k, \tag{3}$$

for all $x \in L_n$ and $y \in L_n \setminus \{n\}$.

Finally, let us anticipate that practically all the results of this paper involve the floor and ceiling functions and some properties about them. Next, we recall their definition.

Definition II.6. *Let $x \in \mathbb{R}$. The* floor *and* ceiling *functions are defined, respectively, as the greatest integer less than or equal to x, and the least integer greater than or equal to x, and are represented as*

$$\lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \leqslant x\}, \tag{4}$$
$$\lceil x \rceil = \min\{m \in \mathbb{Z} \mid m \geqslant x\}. \tag{5}$$

Now we present a technical lemma about some useful properties about floor and ceiling functions.

Lemma II.1. *The following statements are satisfied:*

(i) The functions $f(x) = \lfloor x \rfloor$ and $c(x) = \lceil x \rceil$ are increasing.
(ii) For all $x, y \in \mathbb{R}$, $\lfloor x \rfloor - \lfloor y \rfloor - 1 \leqslant \lfloor x - y \rfloor \leqslant \lfloor x \rfloor - \lfloor y \rfloor$.
(iii) For all $x, y \in \mathbb{R}$, $\lceil x \rceil - \lceil y \rceil \leqslant \lceil x - y \rceil \leqslant \lceil x \rceil - \lceil y \rceil + 1$.

Proof. Let us prove each statement.

(i) The proof is straightforward by applying the definition of floor and ceiling.

(ii) Given $x, y \in \mathbb{R}$, we can decompose each number as $x = n + \alpha$ and $y = m + \beta$, with $n, m \in \mathbb{Z}$ and $\alpha, \beta \in [0, 1[$. Then:
- $\lfloor x \rfloor - \lfloor y \rfloor = \lfloor n + \alpha \rfloor - \lfloor m + \beta \rfloor = n - m$.
- $\lfloor x - y \rfloor = \lfloor n + \alpha - m - \beta \rfloor = \lfloor n - m + (\alpha - \beta) \rfloor = n - m + \lfloor \alpha - \beta \rfloor$. When $0 \leqslant \alpha - \beta < 1$, we have that $\lfloor \alpha - \beta \rfloor = 0$; conversely, when $-1 < \alpha - \beta < 0$, it follows that $\lfloor \alpha - \beta \rfloor = -1$.

With these two conditions, $\lfloor x \rfloor - \lfloor y \rfloor - 1 \leqslant \lfloor x - y \rfloor \leqslant \lfloor x \rfloor - \lfloor y \rfloor$.

(iii) Given $x, y \in \mathbb{R}$, using the same decomposition $x = n+\alpha$ and $y = m + \beta$, with $n, m \in \mathbb{Z}$ and $\alpha, \beta \in [0, 1[$:
- $\lceil x \rceil - \lceil y \rceil = \lceil n+\alpha \rceil - \lceil m+\beta \rceil = (n+1)-(m+1) = n - m$.
- $\lceil x - y \rceil = \lceil n + \alpha - m - \beta \rceil = \lceil n - m + (\alpha - \beta) \rceil = n - m + \lceil \alpha - \beta \rceil$. When $-1 < \alpha - \beta \leqslant 0$, we have that $\lceil \alpha - \beta \rceil = 0$; conversely, when $0 < \alpha - \beta < 1$, it follows that $\lceil \alpha - \beta \rceil = 1$.

With these two conditions, $\lceil x \rceil - \lceil y \rceil \leqslant \lceil x - y \rceil \leqslant \lceil x \rceil - \lceil y \rceil + 1$.

□

To conclude the section, we present the following lemma which relates the floor and ceiling functions.

Lemma II.2. *The following statements are satisfied:*

(i) If $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$, then $\lfloor x \rfloor - \lfloor y \rfloor = \lceil x \rceil - \lceil y \rceil$.
(ii) If $x \in \mathbb{R} \setminus \mathbb{Z}$ and $y \in \mathbb{R} \setminus \mathbb{Z}$, then $\lfloor x \rfloor - \lfloor y \rfloor = \lceil x \rceil - \lceil y \rceil$.
(iii) If $x \in \mathbb{Z}$ and $y \in \mathbb{R} \setminus \mathbb{Z}$, then $\lfloor x \rfloor - \lfloor y \rfloor = \lceil x \rceil - \lceil y \rceil + 1$.
(iv) If $x \in \mathbb{R} \setminus \mathbb{Z}$ and $y \in \mathbb{Z}$, then $\lfloor x \rfloor - \lfloor y \rfloor = \lceil x \rceil - \lceil y \rceil - 1$.

Proof. (i) If $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$, then $\lfloor x \rfloor = \lceil x \rceil = x$ and $\lfloor y \rfloor = \lceil y \rceil = y$; directly $\lfloor x \rfloor - \lfloor y \rfloor = \lceil x \rceil - \lceil y \rceil$.

(ii) If $x \in \mathbb{R} \setminus \mathbb{Z}$ and $y \in \mathbb{R} \setminus \mathbb{Z}$, then $\lceil x \rceil = \lfloor x \rfloor + 1$ and $\lceil y \rceil = \lfloor y \rfloor + 1$. Subtracting both expressions, the result follows:

$$\lceil x \rceil - \lceil y \rceil = (\lfloor x \rfloor + 1) - (\lfloor y \rfloor + 1) = \lfloor x \rfloor - \lfloor y \rfloor.$$

(iii) If $x \in \mathbb{Z}$ and $y \in \mathbb{R} \setminus \mathbb{Z}$, then $\lfloor x \rfloor = \lceil x \rceil = x$ and $\lceil y \rceil = \lfloor y \rfloor + 1$. Subtracting both expressions, the result follows:

$$\lceil x \rceil - \lceil y \rceil = \lfloor x \rfloor - (\lfloor y \rfloor + 1) = \lfloor x \rfloor - \lfloor y \rfloor - 1.$$

(iv) If $x \in \mathbb{R} \setminus \mathbb{Z}$ and $y \in \mathbb{Z}$, then $\lceil x \rceil = \lfloor x \rfloor + 1$ and $\lfloor y \rfloor = \lceil y \rceil = y$. Subtracting both expressions, the result follows:

$$\lceil x \rceil - \lceil y \rceil = \lfloor x \rfloor + 1 - \lfloor y \rfloor.$$

□

III. Discretization of fuzzy implication functions and extension of discrete implications

From a fuzzy implication function $I : [0, 1]^2 \to [0, 1]$, it would be interesting to generate a discrete implication $I_n : L_n^2 \to L_n$. Moreover, if I satisfies a certain property, one would expect that the property is preserved and I_n also satisfies it in the discrete framework, obtaining a link between fuzzy implication functions and discrete implications. In order to convert a scalar-valued function to a discrete operator, we will now define the generic upper and lower discretizations with respect to a positive integer using the floor and ceiling functions (see Definition II.6). This process will be called discretization.

Definition III.1. *Let $F : \mathbb{R}^2 \to [0, 1]$ be a scalar-valued function. The* upper *and* lower discretizations *of F with respect to a positive integer $n \geqslant 1$ are defined as the discrete mappings $F_n^{\mathrm{U}}, F_n^{\mathrm{L}} : L_n^2 \to L_n$ whose expressions are, respectively,*

$$F_n^{\mathrm{U}}(i, j) = \left\lceil n \cdot F\left(\frac{i}{n}, \frac{j}{n}\right) \right\rceil, \tag{6}$$

$$F_n^{\mathrm{L}}(i, j) = \left\lfloor n \cdot F\left(\frac{i}{n}, \frac{j}{n}\right) \right\rfloor, \tag{7}$$

for all $i, j \in L_n$.

For the discretization to be an appropriate process, when applying it to a fuzzy implication function I the result must be a discrete implication. The following result shows that when discretizing a fuzzy implication function through these methods, a discrete implication is obtained.

Proposition III.1. *Let $I : [0, 1]^2 \to [0, 1]$ be a fuzzy implication function. Then, the upper and lower discretizations $I_n^{\mathrm{U}}, I_n^{\mathrm{L}} : L_n^2 \to L_n$ of I are discrete implications.*

Proof. First of all, let us see that the upper and lower discretizations are well-defined; that is, that the image of any point of L_n^2 lies inside L_n. Since I is a fuzzy implication function, it satisfies that $0 \leqslant I(x, y) \leqslant 1$ and, therefore, $0 \leqslant n \cdot I(x, y) \leqslant n$. In consequence, $0 \leqslant \lfloor n \cdot I(x, y) \rfloor \leqslant n$ and $0 \leqslant \lceil n \cdot I(x, y) \rceil \leqslant n$ for all $(x, y) \in [0, 1]^2$. In short, the discrete mappings

$$I_n^{\mathrm{U}}(i, j) = \left\lceil n \cdot I\left(\frac{i}{n}, \frac{j}{n}\right) \right\rceil,$$
$$I_n^{\mathrm{L}}(i, j) = \left\lfloor n \cdot I\left(\frac{i}{n}, \frac{j}{n}\right) \right\rfloor,$$

are well-defined. Let us check that both satisfy the axioms of discrete implications. Without loss of generality we will prove only for I_n^{U} since the proof for I_n^{L} is analogous.

- $I_n^{\mathrm{U}}(0, 0) = \lceil n \cdot I(0, 0) \rceil = \lceil n \cdot 1 \rceil = \lceil n \rceil = n$.
- $I_n^{\mathrm{U}}(n, n) = \lceil n \cdot I(1, 1) \rceil = \lceil n \cdot 1 \rceil = \lceil n \rceil = n$.
- $I_n^{\mathrm{U}}(n, 0) = \lceil n \cdot I(1, 0) \rceil = \lceil n \cdot 0 \rceil = 0$.

Now, we know that I is decreasing in the first argument and increasing in the second. Applying statement (i) of Lemma II.1, $I_n^{\mathrm{U}}, I_n^{\mathrm{L}}$ are also decreasing in the first argument and increasing in the second. □

Throughout the paper we will consider both discretizations as they are different operators and may not coincide in some cases, as shown in the following example.

Example III.1. *The discretizations I_n^{L} and I_n^{U} do not coincide in general. If we consider the Reichenbach implication, given by $I_{\mathrm{RC}}(x, y) = 1 - x + xy$ for all $x, y \in [0, 1]$, the* *discretizations evaluated at* $(n-1, n-1)$ *do not coincide because*

$$I_n^{\mathrm{L}}(n-1,n-1) = \left\lfloor n-1+\frac{1}{n} \right\rfloor = n-1,$$
$$I_n^{\mathrm{U}}(n-1,n-1) = \left\lceil n-1+\frac{1}{n} \right\rceil = n,$$

for all $n \geqslant 1$.

So far, we have seen that it is possible to convert a fuzzy implication function in $[0,1]$ to a discrete implication. Let us tackle the inverse problem. Given a discrete implication, we ask whether it is possible to find a fuzzy implication function whose discretizations coincide and are the initial discrete implication. As the reader should be foreseeing, the answer to this problem is affirmative: it is enough to construct an implication I such that $I(x,y) = \frac{I_n(n\cdot x, n\cdot y)}{n}$ when $(x,y) \in \Gamma_n^2$, where $\Gamma_n = \{ i/n \mid i \in L_n \}$; for the points in the remaining space $[0,1]^2 \setminus \Gamma_n^2$, we can perform a piece-wise linear interpolation.

Theorem III.1. *Let* $I_n : L_n^2 \to L_n$ *be a discrete implication. Then,* I_n *is the upper and lower discretization of some continuous implication* $I : [0,1]^2 \to [0,1]$.

Proof. From the given discrete implication I_n, let us construct a continuous implication. Let us consider the set $\Gamma_n \subset [0,1]$, such that Γ_n^2 defines a grid within $[0,1]^2$. For each $i,j \in L_n \setminus \{n\}$, let us consider the following points:

$$P_{i,j} = \left(\frac{i}{n}, \frac{j}{n}, \frac{I_n(i,j)}{n} \right),$$
$$P_{i,j+1} = \left(\frac{i}{n}, \frac{j+1}{n}, \frac{I_n(i,j+1)}{n} \right),$$
$$P_{i+1,j} = \left(\frac{i+1}{n}, \frac{j}{n}, \frac{I_n(i+1,j)}{n} \right),$$
$$P_{i+1,j+1} = \left(\frac{i+1}{n}, \frac{j+1}{n}, \frac{I_n(i+1,j+1)}{n} \right).$$

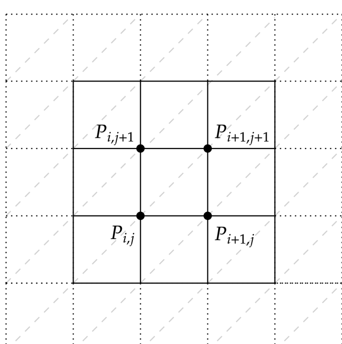


Fig. 1. Scheme for the construction of the planes that will define the continuous implication. Within each triangle in the XY plane generated by $P_{i,j}, P_{i,j+1}, P_{i+1,j+1}$ and $P_{i,j}, P_{i+1,j}, P_{i+1,j+1}$, a plane passing through these three points will be constructed.

The plane passing through the points $P_{i,j}, P_{i,j+1}, P_{i+1,j+1}$ has $\boldsymbol{v}_1 = \overrightarrow{P_{i,j}P_{i,j+1}}$ and $\boldsymbol{v}_2 = \overrightarrow{P_{i,j}P_{i+1,j+1}}$ as director vectors, and passes through the point $P_{i,j}$. Likewise, the plane that passes through the points $P_{i,j}, P_{i+1,j}, P_{i+1,j+1}$ has as director vectors the same $\boldsymbol{v}_2$, and also $\boldsymbol{v}_3 = \overrightarrow{P_{i,j}P_{i+1,j}}$. Because both planes pass through the same point $P_{i,j}$ and share the director vector of the diagonal, continuity is assured on the diagonal of each square (see Figure 1). Now, each triangle named above defines a domain in $[0,1]^2$, and the expression of the continuous implication in each domain is the plane passing through the three points of the triangle. Let us calculate the expression in each domain.

- Domain of the plane generated by $P_{i,j}, P_{i,j+1}, P_{i+1,j+1}$:

$$\mathcal{R}_{i,j}^1 = \left\{ (x,y) \in \left[\frac{i}{n}, \frac{i+1}{n} \right] \times \left[\frac{j}{n}, \frac{j+1}{n} \right] \;\middle|\; \left(y - \frac{j}{n} \right) \geqslant \left(x - \frac{i}{n} \right) \right\}.$$

The expression of the plane is:

$$z_{ij}(x,y) = \left(x - \frac{i}{n} \right) \left(\frac{I_n(i+1,j+1)}{n} - \frac{I_n(i,j+1)}{n} \right) + \left(y - \frac{j}{n} \right) \left(\frac{I_n(i,j+1)}{n} - \frac{I_n(i,j)}{n} \right) + \frac{I_n(i,j)}{n}.$$

- Domain of the plane generated by $P_{i,j}, P_{i+1,j}, P_{i+1,j+1}$:

$$\mathcal{R}_{i,j}^2 = \left\{ (x,y) \in \left[\frac{i}{n}, \frac{i+1}{n} \right] \times \left[\frac{j}{n}, \frac{j+1}{n} \right] \;\middle|\; \left(y - \frac{j}{n} \right) \leqslant \left(x - \frac{i}{n} \right) \right\}.$$

The expression of the plane is:

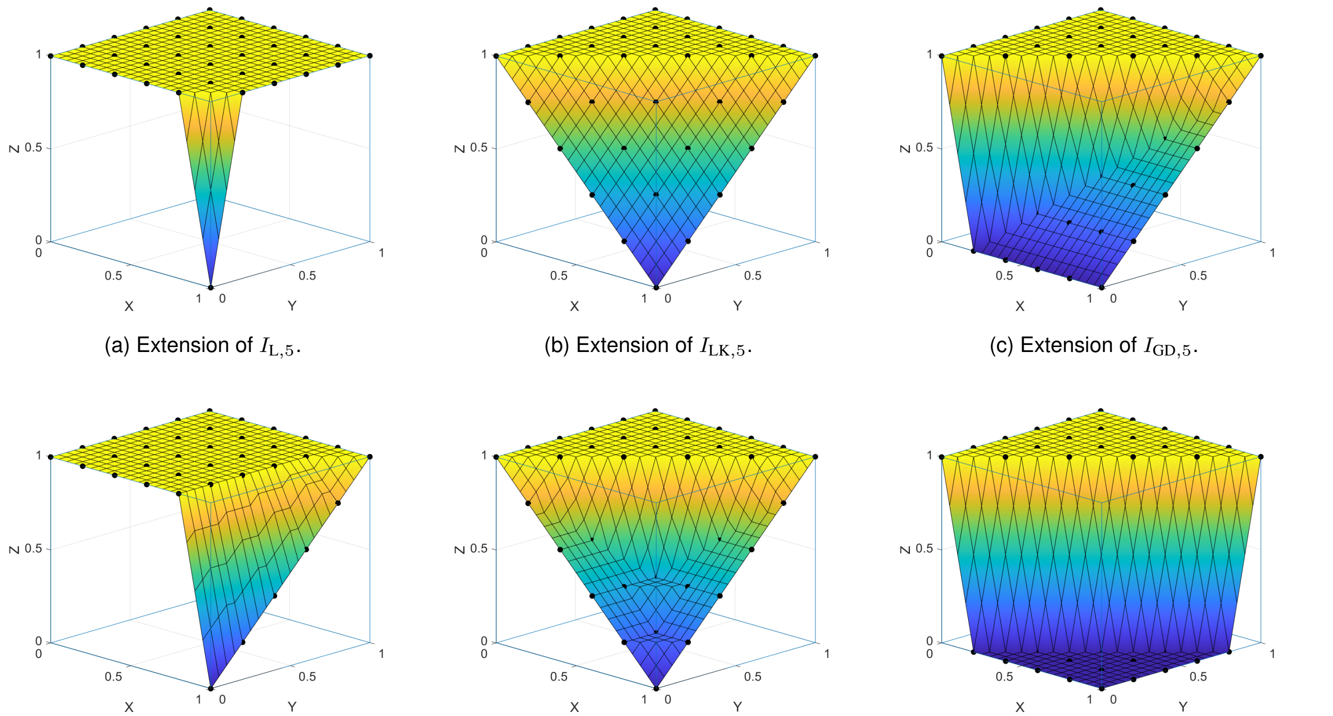
$$w_{ij}(x,y) = \left(x - \frac{i}{n} \right) \left(\frac{I_n(i+1,j)}{n} - \frac{I_n(i,j)}{n} \right) + \left(y - \frac{j}{n} \right) \left(\frac{I_n(i+1,j+1)}{n} - \frac{I_n(i+1,j)}{n} \right) + \frac{I_n(i,j)}{n}.$$

Then, for all $i,j \in L_n \setminus \{n\}$, the continuous implication $I : [0,1]^2 \to [0,1]$ is defined as:

$$I(x,y) = \begin{cases} z_{ij}(x,y), & \text{if } (x,y) \in \mathcal{R}_{i,j}^1 \text{ for some } i,j \in L_n \setminus \{n\}, \\ w_{ij}(x,y), & \text{if } (x,y) \in \mathcal{R}_{i,j}^2 \text{ for some } i,j \in L_n \setminus \{n\}. \end{cases} \tag{8}$$

It is direct to show that this binary function satisfies the boundary conditions of an implication in $[0,1]$ and it is decreasing in the first argument and increasing in the second one, and therefore is a fuzzy implication function. $\square$

Remark III.1. *The continuous implication constructed in Theorem III.1 is not unique. In the proof, a linear piece-wise interpolation has been applied on each of the two triangles generated by the main diagonal of each square* $\left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right]$, *for all* $i,j \in L_n \setminus \{n\}$ *(see Equation* (8) *for its expression). However, an analogous reasoning can be performed to obtain another continuous interpolation. Let us consider all the triangles generated in the same way, and let us also perform the same linear piece-wise interpolation on all of them, except for the lower triangle* $\mathcal{R}_{0,1}^1 \subset \left[0, \frac{1}{n}\right]^2$, *where we will construct a ruled surface.*



(a) Extension of $I_{\mathrm{L},5}$. (b) Extension of $I_{\mathrm{LK},5}$. (c) Extension of $I_{\mathrm{GD},5}$.

(d) Extension of $I_{\mathrm{WB},5}$. (e) Extension of $I_{\mathrm{FD},5}$. (f) Extension of $I_{\mathrm{RS},5}$.

Fig. 2. Representation of the continuous extension of the discrete implications defined in Example III.2 with $n = 5$. The black dots represent the discrete implication scaled in Γ_5 to fit in the unit cube. The piece-wise appearance can be observed using the planes over each domain $\mathcal{R}^1_{i,j}$ and $\mathcal{R}^2_{i,j}$.

Indeed, let us consider the function $f(x) = -ax^2 + 1$*, which will represent the values of* I *on the lower boundary, i.e.,* $I(x, 0) = f(x)$*, for all* $x \in \left[0, \frac{1}{n}\right]$*. To be an interpolation of the discrete implication* I_n*, we have to determine the value of* a *with the relation*

$$f\left(\frac{1}{n}\right) = \frac{I_n(1,0)}{n},$$

that leads us to get

$$a = n\left(n - I_n(1,0)\right).$$

Therefore, it only remains to generate the ruled surface that has as directrices functions the straight line that joins the point $(0, 0, 1)$ *with* $\left(\frac{1}{n}, \frac{1}{n}, \frac{I_n(1,1)}{n}\right)$*, and the function* f*. It is straightforward to observe that the generated surface is decreasing in the first argument, since* f *is decreasing, and increasing in the second argument, since each line of the sheaf of straight lines is increasing.*

Example III.2. *Let us consider the following discrete implications. In order of appearance: largest, Łukasiewicz, Gödel, Weber, Fodor and Rescher, whose expressions are, respectively,*

$$I_{\mathrm{L},n}(i,j) = \begin{cases} 0, & \text{if } (i,j) = (n,0), \\ n, & \text{otherwise,} \end{cases}$$

$$I_{\mathrm{LK},n}(i,j) = \min(n, n - i + j),$$

$$I_{\mathrm{GD},n}(i,j) = \begin{cases} n, & \text{if } i \leqslant j, \\ j, & \text{otherwise,} \end{cases}$$

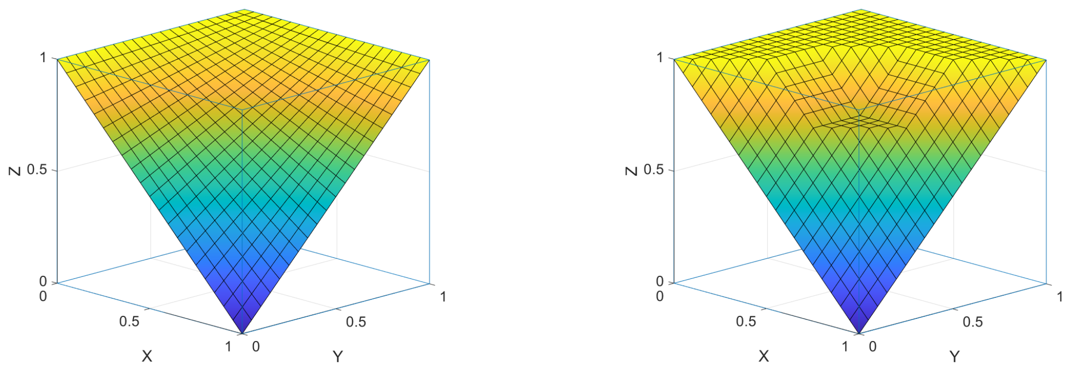
$$I_{\mathrm{WB},n}(i,j) = \begin{cases} n, & \text{if } i < n, \\ j, & \text{otherwise,} \end{cases}$$

$$I_{\mathrm{FD},n}(i,j) = \begin{cases} n, & \text{if } i \leqslant j, \\ \max(n-i, j), & \text{otherwise,} \end{cases}$$

$$I_{\mathrm{RS},n}(i,j) = \begin{cases} n, & \text{if } i \leqslant j, \\ 0, & \text{otherwise.} \end{cases}$$

Applying Theorem III.1, in particular Equation (8)*, in Figure 2 we have the plot of the continuous extensions of these discrete implications such that their discretizations yield the original discrete implications. It can be seen that the extension of the Łukasiewicz discrete implication* $I_{\mathrm{LK},n}$ *results in Łukasiewicz implication* I_{LK}*, since it is continuous and linearly piece-wise defined. For the other discrete implications, it is not satisfied that their extension is their corresponding version in* $[0, 1]$*. Since* I_{L}*,* I_{GD}*,* I_{WB}*,* I_{FD} *and* I_{RS} *are non-continuous fuzzy implication functions, when extending their discrete version in* L_n *they cannot be recovered because the extension procedure generates continuous implications.*

Moreover, if we consider a continuous fuzzy implication function, we discretize it and then we construct its continuous extension, we may not recover the original fuzzy implication function even though it is continuous. For instance, considering the Reichenbach implication I_{RC}*, this behavior is illustrated in Figure 3.*



(a) Plot of I_{RC}. (b) Extension of the upper discretization of I_{RC}.

Fig. 3. Graphical representation of the Reichenbach implication (Figure 3a), and the continuous extension of its upper discretization (Figure 3b). It can be seen that the original implication is not recovered, even if it is continuous.

IV. Preservation of properties through the discretization process

We start this section by returning to the problem raised at the beginning of Section III. Given a fuzzy implication function I which satisfies a certain property, by means of the discretization process, the aim of this section is to study whether I_n^{U} and I_n^{L} also satisfy it in its discrete counterpart. If the answer is affirmative, it would not be necessary to carry out a separate study of the properties of the discrete implications, since it would be sufficient to discretize I and conclude that I_n^{U} and I_n^{L} also satisfy it. However, if the answer is negative, it is interesting to establish under which conditions the properties are preserved.

Let us start studying the conversion between continuity in $[0,1]$ and smoothness in L_n. The following example shows that continuity is not necessarily converted into smoothness, and therefore property preservation is not always satisfied.

Example IV.1. *The fuzzy implication function* $I : [0,1]^2 \to [0,1]$ *given by*

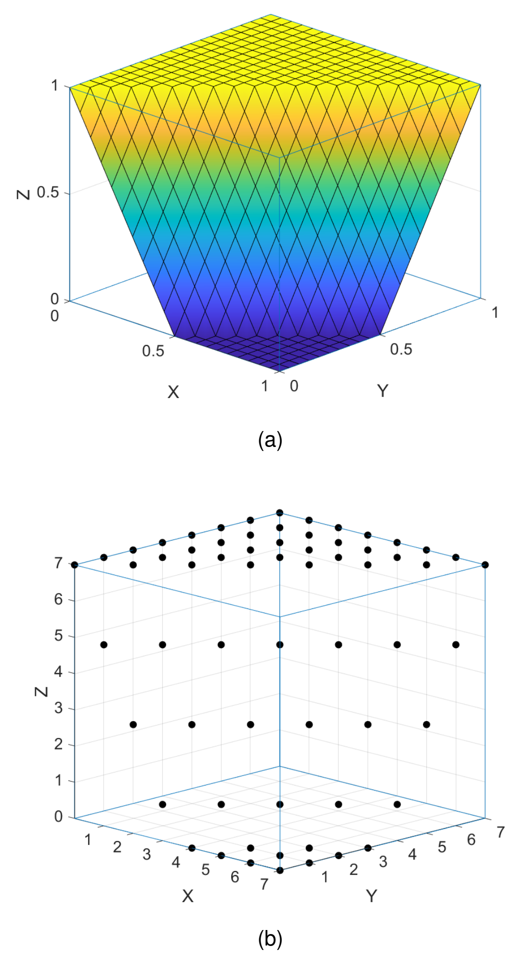
$$I(x,y) = \begin{cases} 1, & \text{if } x \leqslant y, \\ 1 + 2y - 2x, & \text{if } x - \frac{1}{2} \leqslant y \leqslant x, \\ 0, & \text{otherwise,} \end{cases} \tag{9}$$

is a continuous function. However, both discretizations are not smooth because

$$\begin{aligned} I_n^{\mathrm{L}}(0,0) - I_n^{\mathrm{L}}(1,0) &= I_n^{\mathrm{U}}(0,0) - I_n^{\mathrm{U}}(1,0) \\ &= n - \left\lceil n \cdot I\left(\frac{1}{n}, 0\right) \right\rceil \\ &= n - \left\lceil n - 2 \right\rceil = 2, \end{aligned}$$

for all $n \geqslant 2$. *The plot of the fuzzy implication function and the plot of its upper discretization are illustrated in Figure 4.*

In Example IV.1, the smoothness fails because the fuzzy implication function I presents a too sharp growth over a region of Γ_n^2. For the discretizations to be smooth, each section of I restricted to Γ_n^2 must not present sharp differences between consecutive values. This condition is sufficient to ensure smoothness.



(a)

(b)

Fig. 4. Plot of the continuous implication given in Equation (9) (Figure 4a) and its upper discretization with $n = 7$ (Figure 4b), which is not smooth.

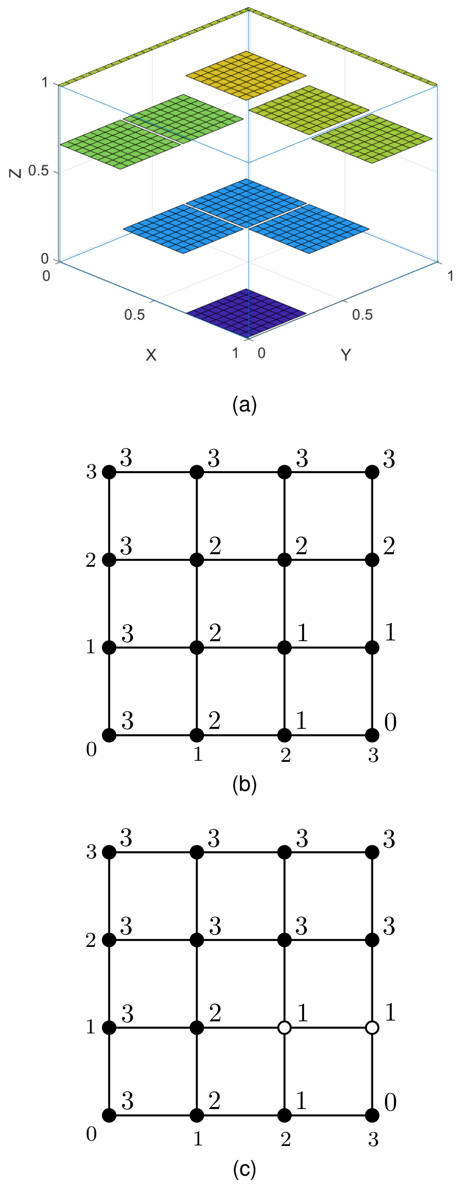


Fig. 5. Plot of the implication presented in Equation (12) (Figure 5a), its lower discretization I_n^{L} (Figure 5b) and its upper discretization I_n^{U} (Figure 5c). For each point $(i,j) \in L_n^2$, the image of the point of each discretization is shown. The white dots represent the points where smoothness fails.

Proposition IV.1. *Let $I : [0,1]^2 \to [0,1]$ be a fuzzy implication function, and $\Delta_1 : (L_n \backslash \{n\}) \times L_n \to [0,1]$ and $\Delta_2 : L_n \times (L_n \backslash \{n\}) \to [0,1]$ be the forward difference operations in each argument of I, given by*

$$\Delta_1(i,j) = I\left(\frac{i}{n}, \frac{j}{n}\right) - I\left(\frac{i+1}{n}, \frac{j}{n}\right), \tag{10}$$

$$\Delta_2(i,j) = I\left(\frac{i}{n}, \frac{j+1}{n}\right) - I\left(\frac{i}{n}, \frac{j}{n}\right). \tag{11}$$

If $0 \leqslant \Delta_1(i,j) \leqslant \frac{1}{n}$, for all $(i,j) \in (L_n \backslash \{n\}) \times L_n$, and $0 \leqslant \Delta_2(i,j) \leqslant \frac{1}{n}$ for all $(i,j) \in L_n \times (L_n \backslash \{n\})$, the lower and upper discretizations $I_n^{\mathrm{U}}, I_n^{\mathrm{L}} : L_n^2 \to L_n$ of I are smooth.

Proof. Let us prove the smoothness of I_n^{L} in the first argument by considering the inequalities proved in Lemma II.1 and considering two cases:

- If $0 \leqslant \Delta_1(i,j) < \frac{1}{n}$ for some $(i,j) \in (L_n \backslash \{n\}) \times L_n$, applying that $\lfloor x \rfloor - \lfloor y \rfloor - 1 \leqslant \lfloor x - y \rfloor$ for all $x, y \in \mathbb{R}$:

$$\begin{aligned} I_n^{\mathrm{L}}(i,j) - I_n^{\mathrm{L}}(i+1,j) - 1 = \\ &= \left\lfloor n \cdot I\left(\frac{i}{n}, \frac{j}{n}\right) \right\rfloor - \left\lfloor n \cdot I\left(\frac{i+1}{n}, \frac{j}{n}\right) \right\rfloor - 1 \\ &\leqslant \left\lfloor n \cdot \left(I\left(\frac{i}{n}, \frac{j}{n}\right) - I\left(\frac{i+1}{n}, \frac{j}{n}\right) \right) \right\rfloor \\ &= \lfloor n \cdot \Delta_1(i,j) \rfloor = 0. \end{aligned}$$

- If $\Delta_1 = \frac{1}{n}$ for some $(i,j) \in (L_n \backslash \{n\}) \times L_n$, using that $\lfloor x + k \rfloor = \lfloor x \rfloor + k$ for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$:

$$\begin{aligned} I_n^{\mathrm{L}}(i,j) - I_n^{\mathrm{L}}(i+1,j) = \\ &= \left\lfloor n \cdot I\left(\frac{i}{n}, \frac{j}{n}\right) \right\rfloor - \left\lfloor n \cdot I\left(\frac{i+1}{n}, \frac{j}{n}\right) \right\rfloor \\ &= \left\lfloor 1 + n \cdot I\left(\frac{i+1}{n}, \frac{j}{n}\right) \right\rfloor - \left\lfloor n \cdot I\left(\frac{i+1}{n}, \frac{j}{n}\right) \right\rfloor \\ &= 1 + \left\lfloor n \cdot I\left(\frac{i+1}{n}, \frac{j}{n}\right) \right\rfloor - \left\lfloor n \cdot I\left(\frac{i+1}{n}, \frac{j}{n}\right) \right\rfloor \\ &= 1. \end{aligned}$$

In any case, we have proved that $I_n^{\mathrm{L}}(i,j) - I_n^{\mathrm{L}}(i+1,j) \leqslant 1$ for all $(i,j) \in (L_n \backslash \{n\}) \times L_n$. Smoothness in the second argument can be proved with an analogous argument considering two cases with Δ_2. Now, let us prove the smoothness of I_n^{U} in the first argument.

$$\begin{aligned} I_n^{\mathrm{U}}(i,j) - I_n^{\mathrm{U}}(i+1,j) &= \left\lceil n \cdot I\left(\frac{i}{n}, \frac{j}{n}\right) \right\rceil - \left\lceil n \cdot I\left(\frac{i+1}{n}, \frac{j}{n}\right) \right\rceil \\ &\leqslant \left\lceil n \cdot \left(I\left(\frac{i}{n}, \frac{j}{n}\right) - I\left(\frac{i+1}{n}, \frac{j}{n}\right) \right) \right\rceil \\ &= \lceil n \cdot \Delta_1(i,j) \rceil = 1. \end{aligned}$$

Obtaining that $I_n^{\mathrm{U}}(i,j) - I_n^{\mathrm{U}}(i+1,j) \leqslant 1$ for all $(i,j) \in (L_n \backslash \{n\}) \times L_n$. The smoothness in the second argument can be proved with an analogous argument. □

Example IV.2 shows that the smoothness of I_n^{L} does not imply the smoothness of I_n^{U}. Furthermore, it is also shown that the condition stated in Proposition IV.1 is not a necessary condition.

Example IV.2. *Let us consider the non-continuous piece-wise implication $I : [0,1]^2 \to [0,1]$ given by*

$$I(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1, \\ \frac{2}{3}, & \text{if } x \in \left]0, \frac{1}{3}\right] \text{ and } y \in \left[0, \frac{2}{3}\right[, \\ \frac{23}{30}, & \text{if } x \in \left]0, \frac{1}{3}\right] \text{ and } y \in \left[\frac{2}{3}, 1\right[, \\ \frac{1}{3}, & \text{if } x \in \left]\frac{1}{3}, \frac{2}{3}\right] \text{ and } y \in \left[0, \frac{2}{3}\right[, \\ \frac{43}{60}, & \text{if } x \in \left]\frac{1}{3}, \frac{2}{3}\right] \text{ and } y \in \left[\frac{2}{3}, 1\right[, \\ 0, & \text{if } x \in \left]\frac{2}{3}, 1\right] \text{ and } y \in \left[0, \frac{1}{3}\right[, \\ \frac{1}{3}, & \text{if } x \in \left]\frac{2}{3}, 1\right] \text{ and } y \in \left[\frac{1}{3}, \frac{2}{3}\right[, \\ \frac{7}{10}, & \text{if } x \in \left]\frac{2}{3}, 1\right] \text{ and } y \in \left[\frac{2}{3}, 1\right[. \end{cases} \tag{12}$$

Setting $n = 3$, although the lower discretization is smooth, the upper discretization is not; see diagram in Figure 5.

Furthermore, it can be observed that the conditions of Proposition IV.1 are not satisfied since

$$\Delta_2(2,1) = I\left(\frac{2}{3},\frac{2}{3}\right) - I\left(\frac{2}{3},\frac{1}{3}\right) = \frac{43}{60} - \frac{1}{3} > \frac{1}{3} = \frac{1}{n},$$

but the lower discretization is still smooth. The reason why smoothness is not met in the upper discretization is due to two facts. On the one hand, since I_3^{L} is smooth at point $(3,1)$, it follows that $I_3^{\mathrm{L}}(3,2) - I_3^{\mathrm{L}}(3,1) \leqslant 1$. Because $3 \cdot I\left(\frac{3}{3},\frac{2}{3}\right) \notin L_3$ and $3 \cdot I\left(\frac{3}{3},\frac{1}{3}\right) \in L_3$, applying Lemma II.2, we get

$$\begin{aligned}\left\lfloor 3 \cdot I\left(\frac{3}{3},\frac{2}{3}\right)\right\rfloor - \left\lfloor 3 \cdot I\left(\frac{3}{3},\frac{1}{3}\right)\right\rfloor &= \\ = \left\lceil 3 \cdot I\left(\frac{3}{3},\frac{2}{3}\right)\right\rceil - \left\lceil 3 \cdot I\left(\frac{3}{3},\frac{1}{3}\right)\right\rceil - 1 &\leqslant 1.\end{aligned}$$

Therefore $I_3^{\mathrm{U}}(3,2) - I_3^{\mathrm{U}}(3,1) \leqslant 2$. On the other hand, $I\left(\frac{3}{3},\frac{2}{3}\right) - I\left(\frac{3}{3},\frac{1}{3}\right) > \frac{1}{3}$, and this causes that $3 \cdot I\left(\frac{3}{3},\frac{2}{3}\right) - 3 \cdot I\left(\frac{3}{3},\frac{1}{3}\right) > 1$. Applying ceiling function at both sides of the inequality and using that $\lceil x+k \rceil = \lceil x \rceil + k$, for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$, we get

$$\left\lceil 3 \cdot I\left(\frac{3}{3},\frac{2}{3}\right)\right\rceil - \left\lceil 3 \cdot I\left(\frac{3}{3},\frac{1}{3}\right)\right\rceil > 1.$$

With all this, it must necessarily happen that $I_3^{\mathrm{U}}(3,2) - I_3^{\mathrm{U}}(3,1) = 2$.

Next, we consider when it is that the smoothness of I_n^{U} is equivalent to the smoothness of I_n^{L}.

Proposition IV.2. *Let $I : [0,1]^2 \to [0,1]$ be a fuzzy implication function and $I_n^{\mathrm{U}}, I_n^{\mathrm{L}} : L_n^2 \to L_n$ be its discretizations. Let us suppose that for each $(i,j) \in (L_n \backslash \{n\})^2$ one, and only one of the following conditions is satisfied:*

(i) $I\left(\frac{i}{n},\frac{j}{n}\right) \in \Gamma_n$, $I\left(\frac{i+1}{n},\frac{j}{n}\right) \in \Gamma_n$ *and* $I\left(\frac{i}{n},\frac{j+1}{n}\right) \in \Gamma_n$.
(ii) $I\left(\frac{i}{n},\frac{j}{n}\right) \in [0,1] \setminus \Gamma_n$, $I\left(\frac{i+1}{n},\frac{j}{n}\right) \in [0,1] \setminus \Gamma_n$ *and* $I\left(\frac{i}{n},\frac{j+1}{n}\right) \in [0,1] \setminus \Gamma_n$.

Then, I_n^{L} is smooth if and only if I_n^{U} is smooth.

Proof. Given $(i,j) \in (L_n \backslash \{n\})^2$, let us prove each case.

- If I satisfies *(i)*, applying statement *(i)* of Lemma II.2, we have that I_n^{L} is smooth if, and only if,

$$\left\{\begin{array}{l}\left\lfloor n \cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rfloor - \left\lfloor n \cdot I\left(\frac{i+1}{n},\frac{j}{n}\right)\right\rfloor \leqslant 1 \\ \left\lfloor n \cdot I\left(\frac{i}{n},\frac{j+1}{n}\right)\right\rfloor - \left\lfloor n \cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rfloor \leqslant 1\end{array}\right\}$$
$$\Leftrightarrow \left\{\begin{array}{l}\left\lceil n \cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rceil - \left\lceil n \cdot I\left(\frac{i+1}{n},\frac{j}{n}\right)\right\rceil \leqslant 1 \\ \left\lceil n \cdot I\left(\frac{i}{n},\frac{j+1}{n}\right)\right\rceil - \left\lceil n \cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rceil \leqslant 1\end{array}\right\},$$

  and this occurs if, and only if, I_n^{U} is smooth.

- If I satisfies *(ii)*, applying statement *(ii)* of Lemma II.2, we have that I_n^{L} is smooth if, and only if,

$$\left\{\begin{array}{l}\left\lfloor n \cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rfloor - \left\lfloor n \cdot I\left(\frac{i+1}{n},\frac{j}{n}\right)\right\rfloor \leqslant 1 \\ \left\lfloor n \cdot I\left(\frac{i}{n},\frac{j+1}{n}\right)\right\rfloor - \left\lfloor n \cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rfloor \leqslant 1\end{array}\right\}$$
$$\Leftrightarrow \left\{\begin{array}{l}\left\lceil n \cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rceil - \left\lceil n \cdot I\left(\frac{i+1}{n},\frac{j}{n}\right)\right\rceil \leqslant 1 \\ \left\lceil n \cdot I\left(\frac{i}{n},\frac{j+1}{n}\right)\right\rceil - \left\lceil n \cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rceil \leqslant 1\end{array}\right\},$$

  and this occurs if, and only if, I_n^{U} is smooth.

□

This proposition establishes an equivalence between the smoothness of I_n^{L} and the smoothness of I_n^{U} assuming only two conditions about the range of the implication I. However, the reader will have noted that these two conditions are quite restrictive. In the next two propositions, we will study when the smoothness of I_n^{L} implies the smoothness of I_n^{U} and vice versa, adding more possibilities about the ranges of I.

Proposition IV.3. *Let $I : [0,1]^2 \to [0,1]$ be a fuzzy implication function and $I_n^{\mathrm{U}}, I_n^{\mathrm{L}} : L_n^2 \to L_n$ be its discretizations. Let us suppose that for each $(i,j) \in (L_n \backslash \{n\})^2$ one, and only one of the following conditions is satisfied:*

(i) $I\left(\frac{i}{n},\frac{j}{n}\right) \in \Gamma_n$, $I\left(\frac{i+1}{n},\frac{j}{n}\right) \in \Gamma_n$ *and* $I\left(\frac{i}{n},\frac{j+1}{n}\right) \in \Gamma_n$.
(ii) $I\left(\frac{i}{n},\frac{j}{n}\right) \in \Gamma_n$, $I\left(\frac{i+1}{n},\frac{j}{n}\right) \in [0,1]\backslash\Gamma_n$ *and* $I\left(\frac{i}{n},\frac{j+1}{n}\right) \in \Gamma_n$.
(iii) $I\left(\frac{i}{n},\frac{j}{n}\right) \in [0,1] \setminus \Gamma_n$, $I\left(\frac{i+1}{n},\frac{j}{n}\right) \in [0,1] \setminus \Gamma_n$ *and* $I\left(\frac{i}{n},\frac{j+1}{n}\right) \in \Gamma_n$.
(iv) $I\left(\frac{i}{n},\frac{j}{n}\right) \in [0,1] \setminus \Gamma_n$, $I\left(\frac{i+1}{n},\frac{j}{n}\right) \in [0,1] \setminus \Gamma_n$ *and* $I\left(\frac{i}{n},\frac{j+1}{n}\right) \in [0,1] \setminus \Gamma_n$.

Then, the smoothness of I_n^{L} implies the smoothness of I_n^{U}.

Proof. Given $(i,j) \in (L_n \backslash \{n\})^2$, let us prove each case.

- If I satisfies condition *(i)*, the smoothness of I_n^{L} implies the smoothness of I_n^{U} as a consequence of Proposition IV.2.
- If I satisfies condition *(ii)*, applying statements *(i)* and *(iii)* of Lemma II.2, we have that I_n^{L} is smooth is smooth if and only if,

$$\left\{\begin{array}{l}\left\lfloor n \cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rfloor - \left\lfloor n \cdot I\left(\frac{i+1}{n},\frac{j}{n}\right)\right\rfloor \leqslant 1 \\ \left\lfloor n \cdot I\left(\frac{i}{n},\frac{j+1}{n}\right)\right\rfloor - \left\lfloor n \cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rfloor \leqslant 1\end{array}\right\}$$
$$\Leftrightarrow \left\{\begin{array}{r}\left\lceil n \cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rceil - \left\lceil n \cdot I\left(\frac{i+1}{n},\frac{j}{n}\right)\right\rceil + 1 \leqslant 1 \\ \left\lceil n \cdot I\left(\frac{i}{n},\frac{j+1}{n}\right)\right\rceil - \left\lceil n \cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rceil \leqslant 1\end{array}\right\}$$
$$\Leftrightarrow \left\{\begin{array}{l}\left\lceil n \cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rceil - \left\lceil n \cdot I\left(\frac{i+1}{n},\frac{j}{n}\right)\right\rceil \leqslant 0 \\ \left\lceil n \cdot I\left(\frac{i}{n},\frac{j+1}{n}\right)\right\rceil - \left\lceil n \cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rceil \leqslant 1\end{array}\right\},$$

  and this implies that I_n^{U} is smooth.

- If I satisfies condition *(iii)*, applying statements *(ii)* and *(iii)* of Lemma II.2, we have that I_n^{L} is smooth if and only if,

$$\left\{\begin{array}{l}\left\lfloor n\cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rfloor-\left\lfloor n\cdot I\left(\frac{i+1}{n},\frac{j}{n}\right)\right\rfloor\leqslant 1\\ \left\lfloor n\cdot I\left(\frac{i}{n},\frac{j+1}{n}\right)\right\rfloor-\left\lfloor n\cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rfloor\leqslant 1\end{array}\right\}$$
$$\Leftrightarrow\left\{\begin{array}{r}\left\lceil n\cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rceil-\left\lceil n\cdot I\left(\frac{i+1}{n},\frac{j}{n}\right)\right\rceil+1\leqslant 1\\ \left\lceil n\cdot I\left(\frac{i}{n},\frac{j+1}{n}\right)\right\rceil-\left\lceil n\cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rceil\leqslant 1\end{array}\right\}$$
$$\Leftrightarrow\left\{\begin{array}{l}\left\lceil n\cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rceil-\left\lceil n\cdot I\left(\frac{i+1}{n},\frac{j}{n}\right)\right\rceil\leqslant 0\\ \left\lceil n\cdot I\left(\frac{i}{n},\frac{j+1}{n}\right)\right\rceil-\left\lceil n\cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rceil\leqslant 1\end{array}\right\},$$

  and this implies that I_n^{U} is smooth.
- If I satisfies condition *(iv)*, the smoothness of I_n^{L} implies the smoothness of I_n^{U} as a consequence of Proposition IV.2.

□

Proposition IV.4. *Let $I:[0,1]^2\to[0,1]$ be a fuzzy implication function and $I_n^{\mathrm{U}},I_n^{\mathrm{L}}:L_n^2\to L_n$ be its discretizations. Let us suppose that for each $(i,j)\in(L_n\backslash\{n\})^2$ one, and only one of the following conditions is satisfied:*

(i) $I\left(\frac{i}{n},\frac{j}{n}\right)\in\Gamma_n$, $I\left(\frac{i+1}{n},\frac{j}{n}\right)\in\Gamma_n$ *and* $I\left(\frac{i}{n},\frac{j+1}{n}\right)\in\Gamma_n$.
(ii) $I\left(\frac{i}{n},\frac{j}{n}\right)\in\Gamma_n$, $I\left(\frac{i+1}{n},\frac{j}{n}\right)\in\Gamma_n$ *and* $I\left(\frac{i}{n},\frac{j+1}{n}\right)\in[0,1]\backslash\Gamma_n$.
(iii) $I\left(\frac{i}{n},\frac{j}{n}\right)\in[0,1]\backslash\Gamma_n$, $I\left(\frac{i+1}{n},\frac{j}{n}\right)\in\Gamma_n$ *and* $I\left(\frac{i}{n},\frac{j+1}{n}\right)\in[0,1]\setminus\Gamma_n$.
(iv) $I\left(\frac{i}{n},\frac{j}{n}\right)\in[0,1]\setminus\Gamma_n$, $I\left(\frac{i+1}{n},\frac{j}{n}\right)\in[0,1]\setminus\Gamma_n$ *and* $I\left(\frac{i}{n},\frac{j+1}{n}\right)\in[0,1]\setminus\Gamma_n$.

Then, the smoothness of I_n^{U} implies the smoothness of I_n^{L}.

Proof. Given $(i,j)\in(L_n\backslash\{n\})^2$, let us prove each case.

- If I satisfies condition *(i)*, the smoothness of I_n^{U} implies the smoothness of I_n^{L} as a consequence of Proposition IV.2.
- If I satisfies condition *(ii)*, applying statements *(i)* and *(iv)* of Lemma II.2, we have that I_n^{U} is smooth if and only if,

$$\left\{\begin{array}{l}\left\lceil n\cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rceil-\left\lceil n\cdot I\left(\frac{i+1}{n},\frac{j}{n}\right)\right\rceil\leqslant 1\\ \left\lceil n\cdot I\left(\frac{i}{n},\frac{j+1}{n}\right)\right\rceil-\left\lceil n\cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rceil\leqslant 1\end{array}\right\}$$
$$\Leftrightarrow\left\{\begin{array}{r}\left\lfloor n\cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rfloor-\left\lfloor n\cdot I\left(\frac{i+1}{n},\frac{j}{n}\right)\right\rfloor\leqslant 1\\ \left\lfloor n\cdot I\left(\frac{i}{n},\frac{j+1}{n}\right)\right\rfloor-\left\lfloor n\cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rfloor+1\leqslant 1\end{array}\right\}$$
$$\Leftrightarrow\left\{\begin{array}{l}\left\lfloor n\cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rfloor-\left\lfloor n\cdot I\left(\frac{i+1}{n},\frac{j}{n}\right)\right\rfloor\leqslant 1\\ \left\lfloor n\cdot I\left(\frac{i}{n},\frac{j+1}{n}\right)\right\rfloor-\left\lfloor n\cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rfloor\leqslant 0\end{array}\right\},$$

  and this implies that I_n^{L} is smooth.
- If I satisfies condition *(iii)*, applying statements *(ii)* and *(iv)* of Lemma II.2, we have that I_n^{U} is smooth if and only if,

$$\left\{\begin{array}{l}\left\lceil n\cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rceil-\left\lceil n\cdot I\left(\frac{i+1}{n},\frac{j}{n}\right)\right\rceil\leqslant 1\\ \left\lceil n\cdot I\left(\frac{i}{n},\frac{j+1}{n}\right)\right\rceil-\left\lceil n\cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rceil\leqslant 1\end{array}\right\}$$
$$\Leftrightarrow\left\{\begin{array}{r}\left\lfloor n\cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rfloor-\left\lfloor n\cdot I\left(\frac{i+1}{n},\frac{j}{n}\right)\right\rfloor+1\leqslant 1\\ \left\lfloor n\cdot I\left(\frac{i}{n},\frac{j+1}{n}\right)\right\rfloor-\left\lfloor n\cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rfloor\leqslant 1\end{array}\right\}$$
$$\Leftrightarrow\left\{\begin{array}{l}\left\lfloor n\cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rfloor-\left\lfloor n\cdot I\left(\frac{i+1}{n},\frac{j}{n}\right)\right\rfloor\leqslant 0\\ \left\lfloor n\cdot I\left(\frac{i}{n},\frac{j+1}{n}\right)\right\rfloor-\left\lfloor n\cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rfloor\leqslant 1\end{array}\right\},$$

  and this implies that I_n^{L} is smooth.
- If I satisfies condition *(iv)*, the smoothness of I_n^{U} implies the smoothness of I_n^{L} as a consequence of Proposition IV.2.

□

Let us start studying other additional properties presented in Section II. The first property is the exchange principle (**EP**), which is not generally preserved through the discretization. For instance, given the Reichenbach implication I_{RC}, that satisfies the exchange principle [7], when we discretize it we obtain that

$$I_n^{\mathrm{U}}(i,j)=\left\lceil n-i+\frac{ij}{n}\right\rceil,$$
$$I_n^{\mathrm{L}}(i,j)=\left\lfloor n-i+\frac{ij}{n}\right\rfloor,$$

and considering $n=5$, we have that

$$I_n^{\mathrm{L}}\left(2,I_n^{\mathrm{L}}(3,2)\right)\neq I_n^{\mathrm{L}}\left(3,I_n^{\mathrm{L}}(2,2)\right),$$
$$I_n^{\mathrm{U}}\left(2,I_n^{\mathrm{U}}(4,1)\right)\neq I_n^{\mathrm{U}}\left(4,I_n^{\mathrm{U}}(2,1)\right).$$

Remark IV.1. *The Reichenbach implication I_{RC} is an (S,N)-implication generated by the probabilistic sum t-conorm $S_{\mathrm{P}}(x,y)=x+y-xy$, for all $x,y\in[0,1]$ and the classical negation $N_{\mathrm{C}}(x)=1-x$, for all $x\in[0,1]$. It is known that every (S,N)-implication, in the $[0,1]$ or discrete framework, satisfies (**EP**) (see [7]). Therefore, since I_n^{U} and I_n^{L} may not satisfy (**EP**), the discretizations may not belong to the discrete counterpart of the family.*

However, when I satisfies (**EP**) and $\mathrm{Ran}\, I|_{\Gamma_n^2}\subseteq\Gamma_n$, the property is preserved. This direct result is shown in the following proposition.

Proposition IV.5. *Let $I:[0,1]^2\to[0,1]$ be a fuzzy implication function, and let $I_n^{\mathrm{U}},I_n^{\mathrm{L}}:L_n^2\to L_n$ be its upper and lower discretizations, respectively. If I satisfies (**EP**) and $\mathrm{Ran}\, I|_{\Gamma_n^2}\subseteq\Gamma_n$, then I_n^{L} and I_n^{U} also satisfy (**EP**).*

Proof. If I satisfies (**EP**), then $I(x,I(y,z))=I(y,I(x,z))$ for all $x,y,z\in[0,1]$; in particular, the equality is still true for all $x, y, z \in \Gamma_n$. Setting $x = \frac{i}{n}$, $y = \frac{j}{n}$ and $z = \frac{k}{n}$, we have that

$$I\left(\frac{i}{n}, I\left(\frac{j}{n}, \frac{k}{n}\right)\right) = I\left(\frac{j}{n}, I\left(\frac{i}{n}, \frac{k}{n}\right)\right),$$

for all $i, j, k \in L_n$. From this equality, multiplying by n and applying the floor and ceiling functions, we get

$$\begin{cases} \left\lceil n \cdot I\left(\frac{i}{n}, I\left(\frac{j}{n}, \frac{k}{n}\right)\right) \right\rceil = \left\lceil n \cdot I\left(\frac{j}{n}, I\left(\frac{i}{n}, \frac{k}{n}\right)\right) \right\rceil, \\ \left\lfloor n \cdot I\left(\frac{i}{n}, I\left(\frac{j}{n}, \frac{k}{n}\right)\right) \right\rfloor = \left\lfloor n \cdot I\left(\frac{j}{n}, I\left(\frac{i}{n}, \frac{k}{n}\right)\right) \right\rfloor. \end{cases} \tag{13}$$

Since $\operatorname{Ran} I|_{\Gamma_n^2} \subseteq \Gamma_n$, the compositions of these equalities are well defined. Moreover, using this condition, it is true that

$$I\left(\frac{i}{n}, \frac{j}{n}\right) = \frac{\left\lfloor n \cdot I\left(\frac{i}{n}, \frac{j}{n}\right) \right\rfloor}{n} = \frac{\left\lceil n \cdot I\left(\frac{i}{n}, \frac{j}{n}\right) \right\rceil}{n},$$

for all $i, j \in L_n$. Substituting this equality in Equation (13), the result holds. □

Note that the conditions set out by this result are not necessary for (**EP**) to be preserved, as it is shown in the following example.

Example IV.3. *The Goguen implication, given by*

$$I_{\mathrm{GG}}(x, y) = \begin{cases} 1, & \text{if } x \leqslant y, \\ \frac{y}{x}, & \text{if } x > y, \end{cases} \tag{14}$$

*for all $x, y \in [0,1]$ satisfies (**EP**). Setting $n = 3$, although $\operatorname{Ran} I|_{\Gamma_3^2} \not\subseteq \Gamma_3$ since $I_{\mathrm{GG}}\left(\frac{2}{3}, \frac{1}{3}\right) = \frac{1}{2}$, it is straightforward to check that both discretizations satisfy (**EP**).*

Let us now look at the left neutrality principle (**NP**).

Proposition IV.6. *Let $I : [0,1]^2 \to [0,1]$ be a fuzzy implication function, and let $I_n^{\mathrm{U}}, I_n^{\mathrm{L}} : L_n^2 \to L_n$ be its upper and lower discretizations, respectively. Then,*

*(i) I_n^{U} satisfies (**NP**) if, and only if, $I\left(1, \frac{j}{n}\right) \in \left]\frac{j-1}{n}, \frac{j}{n}\right]$ for all $j \in L_n$.*
*(ii) I_n^{L} satisfies (**NP**) if, and only if, $I\left(1, \frac{j}{n}\right) \in \left[\frac{j}{n}, \frac{j+1}{n}\right[$ for all $j \in L_n$.*

Proof. Let us prove first for I_n^{U}. We have that:

$$\begin{aligned} I_n^{\mathrm{U}}(n, j) = j &\Leftrightarrow \left\lceil n \cdot I\left(\frac{n}{n}, \frac{j}{n}\right) \right\rceil = j \\ &\Leftrightarrow j - 1 < n \cdot I\left(\frac{n}{n}, \frac{j}{n}\right) \leqslant j \\ &\Leftrightarrow \frac{j-1}{n} < I\left(1, \frac{j}{n}\right) \leqslant \frac{j}{n}, \end{aligned}$$

for all $j \in L_n$. This last expression is equivalent to $I\left(1, \frac{j}{n}\right) \in \left]\frac{j-1}{n}, \frac{j}{n}\right]$. Let us prove it for I_n^{L}.

$$\begin{aligned} I_n^{\mathrm{L}}(n, j) = j &\Leftrightarrow \left\lfloor n \cdot I\left(\frac{n}{n}, \frac{j}{n}\right) \right\rfloor = j \\ &\Leftrightarrow j \leqslant n \cdot I\left(\frac{n}{n}, \frac{j}{n}\right) < j + 1 \\ &\Leftrightarrow \frac{j}{n} \leqslant I\left(1, \frac{j}{n}\right) < \frac{j+1}{n}, \end{aligned}$$

for all $j \in L_n$. This last expression is equivalent to $I\left(1, \frac{j}{n}\right) \in \left[\frac{j}{n}, \frac{j+1}{n}\right[$, as we want to prove. □

Corollary IV.6.1. *Let $I : [0,1]^2 \to [0,1]$ be a fuzzy implication function, and let $I_n^{\mathrm{U}}, I_n^{\mathrm{L}} : L_n^2 \to L_n$ be its upper and lower discretizations, respectively. If I satisfies (**NP**), I_n^{U} and I_n^{L} also satisfy (**NP**) in the discrete framework.*

Proof. Since I satisfies (**NP**), we have that $I(1, y) = y$ for all $y \in [0,1]$; in particular, for all $y \in \Gamma_n$. Setting $y = \frac{j}{n}$ for all $j \in L_n$, it holds that $I\left(1, \frac{j}{n}\right) = \frac{j}{n}$ and applying Proposition IV.6 both I_n^{U} and I_n^{L} satisfy (**NP**). □

Based on Proposition IV.6 and Corollary IV.6.1, it can be deduced that the fact that the implication function I satisfies (**NP**) is not a necessary condition for the discretizations I_n^{U} and I_n^{L} to satisfy (**NP**) in the discrete framework, as the example below shows.

Example IV.4. *Given $n \geqslant 1$, let us consider the piece-wise implication $I : [0,1]^2 \to [0,1]$ given by:*

$$I(x, y) = \begin{cases} 1, & \text{if } (x, y) \in [0, 0.5] \times [0,1] \text{ or if } y = 1, \\ \frac{j}{n}, & \text{if } (x, y) \in [0.5, 1] \times \left[\frac{j}{n}, \frac{j+1}{n}\right[\text{ for} \\ & \text{some } j \in L_n \setminus \{n\}. \end{cases}$$

*Clearly, I does not satisfy (**NP**). However, it is straightforward to check that the conditions of Proposition IV.6 are fulfilled, and therefore both discretizations I_n^{L} and I_n^{U} satisfy (**NP**) in the discrete framework.*

Next, we consider the identity principle (**IP**) and the ordering principle (**OP**).

Proposition IV.7. *Let $I : [0,1]^2 \to [0,1]$ be a fuzzy implication function, and let $I_n^{\mathrm{U}}, I_n^{\mathrm{L}} : L_n^2 \to L_n$ be its upper and lower discretizations, respectively. Then,*

*(i) I_n^{U} satisfies (**IP**) if, and only if, $I\left(\frac{i}{n}, \frac{i}{n}\right) \in \left]\frac{n-1}{n}, 1\right]$.*
*(ii) I_n^{L} satisfies (**IP**) if, and only if, $I\left(\frac{i}{n}, \frac{i}{n}\right) = 1$ for all $i \in L_n$.*

Proof. Let us prove *(i)*. I_n^{U} satisfies the identity principle if and only if $I_n^{\mathrm{U}}(i, i) = n$, and this occurs if and only if $n - 1 < n \cdot I\left(\frac{i}{n}, \frac{i}{n}\right) \leqslant n$ or, equivalently, $I\left(\frac{i}{n}, \frac{i}{n}\right) \in \left]\frac{n-1}{n}, 1\right]$.

Now, we prove *(ii)*. I_n^{L} satisfies the identity principle if and only if $\left\lfloor n \cdot I\left(\frac{i}{n}, \frac{i}{n}\right) \right\rfloor = n$. Since the maximum value of the left side of the equality is n, this occurs if and only if $n \cdot I\left(\frac{i}{n}, \frac{i}{n}\right) = n$ or, equivalently, $I\left(\frac{i}{n}, \frac{i}{n}\right) = 1$. □

Corollary IV.7.1. *Let $I : [0,1]^2 \to [0,1]$ be a fuzzy implication function, and let $I_n^{\mathrm{U}}, I_n^{\mathrm{L}} : L_n^2 \to L_n$ be its upper and lower discretizations, respectively. If I satisfies (**IP**), I_n^{U} and I_n^{L} also satisfy (**IP**).*

Proof. Since I satisfies the identity principle, $I(x, x) = 1$ for all $x \in [0,1]$; in particular, setting $x = \frac{i}{n}$ for all $i \in L_n$. With this, $I\left(\frac{i}{n}, \frac{i}{n}\right) = 1$ and applying Proposition IV.7 the result holds. □

Proposition IV.8. *Let $I : [0,1]^2 \to [0,1]$ be a fuzzy implication function, and let $I_n^{\mathrm{U}}, I_n^{\mathrm{L}} : L_n^2 \to L_n$ be its upper and lower discretizations, respectively. Then,* *(i)* I_n^{L} *satisfies* ***(OP)*** *if and only if* $I\left(\frac{i}{n},\frac{j}{n}\right)=1$ *with* $i\leqslant j$, *and* $I\left(\frac{i}{n},\frac{j}{n}\right)<1$ *with* $i>j$, *for all* $i,j\in L_n$.

(ii) I_n^{U} *satisfies* ***(OP)*** *if and only if* $I\left(\frac{i}{n},\frac{j}{n}\right)>\frac{n-1}{n}$ *with* $i\leqslant j$, *and* $I\left(\frac{i}{n},\frac{j}{n}\right)\leqslant\frac{n-1}{n}$ *with* $i>j$, *for all* $i,j\in L_n$.

Proof. Let us prove *(i)*. We obtain that

$$I_n^{\mathrm{L}}(i,j)=n\Leftrightarrow\left\lfloor n\cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rfloor=n \\ \Leftrightarrow n\cdot I\left(\frac{i}{n},\frac{j}{n}\right)=n \\ \Leftrightarrow I\left(\frac{i}{n},\frac{j}{n}\right)=1,$$

for all $i,j\in L_n$ such that $i\leqslant j$. Using that $I\left(\frac{i}{n},\frac{j}{n}\right)<1$ if and only if $i>j$, the result follows.

Let us prove *(ii)*. We have the following equivalences:

$$\{I_n^{\mathrm{U}}(i,j)=n\ \Leftrightarrow\ i\leqslant j\}\Leftrightarrow\left\{\left\lceil n\cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rceil=n\ \Leftrightarrow\ i\leqslant j\right\} \\ \Leftrightarrow\left\{n-1<n\cdot I\left(\frac{i}{n},\frac{j}{n}\right)\leqslant n\ \Leftrightarrow\ i\leqslant j\right\} \\ \Leftrightarrow\left\{I\left(\frac{i}{n},\frac{j}{n}\right)\in\left]\frac{n-1}{n},1\right]\ \Leftrightarrow\ i\leqslant j\right\},$$

where $i,j\in L_n$. Using that $I\left(\frac{i}{n},\frac{j}{n}\right)<1$ if and only if $i>j$, the result follows. □

Corollary IV.8.1. *Let* $I:[0,1]^2\to[0,1]$ *be a fuzzy implication function, and let* $I_n^{\mathrm{U}},I_n^{\mathrm{L}}:L_n^2\to L_n$ *be its upper and lower discretizations, respectively. Then,*

(i) If I *satisfies* ***(OP)***, I_n^{L} *also satisfies* ***(OP)***.

(ii) If I *satisfies* ***(OP)*** *and* $I(x,y)\leqslant\frac{n-1}{n}$ *for all* $x,y\in[0,1]$ *such that* $x>y$, I_n^{U} *also satisfies* ***(OP)***.

Proof. Let us prove *(i)*. If I satisfies the ordering principle, then $I(x,y)=1$ if and only if $x\leqslant y$; in particular, setting $x=\frac{i}{n}$ and $y=\frac{j}{n}$, with $i,j\in L_n$. Applying Proposition IV.8 the result holds.

Now, let us prove *(ii)*. Since $I(x,y)=1>\frac{n}{n-1}$ if and only if $x\leqslant y$ for all $x,y\in[0,1]$, applying Proposition IV.8 the result holds. □

As in the discussion in Example IV.4, it can be derived that the fact that the implication function I satisfies (**OP**) or (**IP**) are not necessary conditions for the discretizations I_n^{L} and I_n^{U} to satisfy (**OP**) or (**IP**), respectively, in the discrete framework, as the following example shows.

Example IV.5. *Given* $n\geqslant 1$, *let us consider the piece-wise implication* $I:[0,1]^2\to[0,1]$ *given by:*

$$I(x,y)=\begin{cases}1, & \text{if } (x,y)\in\bigcup_{i=0}^{n-1}\left[\frac{i}{n},\frac{i+1}{n}\right]\times\left[\frac{i+1}{n},1\right],\\ 0, & \text{otherwise.}\end{cases} \tag{15}$$

Clearly, I *does not satisfy* ***(IP)***, *since* $I(x,x)=1$ *only when* $x\in\Gamma_n$. *However, it is straightforward to check that the conditions of Proposition IV.7 are fulfilled, and therefore both discretizations* I_n^{L} *and* I_n^{U} *satisfy* ***(IP)*** *in the discrete framework. Moreover,* I *does not satisfy* ***(OP)***, *since* $I(x,y)\neq 1$ *for all* $(x,y)\in\bigcup_{i=0}^{n-1}R_i$, *where*

$$R_i=\left\{(x,y)\in\left[\frac{i}{n},\frac{i+1}{n}\right]\times\left[\frac{i}{n},\frac{i+1}{n}\right]\ \middle|\ x\leqslant y\right\}.$$

However, conditions of Proposition IV.8 are satisfied, and therefore both discretizations I_n^{L} *and* I_n^{U} *satisfy* ***(OP)*** *in the discrete framework.*

Let us now study the preservation of the consequent boundary (**CB**).

Proposition IV.9. *Let* $I:[0,1]^2\to[0,1]$ *be a fuzzy implication function, and let* $I_n^{\mathrm{U}},I_n^{\mathrm{L}}:L_n^2\to L_n$ *be its upper and lower discretizations, respectively. Then,*

(i) I_n^{U} *satisfies* ***(CB)*** *if and only if* $I\left(\frac{i}{n},\frac{j}{n}\right)>\frac{j-1}{n}$ *for all* $i,j\in L_n$.

(ii) I_n^{L} *satisfies* ***(CB)*** *if and only if* $I\left(\frac{i}{n},\frac{j}{n}\right)\geqslant\frac{j}{n}$ *for all* $i,j\in L_n$.

Proof. Let us prove *(i)*, observing that

$$I_n^{\mathrm{U}}(i,j)\geqslant j\Leftrightarrow\left\lceil n\cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rceil\geqslant j \\ \Leftrightarrow n\cdot I\left(\frac{i}{n},\frac{j}{n}\right)>j-1 \\ \Leftrightarrow I\left(\frac{i}{n},\frac{j}{n}\right)>\frac{j-1}{n},$$

for all $i,j\in L_n$. With a similar argument, let us prove *(ii)*:

$$I_n^{\mathrm{L}}(i,j)\geqslant j\Leftrightarrow\left\lfloor n\cdot I\left(\frac{i}{n},\frac{j}{n}\right)\right\rfloor\geqslant j \\ \Leftrightarrow n\cdot I\left(\frac{i}{n},\frac{j}{n}\right)\geqslant j \\ \Leftrightarrow I\left(\frac{i}{n},\frac{j}{n}\right)\geqslant\frac{j}{n},$$

for all $i,j\in L_n$. □

Corollary IV.9.1. *Let* $I:[0,1]^2\to[0,1]$ *be a fuzzy implication function, and let* $I_n^{\mathrm{U}},I_n^{\mathrm{L}}:L_n^2\to L_n$ *be its upper and lower discretizations, respectively. Then, if* I *satisfies* ***(CB)***, I_n^{U} *and* I_n^{L} *also satisfy* ***(CB)***.

Proof. Since I satisfies the consequent boundary, $I(x,y)\geqslant y$ for all $x,y\in[0,1]$; in particular, setting $x,y\in\Gamma_n$. Therefore, $I\left(\frac{i}{n},\frac{j}{n}\right)\geqslant\frac{j}{n}$ for all $i,j\in L_n$, and the consequent boundary for I_n^{L} is proved using Proposition IV.9. Also, from this inequality, we have that $I\left(\frac{i}{n},\frac{j}{n}\right)\geqslant\frac{j}{n}>\frac{j-1}{n}$, and consequently I_n^{U} satisfies (**CB**). □

Taking into consideration the results obtained in Proposition IV.9 and Corollary IV.9.1, and following the constructions shown in Example IV.4 and Example IV.5, it is easy to conclude that the fulfillment of (**CB**) by the implication function I is also not necessary for the discretizations I_n^{L} and I_n^{U} to satisfy (**CB**) in the discrete framework.

We now turn to study the contrapositive symmetry (**CP(**N**)**). As in the study of (**EP**) in Proposition IV.5, assuming some conditions of $\operatorname{Ran} N$, we can obtain when the property is preserved.

Proposition IV.10. *Let $I : [0,1]^2 \to [0,1]$ be a fuzzy implication function which satisfies (**CP**(N)) with a fuzzy negation $N : [0,1] \to [0,1]$, and let $I_n^{\mathrm{U}}, I_n^{\mathrm{L}} : L_n^2 \to L_n$ be its upper and lower discretizations, respectively. If $\operatorname{Ran} N \subseteq \Gamma_n$, then I_n^{U} and I_n^{L} satisfy (**CP**(N)) with the discrete negation $N_n : L_n \to L_n$ given by $N_n(i) = n \cdot N\left(\frac{i}{n}\right)$.*

Proof. The fuzzy implication function I satisfies the contrapositive symmetry with the fuzzy negation N, and therefore $I(x,y) = I(N(y), N(x))$ for all $x, y \in [0,1]$. In particular, setting $x, y \in \Gamma_n$; in this context, say $x = \frac{i}{n}$ and $y = \frac{j}{n}$, with $i, j \in L_n$, we have

$$I\left(\frac{i}{n}, \frac{j}{n}\right) = I\left(N\left(\frac{j}{n}\right), N\left(\frac{i}{n}\right)\right),$$

for all $i, j \in L_n$. Multiplying both sides of the equality and applying the floor and ceiling functions, we have

$$\left\lceil n \cdot I\left(\frac{i}{n}, \frac{j}{n}\right)\right\rceil = \left\lceil n \cdot I\left(N\left(\frac{j}{n}\right), N\left(\frac{i}{n}\right)\right)\right\rceil,$$
$$\left\lfloor n \cdot I\left(\frac{i}{n}, \frac{j}{n}\right)\right\rfloor = \left\lfloor n \cdot I\left(N\left(\frac{j}{n}\right), N\left(\frac{i}{n}\right)\right)\right\rfloor.$$

Since $\operatorname{Ran} N \subseteq \Gamma_n$ by hypothesis, $\operatorname{Ran} N_n \subseteq L_n$ and the operator $N_n(i) = n \cdot N\left(\frac{i}{n}\right)$ is well defined, and it is a discrete negation which makes I_n^{U} and I_n^{L} satisfy the contrapositive symmetry. □

We finish the study of some distinguished properties of implications by analyzing the law of importation (**LI**(T)), which has been deeply studied in recent years (see [24], [25] for further details). Assuming certain assumptions on the ranges that the fuzzy implication function I and the t-norm T can take, we obtain that the property is preserved. However, although the discretization of any implication is a discrete implication as shown in Proposition III.1, the discretization of a t-norm need not be a discrete t-norm. For this, we recall the following result.

Lemma IV.1. *[2] Let $T : [0,1]^2 \to [0,1]$ be a t-norm. Then, $T|_{\Gamma_n^2}$ is a discrete t-norm over Γ_n if and only if*

$$T\left(\frac{i}{n}, \frac{j}{n}\right) \in \left\{0, \frac{1}{n}, \ldots, \frac{\min\{i,j\}}{n}\right\},$$

for all $i, j \in L_n$.

Remark IV.2. *When a t-norm T satisfies the condition of Lemma IV.1, we will say that the t-norm T is discretizable and the mapping $T_n : L_n^2 \to L_n$ given by $T_n(i,j) = n \cdot T\left(\frac{i}{n}, \frac{j}{n}\right)$ is a discrete t-norm over L_n such that $T_n(i,j) = T_n^{\mathrm{U}}(i,j) = T_n^{\mathrm{L}}(i,j)$ for all $i, j \in L_n$.*

Thanks to the previous lemma, we can formulate the preservation of the law of importation.

Proposition IV.11. *Let $I : [0,1]^2 \to [0,1]$ be a fuzzy implication function and let $I_n^{\mathrm{U}}, I_n^{\mathrm{L}} : L_n^2 \to L_n$ be its upper and lower discretizations, respectively. If I satisfies (**LI**(T)) with respect to a t-norm $T : [0,1]^2 \to [0,1]$, $T|_{\Gamma_n^2}$ is a discrete t-norm over Γ_n and $\operatorname{Ran} I|_{\Gamma_n^2} \subseteq \Gamma_n$, then I_n^{U} and I_n^{L} also satisfy (**LI**(T)) with the discrete t-norm $T_n(i,j) = n \cdot T\left(\frac{i}{n}, \frac{j}{n}\right)$ for all $i, j \in L_n$.*

Proof. If I satisfies **LI**(T), then $I(T(x,y), z) = I(x, I(y,z))$ for all $x, y, z \in [0,1]$; in particular, the equality is still true for all $x, y, z \in \Gamma_n$. Setting $x = \frac{i}{n}$, $y = \frac{j}{n}$ and $z = \frac{k}{n}$, we have that

$$I\left(T\left(\frac{i}{n}, \frac{j}{n}\right), \frac{k}{n}\right) = I\left(\frac{i}{n}, I\left(\frac{j}{n}, \frac{k}{n}\right)\right), \tag{16}$$

for all $i, j, k \in L_n$. Now, since $T|_{\Gamma_n^2}$ is a discrete t-norm over Γ_n, from Lemma IV.1 the operator $T_n(i,j) = n \cdot T\left(\frac{i}{n}, \frac{j}{n}\right)$ is a discrete t-norm. Also, applying that $\operatorname{Ran} I|_{\Gamma_n^2} \subseteq \Gamma_n$, T and I can be rewritten, respectively, as

$$T\left(\frac{i}{n}, \frac{j}{n}\right) = \frac{\left\lfloor n \cdot T\left(\frac{i}{n}, \frac{j}{n}\right)\right\rfloor}{n} = \frac{\left\lceil n \cdot T\left(\frac{i}{n}, \frac{j}{n}\right)\right\rceil}{n} = \frac{T_n(i,j)}{n},$$
$$I\left(\frac{i}{n}, \frac{j}{n}\right) = \frac{\left\lfloor n \cdot I\left(\frac{i}{n}, \frac{j}{n}\right)\right\rfloor}{n} = \frac{\left\lceil n \cdot I\left(\frac{i}{n}, \frac{j}{n}\right)\right\rceil}{n}.$$

Replacing these expressions in Equation (16), we get

$$I_n^{\mathrm{U}}\left(T_n(i,j), k\right) = I_n^{\mathrm{U}}\left(i, I_n^{\mathrm{U}}(j,k)\right),$$
$$I_n^{\mathrm{L}}\left(T_n(i,j), k\right) = I_n^{\mathrm{L}}\left(i, I_n^{\mathrm{L}}(j,k)\right),$$

and therefore the discretizations $I_n^{\mathrm{U}}, I_n^{\mathrm{L}}$ satisfy the law of importation with the discrete t-norm T_n. □

Example IV.6. *The Kleene-Dienes implication $I_{\mathrm{KD}}(x,y) = \max\{1-x, y\}$ satisfies the law of importation with the minimum t-norm, $T_{\mathrm{M}}(x,y) = \min\{x,y\}$, for all $x, y \in [0,1]$. Applying Lemma IV.1, the discrete operator $T_n(i,j) = n \cdot T_{\mathrm{M}}(x,y)$ is a discrete t-norm which is the minimum discrete t-norm. Moreover, since $\operatorname{Ran} I_{\mathrm{KD}}|_{\Gamma_n^2} \subseteq \Gamma_n$, applying Proposition IV.11, the discretizations of I coincide, and satisfy the law of importation with the discrete t-norm T_n.*

Remark IV.3. *The law of importation may not be preserved. For example, the Reichenbach implication I_{RC} satisfies (**LI**(T)) with the product t-norm $T_{\mathrm{P}}(x,y) = xy$, for all $x, y \in [0,1]$. For instance, setting $n = 5$, the number of discrete t-norms is finite and the expression of all of them is known.*

Trying every discrete t-norm T_l defined in L_5, there is no one which satisfies $I_n^{\mathrm{U}}(T_l(i_1, j_1), k_1) = I_n^{\mathrm{U}}(i_1, I_n^{\mathrm{U}}(j_1, k_1))$ for all $i_1, j_1, k_1 \in L_5$, and $I_n^{\mathrm{L}}(T_l(i_2, j_2), k_2) = I_n^{\mathrm{L}}(i_2, I_n^{\mathrm{L}}(j_2, k_2))$ for all $i_1, j_2, k_2 \in L_5$, where I_n^{U} and I_n^{L} represent the upper and lower discretizations, respectively, of I_{RC}.

With all this, we have completed the study for the distinguished properties (**EP**), (**NP**), (**IP**), (**OP**), (**CB**), (**CP**(N)) and (**LI**(T)). Table I summarizes the preservation of the aforementioned properties.

	$I \longrightarrow I_n^{\mathrm{U}}$	$I \longrightarrow I_n^{\mathrm{L}}$	Result
(EP)	✳	✳	Prop. IV.5.
(NP)	✓	✓	Cor. IV.6.1.
(IP)	✓	✓	Cor. IV.7.1.
(OP)	●	✓	Cor. IV.8.1.
(CB)	✓	✓	Cor. IV.9.1.
(**CP**(N))	✳	✳	Prop. IV.10.
(**LI**(T))	✳	✳	Prop. IV.11.

TABLE I. Summary of the preservation of some distinguished properties of an implication through the discretization process. The symbol ✓ denotes that the property is always preserved; ✳ denotes that extra conditions have to be considered to preserve the property, and sufficient conditions have been found; finally, ● denotes that extra conditions have to be considered to preserve the property, and necessary and sufficient conditions have been found.

V. Conclusions and future work

This paper constitutes the first step in the study of the discretization of fuzzy implication functions and the extension of discrete implications. In addition, we have deeply studied under which conditions some important properties such as **(EP)**, **(NP)**, **(IP)**, **(OP)**, **(CB)**, (**CP**(N)) and (**LI**(T)) are preserved through the discretization process using the floor and ceiling functions. Several conclusions can be derived from this study:

1) With Theorem III.1, given a discrete implication we can obtain a fuzzy implication function whose upper and lower discretizations coincide, and they are equal to the given discrete implication. With Proposition III.1, given a fuzzy implication function we can obtain two discrete implications. These two results offer an appropriate method of conversion between operators defined on $[0,1]$ and L_n.
2) The considered two discretization processes do not preserve some important properties of fuzzy implication functions. Specifically, **(EP)**, (**CP**(N)), (**LI**(T)) for both discretizations, and **(OP)** for the lower discretization need additional properties to be preserved. Regarding the relation between continuity and smoothness, continuity does not directly become smoothness; in fact, as noted in Proposition IV.1, the fuzzy implication function need not be continuous for its discretizations to be smooth. All this makes it clear, as the title suggests, that discrete implications cannot be considered solely as the discretization of fuzzy implication functions. They must be studied separately.

As a future work, we plan to do our utmost to further study the discretization process. Concretely:

- Determine necessary and sufficient conditions for the preservation of **(EP)**, (**CP**(N)) and (**LI**(T)) using the proposed discretization method.
- Study the preservation of properties in the extension of discrete implications; that is, study whether a certain property in the discrete framework becomes its corresponding one in $[0,1]$ using the fuzzy implication function obtained in Theorem III.1. If not, study if it is possible to define another extension method which preserves the property.
- Forasmuch as this paper focuses on two particular discretization methods, it should be necessary to consider other discretization processes, performing a study similar to the one carried out in this paper and compare the preservation of properties between different discretization processes.
- It would be interesting to study the preservation of more additional properties that discrete implications can satisfy, taking as a reference the ones studied in [26] and [27] in the $[0,1]$ framework.
- Further to this line of research, in [28] it is proved that any fuzzy implication function I satisfies $I(x,y) = A(1-x,y)$ for all $x,y \in [0,1]$, for some disjunctor A. This relationship between fuzzy implication functions and disjunctors could lead to some relationships between the discretization processes of both operators and the preservation of additional properties among them.

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