Partial quasi-metrics and fixed point theory: an aggregation viewpoint

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ABSTRACT
In many fields of applied sciences the aggregation of numerical values, in order to get a final one which allows to make a decision, plays a central role. Many times these numerical values represent dissimilarities and the merged value can be interpreted as a global dissimilarity. Inspired, on the one hand, by the interest that causes the dissimilarities aggregation problem and, on the other hand, by the utility of generalized dissimilarities in applied sciences, we focus our work on the problem of merging the so-called partial quasi-metrics, which have been introduced in the literature with the aim of developing a framework that allows to unify the notion of metric, quasi-metric and partial metric under a unique one. Concretely, we characterize those functions that merge a collection of partial quasi-metrics into a new one. Moreover, a few relationships between this kind of functions and those that merge (quasi-)metrics and partial metrics are discussed. Furthermore, a general fixed point result for contractions obtained through aggregation functions is given.

KEYWORDS
Partial quasi-metric; aggregation; contraction; fixed point

1. Introduction

In applied sciences, the problem of merging a collection of data (inputs) into a single one datum (output), which contains information on each of the inputs, plays a central role. Typical fields in which this kind of problem arises in a natural way are robotics, decision making, image processing, medical diagnosis, machine learning, pattern recognition, econometrics or business management. Most of the time, in the indicated fields, the information is coded as numerical data and, therefore, such data must be fused in order to obtain a unique numerical value that helps us make a working decision. Many methods to merge these numerical inputs are based on the so-called aggregation functions. For a deeper treatment of such class of functions see, for instance, (Beliakov et al. 2016; Mesiar et al. 2018). Sometimes the nature of the problem imposes that the aggregation method provides an output preserving the fundamental properties of the inputs. This is the case when a collection of dissimilarities are merged in order

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to obtain a new one which represents any type of global dissimilarity and whose dissimilarity values allow us to make a decision. The utility of dissimilarities in modeling problems in applied sciences has motivated the study of those functions that allow to merge a collection of generalized dissimilarities into a new one. Thus, a characterization of those functions that merge a collection of metrics into a new one was given in (Borsík and Doboš 1981). A general solution to the problem of merging S-metrics and pseudometrics was provided in (Pradera et al. 2000, 2002; Pradera and Trillas 2002). Several linkages between the aggregation operators theory and metric aggregation functions have been given in (Casasnovas and Rosselló 2005), (Mesiar and Pap 2008) and (Yager 2010). In (Mayor and Valero 2010), the original work of Borsík and Doboš was extended to the framework of quasi-metrics (see also (Miñana and Valero 2019)). The aggregation problem for partial metrics and, in addition, relationships between this problem and the (quasi-)metric one were explored in (Massanet and Valero 2012). Recently, a refinement of the original Borsík and Doboš characterization has been yielded in (Mayor and Valero 2019).

Inspired, on the one hand, by the interest that causes the dissimilarities aggregation problem and, on the other hand, by the utility of generalized dissimilarities in applied sciences, we focus our work in the problem of merging the so-called partial quasi-metrics in the sense of (Künzi et al. 2006). This type of general dissimilarities have been introduced in the literature with the aim of developing a framework that allows to unify the metric, quasi-metric and partial metric approach. Concretely we provide a characterization of functions that merge a collection of partial quasi-metrics into a new one and a few relationships between this kind of functions and those that merge (quasi-)metrics and partial metrics are discussed. Inspired by the fact that many applications of quasi-metrics and partial metrics are obtained via fixed point methods, we prove a general fixed point result for contractions obtained through aggregation functions in such a way that the results given in (Martín et al. 2013) and (Alghamdi et al. 2015) are retrieved as a particular case.

2. Aggregation of partial quasi-metric spaces

In this section we motivate the problem of merging partial quasi-metrics and provide a characterization of those functions which are useful for such a goal. Moreover, some relationships between these functions and those that merge (quasi-)metrics and partial metrics are exposed.

2.1. *Basic notions and a motivation*

In the last years quasi-metrics and partial metrics have been used successfully as efficient tools in modeling some processes that arise in a natural way in Computer Science.

A new dissimilarity notion was introduced in (Matthews 1994). Such a notion is known as partial metric and it is useful to provide a quantitative mathematical framework to model, among other, the meaning of recursive specifications in denotational semantics for programming languages. It is recalled below.

Definition 2.1. A *partial metric* on a non-empty set X is a function $p : X \times X \to \mathbb{R}_+$ such that, for all $x, y, z \in X$, the following axioms are fulfilled:

(P1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$;
(P2) $0 \leq p(x,x) \leq p(x,y)$;
(P3) $p(x,y) = p(y,x)$;
(P4) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

Notice that $\mathbb{R}_+$ denotes the set of non-negative real numbers in the preceding definition.

Later on, a new framework, known as complexity space, was introduced in order to develop a quantitative mathematical foundation for the asymptotic complexity analysis of algorithms (see (Schellekens 1995)). This framework is based on the notion of quasi-metric space which, according to (Deza and Deza 2009), is defined as follows.

Definition 2.2. A *quasi-metric* on a (nonempty) set X is a function $q : X \times X \to \mathbb{R}_+$ such that for all $x, y, z \in X$:

(Q1) $q(x,y) = q(y,x) = 0 \Leftrightarrow x = y$.
(Q2) $q(x,z) \leq q(x,y) + q(y,z)$.

Inspired, on the one hand, by the applicability of quasi-metrics and partial metrics in the aforesaid fields of Computer Science and, on the other hand, by the fact that both generalized dissimilarities have been explored independently, a new dissimilarity notion called partial quasi-metric, which unifies under the same framework the quasi-metric and partial metric one, was introduced in (Künzi et al. 2006). We recall such a notion next.

Definition 2.3. A *partial quasi-metric* on a nonempty set X is a function $pq : X \times X \to \mathbb{R}_+$ such that for all $x, y, z \in X$:

(PQ1) $pq(x,x) \leq pq(x,y)$;
(PQ2) $pq(x,x) \leq pq(y,x)$;
(PQ3) $pq(x,y) \leq pq(x,z) + pq(z,y) - pq(z,z)$;
(PQ4) $x = y \Leftrightarrow pq(x,x) = pq(x,y)$ and $pq(y,y) = pq(y,x)$.

Observe that a partial metric on X is a partial quasi-metric pq on X such that, for all $x, y \in X$, $pq(x,y) = pq(y,x)$. Moreover, a quasi-metric on X is a partial quasi-metric pq on X such that, for all $x \in X$, $pq(x,x) = 0$. Furthermore, a metric on X is a partial quasi-metric pq on X such that, for all $x, y \in X$, $pq(x,y) = 0 \Leftrightarrow x = y$ and $pq(x,y) = pq(y,x)$.

In order to study those functions that allow to merge an arbitrary collection of partial quasi-metrics into a new one, let us introduce the notion of partial quasi-metric aggregation function. To this end, let us denote by $\mathbb{N}$ the set of positive integer numbers.

Definition 2.4. Given $n \in \mathbb{N}$, we will say that a function $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ is a *partial quasi-metric aggregation function* provided that the function $PQ_\Phi : X \times X \longrightarrow \mathbb{R}_+$ is a partial quasi-metric for every arbitrary collection of partial quasi-metric spaces $\{(X_i, pq_i)\}_{i=1}^n$, where for all $x, y \in \prod_{i=1}^n X_i$, the function PQ_Φ is defined by $PQ_\Phi(x,y) = \Phi\left(pq_1(x_1,y_1), \ldots, pq_n(x_n,y_n)\right)$.

Observe that $\mathbb{R}_+^n = \{(a_1, \ldots, a_n) : a_i \in \mathbb{R}_+ \text{ for all } i = 1, \ldots, n\}$ and $\prod_{i=1}^{n} X_i = X_1 \times \cdots \times X_n$.

Clearly, when we replace the term partial quasi-metric by metric, quasi-metric or partial metric, we retrieve the notion of metric aggregation function, quasi-metric aggregation function and partial metric aggregation function given in (Borsík and Doboš 1981), (Mayor and Valero 2010) and (Massanet and Valero 2012), respectively.

We end this section providing a motivation for the study of partial quasi-metric aggregation functions.

As exposed before, partial metrics and quasi-metrics have shown to be useful in the study of recursive denotational specifications for programming languages and to discuss the complexity analysis of algorithms, respectively.

In denotational semantics one of the targets is to verify the correctness of recursive algorithms through mathematical models. With this aim, Matthews introduced the Baire partial metric space which consists of the pair (Σ_∞, p_B), where Σ_∞ is the set of finite and infinite sequences over a non-empty alphabet Σ and the partial metric p_B is given by $p_B(v, w) = 2^{-l(v,w)}$ for all $x, y \in \Sigma_\infty$ with $l(v, w)$ denoting the longest common prefix of the words v and w when it exists and $l(v, w) = 0$ otherwise. Of course the convention that $2^{-\infty} = 0$ is adopted (see (Matthews 1994)).

Usually the running time of computing of recursive algorithms is analyzed in conjunction with the correctness. In order to discuss by means of a mathematical model the running time of computing, Schellekens introduced the so-called complexity space, which consists of the pair $(\mathcal{C}, q_\mathcal{C})$, where

$$\mathcal{C} = \{f : \mathbb{N} \longrightarrow (0, \infty] : \sum_{n=1}^{\infty} 2^{-n} \frac{1}{f(n)} < \infty\}$$

and $q_\mathcal{C}$ is the quasi-metric on $\mathcal{C}$ defined by

$$q_\mathcal{C}(f, g) = \sum_{n=1}^{\infty} 2^{-n} \max\left(\frac{1}{g(n)} - \frac{1}{f(n)}, 0\right).$$

Clearly the convention that $\frac{1}{\infty} = 0$ is adopted (see Schellekens (1995)).

The running time of computing of an algorithm can be associated to a function belonging to $\mathcal{C}$. In addition, the numerical value $q_\mathcal{C}(f, g)$ (the complexity distance from f to g) can be interpreted as the relative progress made in lowering the complexity by replacing any program P with complexity function f by any program Q with complexity function g. Moreover, the condition $q_\mathcal{C}(f, g) = 0$ can be understood as f is "at least as efficient" as g on all inputs. Notice that $q_\mathcal{C}(f, g) = 0$ implies that $f(n) \leq g(n)$ for all $n \in \mathbb{N}$. The last fact is crucial when the asymptotic upper bound of the complexity of an algorithm wants to be specified. In fact, $q_\mathcal{C}(f, g) = 0$ implies that $f \in \mathcal{O}(g)$, where $f \in \mathcal{O}(g)$ means that there exist $n_0 \in \mathbb{N}$ and $c \in \mathbb{R}_+$ such that $f(n) \leq cg(n)$ for all $n \in \mathbb{N}$ with $n \geq n_0$. Furthermore, it must be pointed out that the asymmetry of dissimilarity $q_\mathcal{C}$ is key when one wants to give information about the growth in complexity when a program is replaced by another one. A metric would be able to be used in order to provide information on the growth but, nevertheless, it could not yield which program is more efficient.

The exposed mathematical structures were developed and applied separately without any relationship between them. In fact, at first glance, it seems difficult to combine two approaches so that we can build a unique framework which allows us to carry out formally the study simultaneously, on the one hand, of the correctness of a recursive algorithm and, on the other hand, the running time of computing of such an algorithm. However, partial quasi-metric aggregation functions could be useful for such a target. Hence, we could consider a new structure which arise merging the both original ones in such a way that the main properties coming from both different nature inputs are kept and the original dissimilarities (a partial metric and a quasi-metric) are fused in a global dissimilarity. Thus, we could consider the pair $(\mathbb{N}_\infty \times \mathcal{C}, PQ_\Phi)$, where Φ is a partial quasi-metric aggregation function and $PQ_\Phi((v,f),(w,g)) = \Phi(p_B(v,w), q_\mathcal{C}(f,g))$ for all $v,w \in \mathbb{N}_\infty$ and $f,g \in \mathcal{C}$.

Observe that the fact that one input is a partial metric and the other one is a quasi-metric, both are particular cases of partial quasi-metrics, forces us to consider partial quasi-metric aggregation functions because of PQ_Φ is not, in general, either a partial metric or a quasi-metric.

On the one hand, every partial quasi-metric aggregation function Φ satisfies that $\Phi(a_1,a_2) = 0$ implies $a_1 = a_2 = 0$ (as we will show in Proposition 3.5 in Section 3) and, thus, we have that $PQ_\Phi((v,f),(v,f)) = \Phi(p_B(v,v), q_\mathcal{C}(f,f)) = \Phi(2^{-l(v,v)}, 0) > 0$ for all finite word $v \in \mathbb{N}_\infty$. It follows that PQ_Φ cannot be a quasi-metric.

On the other hand, $\Phi(0,0) < \Phi(0, \frac{1}{2f(1)})$ (as shown, again, in Proposition 3.5 later on). Whence we have that $PQ_\Phi((v,f),(w,g)) = \Phi(0,0) < \Phi(0,\frac{1}{2f(1)}) = PQ_\Phi((w,g),(v,f))$ when $f,g \in \mathcal{C}$ such that $q_\mathcal{C}(f,g) = 0$, $q_\mathcal{C}(g,f) = \frac{1}{2f(1)}$ and $v,w \in \mathbb{N}_\infty$ are infinite words with $v = w$. It follows that PQ_Φ is not, in general, a partial metric. Notice that $f,g \in \mathcal{C}$ with $f(1) < \infty = g(1)$ and $g(n) = f(n) = \infty$ for all $n \in \mathbb{N}$ with $n > 1$ provides that $q_\mathcal{C}(f,g) = 0$, $q_\mathcal{C}(g,f) \neq 0$.

The preceding reasoning shows that $p_B + q_\mathcal{C}$, the most natural generalized dissimilarity for this objective, is neither a partial metric nor a quasi-metric on $\mathbb{N}_\infty \times \mathcal{C}$ such as it was proved in (Miñana and Valero 2018).

2.2. *A characterization of partial quasi-metric aggregation functions*

Taking into account the interest aroused by the aggregation of dissimilarities and its potential applicability to many fields, we provide the promised characterization of those functions that are useful for merging a collection of partial quasi-metrics into a new one and, in addition, we provide a few relationships between this type of functions and those that merge (quasi-)metrics and partial metrics.

First, recall the next notions on functions $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$.

Definition 2.5. A function $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ is said to be:

(i) *amenable* provided that $\Phi(x) = 0 \Leftrightarrow x = \bar{0}$, where $\bar{0} \in \mathbb{R}_+^n$ with $\bar{0} = (0,\ldots,0)$.
(ii) *non-decreasing* provided that $\Phi(x) \leq \Phi(y)$ for all $x,y \in \mathbb{R}_+^n$ with $x \preceq y$, where $x \preceq y \Leftrightarrow x_i \leq y_i$ for all $i = 1,\ldots,n$.
(iii) *subadditive* provided that $\Phi(x+y) \leq \Phi(x) + \Phi(y)$ for all for all $x,y \in \mathbb{R}_+^n$.

The next result will be helpful to prove the aforementioned characterization.

Lemma 2.6. *Let $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ be a partial quasi-metric aggregation function. Then, Φ is non-decreasing.*

Proof. Let $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ be a partial quasi-metric aggregation function and let $a,b \in \mathbb{R}_+^n$ such that $a \preceq b$, where $a = (a_1,\ldots,a_n)$ and $b = (b_1,\ldots,b_n)$.

Consider, for each $i \in \{1, \dots, n\}$, the partial (quasi-)metric space $(\mathbb{R}_+, p_{\max})$, where $p_{\max}(x, y) = \max\{x, y\}$ for each $x, y \in \mathbb{R}_+$. Then,

$$\Phi(a) = \Phi(p_{\max}(a_1, a_1), \dots, p_{\max}(a_n, a_n)) = PQ_\Phi(a, a) \leq PQ_\Phi(a, b) =$$

$$= \Phi(p_{\max}(a_1, b_1), \dots, p_{\max}(a_n, b_n)) = \Phi(b).$$

Thus, Φ is non-decreasing. □

In the next result we characterize the partial quasi-metric aggregation functions.

Theorem 2.7. *Let $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ and let $x, y, w, z \in \mathbb{R}_+^n$. The following assertions are equivalent:*

1) Φ is a partial quasi-metric aggregation function.
2) Φ satisfies the following conditions:
(2.1) $\Phi(x) + \Phi(y) \leq \Phi(z) + \Phi(w)$, whenever $x + y \preceq z + w$, $y \preceq z$ and $y \preceq w$.
(2.2) $x = y$ whenever $\Phi(x) = \Phi(y)$ with $y \preceq x$.
3) Φ satisfies condition (2.1) and the following one:
$x = y$ and $z = w$, whenever $\Phi(x) = \Phi(y)$ and $\Phi(z) = \Phi(w)$ with $y \preceq x$, $y \preceq w$, $z \preceq x$ and $z \preceq w$.

Proof. Let $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ and let $x, y, w, z \in \mathbb{R}_+^n$.

1) $\Rightarrow$ 2) Suppose that Φ is a partial quasi-metric aggregation function.

(2.1) Suppose that $x + y \preceq z + w$, $y \preceq z$ and $y \preceq w$. For each $i \in \{1, \dots, n\}$, consider the set $X_i = \{y_i, z_i, w_i\}$ and define p_i on $X_i \times X_i$ as follows:

$$p_i(y_i, y_i) = y_i; \quad p_i(z_i, z_i) = z_i; \quad p_i(w_i, w_i) = w_i;$$

$$p_i(z_i, w_i) = p_i(w_i, z_i) = z_i + w_i - y_i;$$

$$p_i(z_i, y_i) = p_i(y_i, z_i) = z_i; \quad p_i(w_i, y_i) = p_i(y_i, w_i) = w_i.$$

It is not hard to check that p_i is a partial (quasi-)metric on X_i, for each $i \in \{1, \dots, n\}$. Then,

$$\Phi(z+w-y)+\Phi(y) = \Phi(p_1(z_1, w_1), \dots, p_n(z_n, w_n))+\Phi(p_1(y_1, y_1), \dots, p_n(y_n, y_n)) =$$

$$= PQ_\Phi(z, w) + PQ_\Phi(y, y) \leq PQ_\Phi(z, y) + PQ_\Phi(y, w) =$$

$$= \Phi(p_1(z_1, y_1), \dots, p_n(z_n, y_n))+\Phi(p_1(y_1, w_1), \dots, p_n(y_n, w_n)) = \Phi(z)+\Phi(w).$$

In addition, by our assumption $x \preceq z + w - y$ and so, using Lemma 2.6, we obtain

$$\Phi(x) + \Phi(y) \leq \Phi(z + w - y) + \Phi(y) \leq \Phi(z) + \Phi(w).$$

(2.2) Suppose that $y \preceq x$ and $\Phi(x) = \Phi(y)$. For each $i \in \{1, \ldots, n\}$, consider the set $X_i = \{x_i, y_i\}$ and define p_i on $X_i \times X_i$ as follows:

$$p_i(x_i, x_i) = x_i; \quad p_i(y_i, y_i) = y_i; \quad p_i(x_i, y_i) = p_i(y_i, x_i) = x_i.$$

One can verify that, for each $i \in \{1, \ldots, n\}$, (X_i, p_i) is a partial (quasi-)metric space. Then,

$$PQ_\Phi(x, y) = \Phi(x) = PQ_\Phi(x, x)$$

and

$$PQ_\Phi(y, x) = \Phi(x) = \Phi(y) = PQ_\Phi(y, y).$$

So, since Φ is a partial quasi-metric aggregation function then PQ_Φ is a partial quasi-metric and so, we deduce that $x = y$.

$2) \Rightarrow 3)$ We just need to check that (2.2) implies (3.2). So, suppose that (2.2) is satisfied by Φ and suppose $y \preceq x$, $y \preceq w$, $z \preceq x$, $z \preceq w$, $\Phi(x) = \Phi(y)$ and $\Phi(z) = \Phi(w)$.

On the one hand, let $x' = x$ and $y' = y$. Then, by our assumption, $y' \preceq x'$ and $\Phi(x') = \Phi(y')$. Thus, since Φ fulfills (2.2) we have that $x' = y'$ and so $x = y$.

On the other hand, let $x'' = w$ and $y'' = z$. By our assumption again, $y'' \preceq x''$ and $\Phi(x'') = \Phi(y'')$. Thus, since Φ fulfills (2.2) we have that $x'' = y''$ and so $z = w$.

$3) \Rightarrow 1)$ Suppose that Φ satisfies (3.1) and (3.2) and let $\{(X_i, pq_i)\}_{i=1}^n$ be an arbitrary collection of partial quasi-metric spaces. We will see that PQ_Φ is partial quasi-metric on $X = \prod_{i=1}^n X_i$.

First of all, we claim that if Φ satisfies (3.1), then Φ is non-decreasing. Indeed, let $a, b \in \mathbb{R}_+^n$ with $a \preceq b$. Then, $a + a \preceq a + b$, $a \preceq a$ and $a \preceq b$. Then, if Φ satisfies (3.1) then $\Phi(a) + \Phi(a) \leq \Phi(a) + \Phi(b)$ and so $\Phi(a) \leq \Phi(b)$.

Let $x, y \in X$. Since, pq_i is a partial quasi-metric on X_i, for each $i \in \{1, \ldots, n\}$, we have that $pq_i(x_i, x_i) \leq pq_i(x_i, y_i)$, for each $i \in \{1, \ldots, n\}$. So, $(pq_1(x_1, x_1), \ldots, pq_n(x_n, x_n)) \preceq (pq_1(x_1, y_1), \ldots, pq_n(x_n, y_n))$. Since Φ is non-decreasing, we conclude that $PQ_\Phi(x, x) = \Phi(pq_1(x_1, x_1), \ldots, pq_n(x_n, x_n)) \leq \Phi(pq_1(x_1, y_1), \ldots, pq_n(x_n, y_n)) = PQ_\Phi(x, y)$. Thus, PQ_Φ satisfies $(PQ1)$. Analogously, it is proved that PQ_Φ satisfies $(PQ2)$.

We focus now in showing that PQ_Φ satisfies $(PQ3)$. Let $x, y, z \in X$. Let $a = (pq_1(x_1, z_1), \ldots, pq_n(x_n, z_n))$, $b = (pq_1(y_1, y_1), \ldots, pq_n(y_n, y_n))$, $c = (pq_1(x_i, y_i), \ldots, pq_n(x_n, y_n))$ and $d = (pq_1(y_1, z_1), \ldots, pq_n(y_n, z_n))$. Since pq_i is a partial quasi-metric on X_i, for each $i \in \{1, \ldots, n\}$, then $a + b \preceq c + d$, $b \preceq c$ and $b \preceq d$. Thus, since Φ satisfies (3.1) we have that $\Phi(a) + \Phi(b) \leq \Phi(c) + \Phi(d)$ and so

$$PQ_\Phi(x, z) + PQ_\Phi(y, y) = \Phi(a) + \Phi(b) \leq \Phi(c) + \Phi(d) = PQ_\Phi(x, y) + PQ_\Phi(y, z).$$

Finally, we will see that PQ_Φ fulfils $(PQ4)$.

Obviously, if $x = y$ then $PQ_\Phi(x, x) = PQ_\Phi(x, y)$ and $PQ_\Phi(y, y) = PQ_\Phi(y, x)$. Conversely, let $x, y \in X$ such that $PQ_\Phi(x, x) = PQ_\Phi(x, y)$ and $PQ_\Phi(y, y) = PQ_\Phi(y, x)$.

Let $a = (pq_1(x_1,x_1),\ldots,pq_n(x_n,x_n))$, $b = (pq_1(x_1,y_1),\ldots,pq_n(x_n,y_n))$, $c = (pq_1(y_1,y_1),\ldots,pq_n(y_n,y_n))$ and $d = (pq_1(y_1,x_1),\ldots,pq_n(y_n,x_n))$. Due to pq_i is a partial quasi-metric on X_i, for each $i \in \{1,\ldots,n\}$, we have that $a \preceq b$, $a \preceq d$, $c \preceq b$ and $c \preceq d$. Besides, by our assumption, $\Phi(a) = PQ_\Phi(x,x) = PQ_\Phi(x,y) = \Phi(b)$ and $\Phi(c) = PQ_\Phi(y,y) = PQ_\Phi(y,x) = \Phi(d)$. Since Φ fulfills (3.2) we obtain that $a = b$ and $c = d$. Thus, for each $i \in \{1,\ldots,n\}$, $pq_i(x_i,x_i) = pq_i(x_i,y_i)$ and $pq_i(y_i,y_i) = pq_i(y_i,x_i)$. The fact that pq_i is a partial quasi-metric on X_i, for each $i \in \{1,\ldots,n\}$, ensures that $x_i = y_i$, for each $i \in \{1,\ldots,n\}$, and so, $x = y$.

□

Let us stress that the problem of how to induce partial quasi-metrics merging just a partial metric and a quasi-metric was posed in (Miñana and Valero 2018). In order to solve such a problem, the notion of partial quasi-metric generating function was introduced. Let us recall that a function $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a partial quasi-metric generating function provided that for each partial metric space (X,p) and each quasi-metric space (Y,q), the function $\Phi_{p,q} : (X \times Y) \times (X \times Y) \to \mathbb{R}_+$ is a partial quasi-metric on $X \times Y$, where $\Phi_{p,q}((x,y),(u,v)) = \Phi(p(x,u), q(y,v))$ for each $(x,y),(u,v) \in X \times Y$. In the light of Theorem 2.7 we immediately get that every partial quasi-metric aggregation function is a partial quasi-metric generating function.

According to Theorem 2.7, the following example provides a technique for generating new partial quasi-metrics from older ones.

Example 2.8. Consider the functions $f_i : \mathbb{R}_+ \to \mathbb{R}_+$, with $i = 1,\ldots,7$, given by:

$f_1(x) = (x+\alpha)^\beta$ with $\beta \in]0,1]$,
$f_2(x) = \alpha x + \beta$ with $\alpha,\beta \in]0,\infty[$,
$f_3(x) = \frac{\alpha x}{1+x}$ with $\alpha \in]0,\infty[$,
$f_4(x) = \frac{1+\alpha x}{2+\alpha x}$ with $\alpha \in]0,\infty[$,
$f_5(x) = \log_\beta(\alpha + x)$ with $\alpha,\beta \in]1,\infty[$,
$f_6(x) = 1 - e^{-\alpha x}$ with $\alpha \in]0,\infty[$,
$f_7(x) = \sqrt{x^2 + \alpha x}$ with $\alpha \in [0,\infty[$.

It is not hard to check that the preceding functions transform a partial quasi-metric into a new one (we refer the reader to Theorem 1 in Miñana and Valero (2020) for a fuller treatment). Then the function $PQ_+ : X \times X \longrightarrow \mathbb{R}_+$ is a partial quasi-metric for every arbitrary collection of partial quasi-metric spaces $\{(X_i, pq_i)\}_{i=1}^n$ with

$$PQ_+(x,y) = \alpha_1 g_1(pq_1(x_1,y_1)) + \ldots + \alpha_n g_n(pq_n(x_n,y_n))$$

for all $x,y \in \prod_{i=1}^n X_i$ and where $\alpha_i \in]0,\infty[$ and $g_i \in \{f_1,\ldots,f_7\}$ for all $i = 1,\ldots,n$.

Taking into account the information yielded by Theorem 2.7 we are able to state the relationship with those functions that merge partial metrics, quasi-metrics and metrics. Before, we recall the next results, which can be found in (Borsík and Doboš 1981), (Mayor and Valero 2010) and (Massanet and Valero 2012). They yield a description of metric, quasi-metric and partial metric aggregation functions.

Proposition 2.9. *Let $\Phi : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+$ be a metric aggregation function. Then Φ is amenable.*

Proposition 2.10. *Let $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ be a function. If Φ is non-decreasing, subadditive and amenable, then Φ is a metric aggregation function.*

Theorem 2.11. *Let $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ be a function. Then the below assertions are equivalent:*

1) *Φ is a quasi-metric aggregation function.*
2) *Φ is subadditive, non-decreasing and amenable.*

Theorem 2.12. *Let $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ and let $x, y, w, z \in \mathbb{R}_+^n$. The following assertions are equivalent:*

1) *Φ is a partial metric aggregation function.*
2) *Φ satisfies condition (2.1) in Theorem 2.7 and the following one:*
   $x = y = z$ whenever $\Phi(x) = \Phi(y) = \Phi(z)$ with $y \preceq x$, $z \preceq x$.

Theorem 2.11 states that every quasi-metric aggregation function is a metric aggregation function. Of course the converse is not true such as it was proved in (Mayor and Valero 2010). Moreover, several examples were given in (Massanet and Valero 2012) with the aim of showing that there are partial metric aggregation functions that are not either metric aggregation functions or quasi-metric aggregation functions and vice versa. Theorems 2.7 and 2.12 warranty that the class of partial quasi-metric aggregation functions matches up with the class of partial metric aggregation functions.

The next example shows that there exist partial quasi-metric aggregation functions that are neither quasi-metric aggregation functions nor metric aggregation functions.

Example 2.13. Consider the partial quasi-metric space $([0,1], p_{\max})$, where $p_{\max}(x,y) = \max\{x,y\}$ for all $x, y \in [0,1]$. Consider the family of partial metric spaces $\{([0,1], p_i)\}_{i=1,2}$ such that $p_1 = p_2 = p_{\max}$. Define the function $\Phi_2 : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ by $\Phi_2(x) = \frac{x_1+x_2}{4} + \frac{1}{2}$ for all $x \in \mathbb{R}_+^2$. It is not hard to see that the function Φ_2 holds assertions (2.1) and (2.2) in statement of Theorem 2.7 and, thus, it is a partial quasi-metric aggregation function. Moreover, it is clear that Φ_2 is not amenable and, thus, by Theorem 2.11 and Proposition 2.9, Φ is neither a quasi-metric aggregation function nor a metric aggregation function.

Since every partial quasi-metric aggregation function is exactly a partial metric aggregation function the next result, given in (Massanet and Valero 2012), is crucial in order to state the relationship between partial quasi-metric aggregation functions and (quasi-)metric aggregation functions.

Proposition 2.14. *Let $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ be a partial metric aggregation function. Then Φ is non-decreasing and subadditive.*

From the preceding result we obtain immediately that every partial quasi-metric aggregation function is always non-decreasing and subadditive. Taking into account the exposed information we can state exactly the desired relationship.

Corollary 2.15. *Let $\Phi : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+$ be a partial quasi-metric aggregation function. Then the following assertions are equivalent:*

1) *Φ is amenable.*
2) *Φ is a quasi-metric aggregation function.*
3) *Φ is a metric aggregation function.*

3. A fixed point theorem via aggregation

The contributions of partial metrics and quasi-metrics to the study of correctness of recursive algorithms and to discuss the complexity analysis of algorithms were possible thanks to fixed point methods for such dissimilarities.

In the case of partial metrics, the aforesaid fixed point methods were based on the so-called Matthews fixed point theorem. In order to recall such a fixed point theorem, let us introduce a few pertinent notions. According to (Matthews 1994), a mapping from a partial metric space (X, p) into itself is said to be a contraction if there exists $c \in [0, 1[$ such that $p(f(x), f(y)) \leq cp(x, y)$ for all $x, y \in X$. The preceding constant c is said to be the contractive constant of the contraction f. Moreover, a sequence $(x_n)_{n\in\mathbb{N}}$ in a partial metric space (X, p) is said to be a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists in $\mathbb{R}_+$. Thus, a partial metric space (X, p) is called complete provided that for every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X there exists a point $x \in X$ such that $\lim_{n\to\infty} x_n = x$ in (X, p), i.e., $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x, x_n)$.

In the light of the above notions, the Matthews fixed point theorem can be stated as follows:

Theorem 3.1. *Let (X, p) be a complete partial metric space and let $f : X \to X$. If f is a contraction from (X, p) into itself, then f has a unique fixed point x_0. Moreover, $p(x_0, x_0) = 0$ and $\lim_{n\to\infty} p(x_0, f^n(x)) = 0$, for each $x \in X$.*

A subclass of Cauchy sequences in a partial metric space (X, p) are the so-called 0-Cauchy sequences (see Romaguera (2010)). Recall that, a sequence $(x_n)_{n\in\mathbb{N}}$ is 0-Cauchy if $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. As is usual, a partial metric space is called 0-complete if every 0-Cauchy sequence is convergent. Obviously, every complete partial metric is 0-complete but the converse is not true as it was observed in (Romaguera 2010). Nevertheless, following the same arguments used by Matthews one can extend Theorem 3.1 to 0-complete partial metric spaces.

Regarding to quasi-metrics, the aforementioned fixed point methods were based on the next fixed point theorem. In order to state it, let us recall the next concepts. Following (Schellekens 1995), a mapping from a (quasi-)metric space (X, q) into itself is said to be a contraction if there exists $c \in [0, 1[$ such that $q(f(x), f(y)) \leq cq(x, y)$ for all $x, y \in X$. As in the partial metric case, the preceding constant c is said to be the contractive constant of the contraction f. Besides, a quasi-metric space (X, d) is said to be bicomplete if the associated metric space (X, d_q) is complete, where the metric d_q on X is defined by $d_q(x, y) = \max\{q(x, y), q(y, x)\}$ for all $x, y \in X$. We refer the reader to (Deza and Deza 2009) for the fundamentals of metric spaces.

Theorem 3.2. *Let (X, q) be a bicomplete quasi-metric space and let $f : X \to X$. If f is a contraction from (X, q) into itself, then f has a unique fixed point x_0.*

Notice that the preceding result is known as Banach's fixed point theorem when the quasi-metric is replaced by a metric in its statement.

Theorems 3.1 and 3.2 were extended to the case of complete partial quasi-metric spaces in (Künzi et al. 2006). Following the same arguments given in (Matthews 1994) (see also (Künzi et al. 2006)), one can also prove the following result for 0-complete partial quasi-metric spaces.

Theorem 3.3. *Let (X, pq) be a 0-complete partial quasi-metric space and let $f : X \to X$. If f is a contraction from (X, pq) into itself, then f has a unique fixed point x_0.*

Moreover, $pq(x_0, x_0) = 0$ *and* $\lim_{n\to\infty} pq(x_0, f^n(x)) = 0$*, for each* $x \in X$.

Of course a contraction from a partial quasi-metric space into itself is defined in the same terms like in the exposed cases above. The notion of 0-completeness is defined, on account of Mohammadi and Valero (2016) (see also (Romaguera 2011)), in the following way: a partial quasi-metric space (X, pq) is said to be 0-complete provided that for each 0-Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ there exists $x_0 \in X$ such that $\lim_{n\to\infty} pq(x_n, x_0) = \lim_{n\to\infty} pq(x_0, x_n) = 0$, where a sequence $(x_n)_{n\in\mathbb{N}}$ is called 0-Cauchy if it satisfies that $\lim_{n,m\to\infty} pq(x_n, x_m) = 0$.

Observe that Theorem 3.3 retrieves Theorems 3.1 and 3.2 when the partial quasi-metric is exactly a partial metric and a quasi-metric, respectively.

Inspired by the fact that many applications of quasi-metrics and partial metrics are obtained via the exposed fixed point results (as the analysis of algorithms case), we prove a general fixed point result for contractions obtained through aggregation functions. Thus we extend Theorem 3.3 to the case in which the contractions are defined between partial quasi-metric spaces obtained through the aggregation of a collection of partial quasi-metrics. To this end, let us introduce the appropriate notion of contraction.

Definition 3.4. Let $(X_i, pq_i)_{i=1}^n$ be a family of arbitrary partial quasi-metric spaces, $X = \prod_{i=1}^{n} X_i$ and let $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ be a partial quasi-metric aggregation function. A mapping $F : X \longrightarrow X$ will be said to be a projective Φ-contraction from (X, PQ_Φ) into itself, provided the existence of (contractive) constants $c_1, \ldots, c_n \in [0, 1]$ such that

$$pq_i(F_i(x), F_i(y)) \leq c_i \Phi\left(pq_1(x_1, y_1), \ldots, pq_n(x_n, y_n)\right)$$

for all $x, y \in X$ and for all $i = 1, \ldots, n$.

It must be stressed that Definition 3.4 retrieves the notion of Φ-contraction, when the contractive constants belong to $[0, 1[$, introduced in the context of partial metric spaces and quasi-metric spaces given in (Alghamdi et al. 2015) and (Martín et al. 2013), respectively. Of course when $n = 1$ and Φ is the identity function in Definition 3.4, then the contraction notion for self-mappings in partial quasi-metric spaces is retrieved as a particular case when the contractive constant belong to $[0, 1[$.

Since Theorem 3.3 (see also Theorems 3.1 and 3.2) needs an appropriate notion of completeness in order to guarantee the existence and uniqueness of fixed point the next result makes sure such a demand for the partial quasi-metric space obtained via aggregation. Before stating such a result, let us recall that, according to (Herburt and Moszyńska 1995), a function $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ is called homogeneous provided that $\Phi(\alpha x) = \alpha\Phi(x)$ for all $x \in \mathbb{R}_+^n$ and $\alpha \in \mathbb{R}_+$. From now on, we will set $1_i = (0, \ldots, 0, \overbrace{1}^{i}, 0, \ldots, 0)$ for all $i = 1, \ldots, n$.

The next result will be crucial for our proposal.

Proposition 3.5. *Let* $\Phi : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+$ *be a partial quasi-metric aggregation function. Then the following assertions are hold:*

1) $\Phi(a) < \Phi(b)$ *whenever* $a \preceq b$ *with* $a \neq b$.
2) $\Phi(a) = 0$ *implies* $a = (0, \ldots, 0)$.

Proof. 1). For the purpose of contradiction we suppose that $\Phi(a) \geq \Phi(b)$ for some $a \preceq b$ with $a \neq b$. The monotony of Φ ensures that $\Phi(a) = \Phi(b)$. So, $\Phi(b) = \Phi(a)$ with $a \preceq b$. By assertion (2.2) in Theorem 2.7 we obtain that $a = b$, a contradiction.

2). Again, for the purpose of contradiction, assume that $\Phi(a) = 0$ for some $a \neq (0, \ldots, 0)$. Then 1) provides that $\Phi(0, \ldots, 0) < \Phi(a) = 0$, a contradiction.

□

With the aim of introducing the new fixed point result we will say that a function $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ belongs to the class PQA provided that it fulfils for all $x \in \mathbb{R}_+$ the following two properties:

PQA1. $x\Phi(1_i) \leq \Phi(x \cdot 1_i)$ for all $i = 1, \ldots, n$.
PQA2. $\Phi(x, \ldots, x) \leq x\Phi(1, \ldots, 1)$.

Observe that, by Proposition 2.14, those partial quasi-metric aggregation functions belonging to PQA for which the inequalities in conditions $PQA1$ and $PQA2$ can be replaced by equalities and, in addition, $\Phi(1, \ldots, 1) = \Phi(1_i) = 1$ for all $i = 1, \ldots, n$ are instances of Aumann functions (see, for instance, (Pokorný 1986)).

The next result provides us the necessary completeness for a fixed point result.

Lemma 3.6. *Let $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ be a partial quasi-metric aggregation function belonging to PQA. Let $\{(X_i, pq_i)\}_{i=1}^n$ be a family of arbitrary partial quasi-metric spaces and $X = \prod_{i=1}^{n} X_i$. Assume that, for each $i = 1, \ldots, n$, the partial quasi-metric space (X_i, pq_i) is 0-complete. Then the partial quasi-metric space (X, PQ_Φ) is 0-complete, where PQ_Φ is the partial quasi-metric induced by aggregation of the family of partial quasi-metric spaces $\{(X_i, pq_i)\}_{i=1}^n$ through Φ.*

Proof. Let $(x^k)_{k\in\mathbb{N}}$ be a 0-Cauchy sequence in (X, PQ_Φ). Then, for each $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $PQ_\Phi(x^k, x^m) < \epsilon$ for each $k, m \geq k_0$.

Fix $i \in \{1, \ldots, n\}$. We will see that the sequence $(x_i^k)_{k\in\mathbb{N}}$ is 0-Cauchy in (X_i, pq_i).

Let $\epsilon > 0$ and consider $\epsilon \cdot \Phi(1_i) > 0$ (assertion 2) in Proposition 3.5 ensures that $\Phi(1_i) > 0$). Since $(x^k)_{k\in\mathbb{N}}$ is 0-Cauchy sequence in (X, PQ_Φ), there exists $k_0 \in \mathbb{N}$ such that $PQ_\Phi(x^k, x^m) < \epsilon \cdot \Phi(1_i)$, for each $k, m \geq k_0$. Now, by monotony of Φ, we obtain, for each $k, m \geq k_0$, the next inequality

$$\begin{aligned} \Phi(0, \ldots, 0, pq_i(x_i^k, x_i^m), 0, \ldots, 0) &\leq \Phi(pq_1(x_1^k, x_1^m), \ldots, pq_n(x_n^k, x_n^m)) \\ &= PQ_\Phi(x^k, x^m) \\ &< \epsilon \cdot \Phi(1_i). \end{aligned}$$

Moreover, the fact that Φ belongs to PQA provides that

$$pq_i(x_i^k, x_i^m) \cdot \Phi(1_i) \leq \Phi(0, \ldots, 0, pq_i(x_i^k, x_i^m), 0, \ldots, 0) < \epsilon \cdot \Phi(1_i),$$

for each $k, m \geq k_0$. Thus, $pq_i(x_i^k, x_i^m) < \epsilon$, for each $k, m \geq k_0$ and so $(x_i^k)_{k\in\mathbb{N}}$ is 0-Cauchy in (X_i, pq_i).

Hence, since $i \in \{1, \ldots, n\}$ is arbitrary, we deduce that, for each $i \in \{1, \ldots, n\}$, the sequence $(x_i^k)_{k\in\mathbb{N}}$ is 0-Cauchy in (X_i, pq_i). Besides, since (X_i, pq_i) is 0-complete, for each $i \in \{1, \ldots, n\}$, there exists $x_i \in X_i$ such that $\lim_{k\to\infty} pq_i(x_i, x_i^k) = \lim_{k\to\infty} pq_i(x_i^k, x_i) = 0$, for each $i \in \{1, \ldots, n\}$. Then, given $\frac{\epsilon}{\Phi(1,\ldots,1)} > 0$ there exists $k_0 \in \mathbb{N}$ such that, for each $i \in \{1,\ldots,n\}$, we have that $pq_i(x_i^k, x_i) < \frac{\epsilon}{\Phi(1,\ldots,1)}$ and $pq_i(x_i, x_i^k) < \frac{\epsilon}{\Phi(1,\ldots,1)}$ for each $k \geq k_0$. Therefore, by the last inequalities, the fact that Φ belongs to PQA and by assertion 1) in Proposition 3.5, we have the following

$$PQ_\Phi(x^k, x) = \Phi(pq_1(x_1^k, x_1), \ldots, pq_n(x_n^k, x_n)) < \Phi\left(\frac{\epsilon}{\Phi(1,\ldots,1)}, \ldots, \frac{\epsilon}{\Phi(1,\ldots,1)}\right) \leq$$

$$= \frac{\epsilon}{\Phi(1,\ldots,1)} \cdot \Phi(1,\ldots,1) = \epsilon$$

and

$$PQ_\Phi(x, x^k) = \Phi(pq_1(x_1, x_1^k), \ldots, pq_n(x_n, x_n^k)) < \Phi\left(\frac{\epsilon}{\Phi(1,\ldots,1)}, \ldots, \frac{\epsilon}{\Phi(1,\ldots,1)}\right) \leq$$

$$\frac{\epsilon}{\Phi(1,\ldots,1)} \cdot \Phi(1,\ldots,1) = \epsilon,$$

for each $k \geq k_0$, where $x = (x_1, \ldots, x_n) \in X$. Hence, $(x^k)_{k\in\mathbb{N}}$ converges to x and so (PQ_Φ, X) is 0-complete. □

Taking into account that every homogeneous function Φ fulfills the requirements in the statement of Lemma 3.6, we obtain the following.

Corollary 3.7. *Let $\Phi : \mathbb{R}_+^n \to \mathbb{R}_+$ be an homogeneous partial quasi-metric aggregation function. Let $\{(X_i, pq_i)\}_{i=1}^n$ be a family of arbitrary partial quasi-metric spaces and $X = \prod_{i=1}^n X_i$. Assume that, for each $i = 1,\ldots,n$, the partial quasi-metric space (X_i, pq_i) is 0-complete. Then the partial quasi-metric space (X, PQ_Φ) is 0-complete, where PQ_Φ is the partial quasi-metric induced by aggregation of the family of partial quasi-metric spaces $\{(X_i, pq_i)\}_{i=1}^n$ through Φ.*

With the help of Lemma 3.6 we can prove that every projective Φ-contraction is a contraction from the partial quasi-metric space obtained through aggregation into itself.

Theorem 3.8. *Let $\{(X_i, pq_i)\}_{i=1}^n$ be a family of arbitrary partial quasi-metric spaces and $X = \prod_{i=1}^n X_i$. If Φ is a partial quasi-metric aggregation function satisfying $PQA2$ and F is a Φ-projective contraction with contractive constants $c_1, \ldots, c_n \in [0,1]$ such that $c\Phi(1,\ldots,1) < 1$ with $c = \max\{c_1, \ldots, c_n\}$, then F is a contraction from the partial quasi-metric space (X, PQ_Φ) into itself where PQ_Φ is the partial quasi-metric induced by aggregation of the family of partial quasi-metric spaces $\{(X_i, pq_i)\}_{i=1}^n$ through Φ.*

Proof. Let $x, y \in X$. Since Φ is non-decreasing and F is a projective Φ-contraction we have the following inequalities

$$PQ_\Phi(F(x), F(y)) = \Phi(pq_1(F_1(x), F_1(y)), \ldots, pq_n(F_n(x), F_n(y))) \leq$$

$$\Phi(c_1\Phi(pq_1(x_1,y_1)),\ldots,pq_n(x_n,y_n)),\ldots,c_n\Phi(pq_1(x_1,y_1)),\ldots,pq_n(x_n,y_n))) \leq$$

$$\Phi(c\Phi(pq_1(x_1,y_1)),\ldots,pq_n(x_n,y_n)),\ldots,c\Phi(pq_1(x_1,y_1)),\ldots,pq_n(x_n,y_n))).$$

Besides, the fact that Φ satisfies $PQA2$ ensures the next inequality

$$\Phi(c\Phi(pq_1(x_1,y_1)),\ldots,pq_n(x_n,y_n)),\ldots,c\Phi(pq_1(x_1,y_1)),\ldots,pq_n(x_n,y_n))) \leq$$

$$c\Phi(1,\ldots,1)\Phi((pq_1(x_1,y_1)),\ldots,pq_n(x_n,y_n))).$$

Therefore we have that

$$\begin{array}{rcl} PQ_\Phi(F(x),F(y)) & \leq & c\Phi(1,\ldots,1)\Phi((pq_1(x_1,y_1)),\ldots,pq_n(x_n,y_n))) \\ & = & c\Phi(1,\ldots,1)PQ_\Phi(x,y). \end{array}$$

Since $c\Phi(1,\ldots,1)<1$ we conclude that F is a contraction from the partial quasi-metric space (X,PQ_Φ) into itself.

□

The next example shows that the hypothesis "Φ satisfies $PQA2$" cannot be deleted in the statement of Theorem 3.8 in order to warranty that a projective Φ-contraction is also a contraction from (X,PQ_Φ) into itself.

Example 3.9. Let $([0,1],p_{\max})$ be the partial quasi-metric space introduced in Example 2.13. Consider the family of partial quasi-metric spaces $\{([0,1],pq_i)\}_{i=1,2}$ such that $pq_1=pq_2=p_{\max}$. Define the function $\Phi_2:\mathbb{R}^2_+\longrightarrow\mathbb{R}_+$ by $\Phi_2(x)=\frac{x_1+x_2}{4}+\frac{1}{2}$ for all $x\in\mathbb{R}^2_+$. It is not hard to verify that the function Φ_2 fulfills assertions (2.1) and (2.2) in statement of Theorem 2.7 and, hence, it is a partial quasi-metric aggregation function. Clearly, Φ_2 does not satisfy condition $PQA2$. Indeed, $\Phi_2(x,x)\leq x\Phi(1,1)\Leftrightarrow x\in[1,\infty[$.

Next, consider the mapping $F:[0,1]^2\longrightarrow[0,1]^2$ defined by $F(x)=(0,0)$ for all $x\in[0,1]^2$. It is clear that F is a projective Φ-contraction. Nevertheless F is not a contraction from $([0,1]^2,PQ_{\Phi_2})$ into itself, where PQ_{Φ_2} is the partial quasi-metric induced by the aggregation of the family of partial quasi-metric spaces $\{([0,1],pq_i)\}_{i=1,2}$ through Φ_2. Indeed, $PQ_{\Phi_2}(F(0,0),F(0,0))=PQ_{\Phi_2}((0,0),(0,0))=\Phi_2(0,0)=\frac{1}{2}$. Therefore does not exist $c\in[0,1[$ such that

$$PQ_{\Phi_2}(F(0,0),F(0,0))\leq cPQ_{\Phi_2}((0,0),(0,0)).$$

In the next example we show that the hypothesis "$c\Phi(1,\ldots,1)<1$" cannot be also deleted in the statement of Theorem 3.8 in order to warranty that a projective Φ-contraction is also a contraction from (X,PQ_Φ) into itself.

Example 3.10. Consider again the collection of complete partial quasi-metric spaces $([0,1],pq_i)_{i=1,2}$ introduced in Example 3.9. Define the function $\Phi_+:\mathbb{R}^2_+\longrightarrow\mathbb{R}_+$ by $\Phi_+(x)=x_1+x_2$ for all $x\in\mathbb{R}^2_+$. Obviously the function Φ_+ is an homogeneous partial quasi-metric aggregation function and, thus, Φ satisfies $PQA2$. Moreover, $\Phi_+(1,1)=2$. Consider the mapping $F:[0,1]^2\longrightarrow[0,1]^2$ defined by $F(x)=(\frac{x_1+x_2}{2},\frac{x_1+x_2}{2})$ for all $x\in[0,1]^2$. Then we have $p_{\max}(F_i(x),F_i(y))=$

$p_{\max}(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}) \leq \frac{1}{2}\Phi_+(p_{\max}(x_1, y_1), p_{\max}(x_2, y_2))$ for all $x, y \in [0,1]^2$ and for $i = 1, 2$. So, F is a projective Φ_+-contraction with contractive constants $c_1 = c_2 = \frac{1}{2}$. Moreover $\frac{1}{2}\Phi_+(1,1) = 1$.

Nevertheless, F is not a contraction from the partial quasi-metric space $([0,1]^2, PQ_{\Phi_+})$ into itself, where PQ_{Φ_+} is the partial quasi-metric induced by the aggregation of the family of partial quasi-metric spaces $\{([0,1], pq_i)\}_{i=1,2}$ through Φ_+. Indeed, take $x, y \in [0,1]^2$ given by $x = (0,0)$ and $y = (1,1)$. Then there does not exist $c \in [0,1[$ such that $PQ_{\Phi_+}(F(0,0), F(1,1)) \leq cPQ_{\Phi_+}((0,0),(1,1))$, since $PQ_{\Phi_+}(F(0,0), F(1,1)) = PQ_{\Phi_+}((0,0),(1,1)) = 2$.

From Theorem 3.8 we obtain the next result.

Corollary 3.11. *Let $\{(X_i, pq_i)\}_{i=1}^n$ be a family of arbitrary partial quasi-metric spaces and $X = \prod_{i=1}^n X_i$. If Φ is an homogeneus partial quasi-metric aggregation function and F is a Φ-projective contraction with contractive constants $c_1, \ldots, c_n \in [0,1]$ such that $c\Phi(1, \ldots, 1) < 1$ with $c = \max\{c_1, \ldots, c_n\}$, then F is a contraction from the partial quasi-metric space (X, PQ_Φ) into itself, where PQ_Φ is the partial quasi-metric induced by aggregation of the family of partial quasi-metric spaces $\{(X_i, pq_i)\}_{i=1}^n$ through Φ.*

The existence and uniqueness of fixed point for Φ-projective contractions is provided by the next result.

Theorem 3.12. *Let $\{(X_i, pq_i)\}_{i=1}^n$ be a family of arbitrary 0-complete partial quasi-metric spaces and $X = \prod_{i=1}^n X_i$. If Φ is a partial quasi-metric aggregation function which belongs to PQA and F is a projective Φ-contraction with contractive constants $c_1, \ldots, c_n \in [0,1]$ such that $c\Phi(1, \ldots, 1) < 1$ with $c = \max\{c_1, \ldots, c_n\}$, then F has a unique fixed point x_0. Moreover, $PQ_\Phi(x_0, x_0) = 0$ and, for each $x \in X$, $\lim_{n\to\infty} F^n(x) = x_0$ in (X, PQ_Φ), where PQ_Φ is the partial quasi-metric induced by the aggregation of the family of partial quasi-metric spaces $\{(X_i, pq_i)\}_{i=1}^n$ through Φ.*

Proof. On the one hand, Φ and F satisfy conditions in Theorem 3.8 and so, such a theorem guarantees that F is a contraction from the partial quasi-metric space (X, PQ_Φ) into itself. On the other hand, $\{(X_i, pq_i)\}_{i=1}^n$ is a family of arbitrary 0-complete partial quasi-metric spaces and Φ belongs to PQA. Then, by Lemma 3.6, we have that (X, PQ_Φ) is 0-complete. Thus, Theorem 3.3 provides the existence and uniqueness of fixed point $x_0 \in X$ for F such that, $PQ_\Phi(x_0, x_0) = 0$ and $\lim_{n\to\infty} F^n(x) = x_0$ in (X, PQ_Φ), for each $x \in X$. □

Notice that in the above fixed point result the collection of partial quasi-metrics $\{(X_i, pq_i)\}_{i=1}^n$ can be mixed, that is, it can be formed by a subcollection of partial metrics $\{(X_i, p_i)\}_{i=1}^k$, a subcollection of (quasi-)metrics $\{(X_i, q_i)\}_{i=k}^m$ and a subcollection of partial quasi-metrics $\{(X_i, pq_i)\}_{i=m}^n$ which are neither a partial metric nor a quasi-metric. In addition, we want to point out that every partial quasi-metric aggregation function belonging to PQA is also amenable and, thus, it is a (quasi-)metric aggregation function. So Theorem 3.8 retrieves the fixed point theorems, when the contractive constants belongs to $[0,1[$, for projective Φ-contractions given in the framework of partial metric spaces and quasi-metric spaces in Alghamdi et al. (2015) and Martín et al. (2013), respectively.

We end the paper with an example that shows that there are contractions from the partial quasi-metric space (X, PQ_Φ) into itself which are not projective Φ-contraction even when the partial quasi-metric aggregation function Φ is in the PQA class. This last fact shows that Theorem 3.12 is not a direct consequence of Theorem 3.3.

Example 3.13. Consider, one more time, the collection of complete partial quasi-metric spaces $([0,1], pq_i)_{i=1,2}$ introduced in Example 3.9. Define the function $\Phi_{\frac{1}{2}} : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ by $\Phi_{\frac{1}{2}}(x) = \frac{x_1+x_2}{2}$ for all $x \in \mathbb{R}_+^2$. Clearly $\Phi_{\frac{1}{2}}$ is an homogeneous partial quasi-metric aggregation function and, thus, it belongs to the PQA class. Consider the mapping $F : [0,1]^2 \longrightarrow [0,1]^2$ defined by $F(x) = (\frac{x_1+x_2}{2}, 0)$ for all $x \in [0,1]^2$. Then one can verify easily that $PQ_{\Phi_{\frac{1}{2}}}(F(x), F(y)) \leq \frac{1}{2} PQ_{\Phi_{\frac{1}{2}}}(x,y)$ for all $x, y \in [0,1]^2$, where $PQ_{\Phi_{\frac{1}{2}}}$ is the partial quasi-metric induced by the aggregation of the family $\{([0,1], p_i)\}_{i=1,2}$ through $\Phi_{\frac{1}{2}}$. Then F is a contraction from the partial quasi-metric space $([0,1]^2, PQ_{\Phi_{\frac{1}{2}}})$ into itself. However, F is not a projective $\Phi_{\frac{1}{2}}$-contraction. Indeed, take $x, y \in [0,1]^2$ such that $x = (0,0)$ and $y = (1,1)$. Hence $p_{\max}(F_1(x), F_1(y)) = p_{\max}(0,1) = 1$ and $p_{\max}(x_1, y_1) = p_{\max}(x_2, y_2) = p_{\max}(1,0) = 1$. Consequently there does not exist $c \in [0,1[$ such that

$$p_{\max}(F_1(x), F_1(y)) \leq c\Phi_{\frac{1}{2}}(p_{\max}(x_1,y_1), p_{\max}(x_2,y_2)),$$

since $\Phi_{\frac{1}{2}}(p_{\max}(x_1,y_1), p_{\max}(x_2,y_2)) = \Phi_{\frac{1}{2}}(1,1) = 1$.

4. Conclusions

We have characterized those functions that merge a collection of partial quasi-metrics into a new one and, in addition, we have given a few relationships between this kind of functions and those that merge (quasi-)metrics and partial metrics. Moreover, a general fixed point result for contractions obtained through aggregation functions has also been provided.

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