A characterization of p-complete fuzzy metric spaces

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Abstract

George and Veeramani characterized complete fuzzy metric spaces $(X, M, *)$ by means of nested sequences $\{A_n\}$ of closed sets of X which have fuzzy diameter zero [3]. In [5] an appropriate concept of p-Cauchy sequence, according to the concept of p-convergence due to D. Mihet [11], was given. In this paper we introduce for $\{A_n\}$ a concept of p-fuzzy diameter zero, which is according to the concept of p-convergence. Then, we characterize by means of certain nested sequences $\{A_n\}$ which have p-fuzzy diameter zero, those fuzzy metric spaces in which p-Cauchy sequences are convergent (p-convergent), called p-complete spaces (w-p-complete spaces). As a consequence of our results we obtain the well-known characterization of a complete metric space (X, d) by means of nested sequences of closed sets of (X, d).

Keywords: Fuzzy metric space, Cauchy sequence, completeness, nested

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1. Introduction

Here we deal with the concept of fuzzy metric space $(X, M, *)$ (defined using a continuous t-norm $*$), due to George and Veeramani [1]. If M is a fuzzy metric on X, then a topology τ_M, deduced of M, is defined on X. In [2, 9] it was proved that τ_M is metrizable. Fuzzy metrics have been successfully used in Engineering and recently, in particular, in perceptual color differences and color image similarity [4, 13] and inconsistency detection in data sets [14, 15].

In a natural way, many topics studied for metrics have been extended to fuzzy metrics [8, 12, 19, 17, 21]. In particular, an area of high activity is fuzzy fixed point theory [10, 6, 16, 20]. In this framework, as in the classical case, (fuzzy) completeness plays a fundamental role. Now, in our context, they have been appeared several (well-motivated) concepts of Cauchy sequence summarized in [7] and, consequently, also, several concepts of completeness (X is complete, in a wide sense, if Cauchy sequences are convergent). The first concept of Cauchy sequence, which we deal with, in our context, was introduced in [3] although it comes from P-metric spaces [18]. So, the interest of completeness for X, is strongly related to fixed point theory, and in [8] it was stated a characterization of a complete fuzzy metric space X by means of families of closed sets which have fuzzy diameter zero (Definition 3.1). On the other hand, while establishing a fuzzy fixed point theorem, D. Mihet [11] introduced the following weaker concept than convergence: A sequence $\{x_n\}$ is p-convergent to x, for $t_0 > 0$, if $\lim_n M(x, x_n, t_0) = 1$. Then, in a natural way, it was defined an appropriate concept of p-Cauchy sequence [11, 7]. ($\{x_n\}$ is p-Cauchy, for $t_0 > 0$, if $\lim_{m,n} M(x_m, x_n, t_0) = 1$).

The aim of this paper is to obtain a characterization of those fuzzy metric spaces where every p-Cauchy sequence is convergent (p-complete spaces) or p-convergent (w-p-complete spaces). This characterization will be done, in a similar way to complete metric spaces, by means of certain nested sequences of sets of X. For it, we will introduce a concept of p-fuzzy diameter zero for a family of sets of X (Definition 3.2), mimicking the corresponding one in [3], and according to the concept of p-convergence. Next paragraph, summarizes, briefly, the contents of the paper.

For a subset A of X, the function $\phi(t) = \inf\{M(x, y, t) : x, y \in A\}$ for $t > 0$, is the fuzzy diameter of A. A nested sequence $\{A_n\}$ of sets of X has p-fuzzy diameter zero if and only if for some $t_0 > 0$, given $r \in]0, 1[$ there exists $n_r \in \mathbb{N}$ such that $M(x, y, t_0) > 1 - r$ for all $x, y \in A_n,\ n \geq n_r$ (Proposition 3.3) or equivalently $\lim_n \phi_{A_n}(t_0) = 1$ (Proposition 3.7). A point x is a p-accumulation point of a set A of X (Definition 4.1), for $t_0 > 0$, if and only if there exists a sequence $\{a_n\}$ in $A - \{x\}$ such that $\{a_n\}$ is p-convergent to x, for t_0 (Proposition 4.2). The p-closure of A for t_0, denoted $\overset{\sim t_0}{A}$, is the set A jointly with their p-accumulation points of A, for t_0. In a principal space X (every p-convergent sequence in X is convergent), obviously $\overset{\sim t}{A} = \overline{A}$ (closure in τ_M of A), for all $t > 0$. Proposition 5.4 states an interesting result (not used in the paper): Every p-Cauchy sequence with a p-cluster point (Definition 4.5) is p-convergent. The relationship among completeness (in the sense of George and Veeramani) and (w-)p-completeness, above referred is summarized in the following diagram of implications

$$\begin{array}{ccc} & & w\text{-}p\text{-complete} \\ & \nearrow & \\ p\text{-complete} & & \\ & \searrow & \\ & & \text{complete} \end{array}$$

It is almost obvious that X is p-complete if and only if X is principal and w-p-complete. Then, our main result is Theorem 5.8: X is w-p-complete if and only if for every nested sequence $\{A_n\}$ which has p-fuzzy diameter zero there exists $t > 0$ such that $\bigcap \overset{\sim t}{A_n} = \{x\}$.

The p-concepts introduced in the paper, become in the ordinary concepts (without the prefix p) in the case of the standard fuzzy metric M_d deduced from a metric d on X (Propositions 6.2 and 6.3). Then we are able to establish, as a corollary of Theorem 5.8, the well-known characterization of a complete metric space (X, d) by means of nested sequences of closed sets of (X, d) (Corollary 6.4). Throughout this paper, appropriate examples illustrate the theory.

The structure of the paper is as follows. After the preliminary section, in Section 3 we study the concept of p-fuzzy diameter zero for a family of sets of X. In Section 4 we study the p-accumulation points of a set A of X. In Section 5 we characterize the (w-)p-completeness of X and Section 6 is devoted to the particular case of the standard fuzzy metric space $(X, M_d, \cdot)$.

2. Preliminaries

We begin this section recalling the concept of fuzzy metric space introduced by George and Veerameni in [1].

Definition 2.1. A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times]0, \infty[$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

(GV1) $M(x, y, t) > 0$

(GV2) $M(x, y, t) = 1$ if and only if $x = y$

(GV3) $M(x, y, t) = M(y, x, t)$

(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$

(GV5) The assignment $M(x, y, \_) :]0, \infty[\to]0, 1]$ is a continuous function.

If $(X, M, *)$ is a fuzzy metric space we say that $(M, *)$, or simply M, is a fuzzy metric on X. Also, we say that (X, M) or, simply, X is a fuzzy metric space, if no confusion arises.

A celebrated example of fuzzy metric space is the so-called standard fuzzy metric, which is constructed from a classical metric. It is defined as follows.

Let (X, d) be a metric space. Denote by $a \cdot b$ the usual product for all $a, b \in [0, 1]$, and let M_d be the fuzzy set defined on $X \times X \times \mathbb{R}^+$ by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

Then $(M_d, \cdot)$ is a fuzzy metric on X called *standard fuzzy metric* induced by d [1].

George and Veeramani proved in [1] that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, \varepsilon, t) : x \in X, \varepsilon \in]0, 1[, t > 0\}$, where $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$ for all $x \in X$, $\varepsilon \in]0, 1[$ and $t > 0$. In the case of the standard fuzzy metric M_d it is well known that the topology $\tau(d)$ on X, deduced from d, satisfies $\tau(d) = \tau_{M_d}$. From now on, we will suppose X endowed with the topology τ_M.

Convergent sequences in X were characterized in [1] by the following result.

Proposition 2.2. *A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ converges to x_0 if and only if $\lim_n M(x_0, x_n, t) = 1$, for all $t > 0$.*

On account of the previous result and based on the notion of Cauchy sequence given in the context of probabilistic metric spaces (see [18]), George and Veeramani introduced, in a natural way, the next definition in [1].

Definition 2.3. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called *Cauchy* if for each $\varepsilon \in]0, 1[$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$ or equivalently $\lim_{m,n} M(x_n, x_m, t) = 1$ for all $t > 0$.

$(X, M, *)$, or simply M, is called *complete* if every Cauchy sequence in X is convergent (with respect to τ_M).

Motivated by the study of fixed point theory in fuzzy metric spaces, D. Mihet introduced in [11] the following weaker notion than convergence.

Definition 2.4. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called p-convergent to x_0, for $t_0 > 0$, if $\lim_n M(x_n, x_0, t_0) = 1$ or, equivalently, given $\varepsilon \in]0, 1[$ there exists n_ε such that $M(x_0, x_n, t_0) > 1 - \varepsilon$ for all $n \geq n_\varepsilon$.

We will say that $\{x_n\}$ is p-convergent to x_0 without mention of t_0 if confusion is not possible. (This simplification will be used in other concepts throughout the paper).

In addition, it was observed in [11] the following two properties of p-convergent sequences:

(a) Subsequences of p-convergent sequences are p-convergent.

(b) If $\{x_n\}$ is p-convergent to x_0 and to y_0 then $x_0 = y_0$.

Obviously, convergent sequences are p-convergent. Nevertheless, there exist p-convergent sequences which are not convergent as it was pointed out in [11] (see also [5]). Indeed, a sequence $\{x_n\}$ is convergent if and only if it is p-convergent, for all $t_0 > 0$.

With the aim of characterizing those fuzzy metric spaces in which p-convergent sequences are convergent, V. Gregori et al. gave [5] the following definition.

Definition 2.5. A fuzzy metric space $(X, M, *)$ is said to be *principal* (or simply, M is principal) if the family $\{B(x, r, t) : r \in]0, 1[\}$ is a local base at $x \in X$, for each $x \in X$ and each $t > 0$.

In particular, the standard fuzzy metric space $(X, M_d, \cdot)$ is principal. In fact, the authors in [5] observed that many fuzzy metric spaces are principal. Moreover, they obtained the following characterization.

Proposition 2.6. *A fuzzy metric space $(X, M, *)$ is principal if and only if every p-convergent sequence in X is convergent (with respect to τ_M).*

The final part of this section is devoted to the notion of diameter of a set in the context of fuzzy metrics.

Recall that in a metric space (X, d) the diameter of a (non-empty) set A of X, denoted $\mathrm{diam}(A)$, is defined as $\mathrm{diam}(A) = \sup\{d(x, y) : x, y \in A\}$.

Recently, Gregori et al. provided in [8] the following adaptation to the fuzzy context of the preceding notion.

Definition 2.7. Let $(X, M, *)$ be a fuzzy metric space. The fuzzy diameter of a (non-empty) set A of X, with respect to t, is the function $\phi_A :]0, +\infty[\to [0, 1]$ given by $\phi_A(t) = \inf\{M(x, y, t) : x, y \in A\}$, for each $t > 0$.

Furthermore, the authors in [8] observed the following immediate properties on the function ϕ_A.

Proposition 2.8. *The function ϕ_A is well-defined and, in addition, it satisfies the following:*

(i) If $s < t$ then $\phi_A(s) \leq \phi_A(t)$

(ii) If $A \subset B$ then $\phi_A(t) \geq \phi_B(t)$

(iii) $\phi_A(t) = 1$ for some t if and only if A is a singleton set.

3. *p*-fuzzy diameter

We start this section recalling a definition introduced by George and Veeramani in [3].

Definition 3.1. Let $(X, M, *)$ be a fuzzy metric space. A collection of non-empty sets $\{A_i\}_{i \in I}$ in X is said to have fuzzy diameter zero if for each $r \in]0, 1[$ and $t > 0$ we can find $i_{r,t} \in I$ (depending on r and t) such that $M(x, y, t) > 1 - r$ for all $x, y \in A_i$.

According to the previous concept we introduce the following weaker definition.

Definition 3.2. Let $(X, M, *)$ be a fuzzy metric space. A collection of non-empty sets $\{A_i\}_{i\in I}$ of X has p-fuzzy diameter zero if there exists $t_0 > 0$ such that for each $r \in]0,1[$ we can find $i_r \in I$ (depending on r) such that $M(x, y, t_0) > 1 - r$ for each $x, y \in A_i$. We also say that $\{A_i\}$ has p-fuzzy diameter zero for t_0.

In the following, by a nested sequence of sets $\{A_n\}$ we mean a sequence of non-empty sets $\{A_n\}$ of X satisfying $A_{n+1} \subset A_n$ for all $n \in \mathbb{N}$. Then, $\{A_n\}$ has fuzzy diameter zero if and only if given $r \in]0,1[$ and $t > 0$, there exists $n_{r,t} \in \mathbb{N}$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A_n$ with $n \geq n_{r,t}$, or equivalently, $\lim_n \phi_{A_n}(t) = 1$ for all $t > 0$ (see [8], Proposition 2).

We will omit the proofs of the following propositions because they are immediate or can be obtained mimicking the coresponding ones in [8].

Proposition 3.3. *Let $\{A_n\}$ be a nested sequence of sets of a fuzzy metric space X. Then $\{A_n\}$ has p-fuzzy diameter zero if and only if there exists $t_0 > 0$ such that for each $r \in]0,1[$ there exists $n_r \in \mathbb{N}$ such that $M(x, y, t_0) > 1 - r$ for all $x, y \in A_n,\ n \geq n_r$.*

Remark 3.4. *If $\{A_n\}$ has p-fuzzy diameter zero for $t_0 > 0$ then, obviously, it has p-fuzzy diameter zero for each $t \geq t_0$.*

Proposition 3.5. *Let $\{A_n\}$ be a nested sequence of sets of a fuzzy metric space X which has p-fuzzy diameter zero, with non-empty intersection. Then, $\bigcap A_n = \{x\}$, for some $x \in X$.*

Proposition 3.6. *Let $\{A_n\}$ be a (nested) eventually constant sequence of sets of a fuzzy metric space X, i.e., there exists $n_0 \in \mathbb{N}$ such that $A_n = A$ for all $n \geq n_0$. Then $\{A_n\}$ has p-fuzzy diameter zero if and only if A is a singleton set.*

Roughly speaking, a nested sequence of sets $\{A_n\}$ has p-fuzzy diameter zero if the sequence contains small sets whose fuzzy diameter, for some $t_0 > 0$, tends to 1. We formalize this in the following proposition.

Proposition 3.7. *Let $\{A_n\}$ be a nested sequence of sets of a fuzzy metric space X. Then they are equivalent:*

(i) $\{A_n\}$ *has p-fuzzy diameter zero for some* $t_0 > 0$.

(ii) $\lim_n \phi_{A_n}(t_0) = 1$ *for some* $t_0 > 0$

It is easy to conclude that every family of sets of X which has fuzzy diameter zero has p-fuzzy diameter zero. The converse is false as we show in the following example introduced by Mihet in [11].

Example 3.8. Let $\{x_n\}$ be a strictly increasing sequence of positive real numbers that converges to 1, in the usual topology of $\mathbb{R}$. Consider the fuzzy metric space $(X, M, \wedge)$ where $X = \{x_1, x_2, \dots, \} \cup \{1\}$, $\wedge$ is the minimum t-norm and M is defined as follows:

$$M(x, x, t) = 1 \text{ for all } x \in X,\ t > 0;$$
$$M(x_n, x_m, t) = M(x_m, x_n, t) = x_n \wedge x_m \text{ for all } t > 0, \text{ if } n \neq m;$$
$$M(x_n, 1, t) = M(1, x_n, t) = x_n \wedge t \text{ for all } t > 0.$$

Let $A_n = \{x_n, x_{n+1}, \dots\} \cup \{1\}$ for each $n \in \mathbb{N}$. Clearly, $\{A_n\}$ is a nested sequence, and it is easy to verify that $\lim_n \phi_{A_n}(1) = 1$ and $\lim_n \phi_{A_n}(\frac{1}{2}) = \frac{1}{2}$. Then, by Proposition 3.7, $\{A_n\}$ has p-fuzzy diameter zero, and from Proposition 2 in [8], it has not fuzzy diameter zero.

4. *p*-accumulation

We start this section with a natural definition.

Definition 4.1. Let A be a non-empty set of a fuzzy metric space X. A point $x \in X$ is called a p-accumulation point (briefly, p-acc point)) of A if there exists $t_0 > 0$ such that for each $r \in]0, 1[$ we have that $(B(x, r, t_0) - \{x\}) \cap A \neq \emptyset$. In such a case, if necessary, we will say that x is a p-acc point for t_0.

Proposition 4.2. *Let* $(X, M, *)$ *be a fuzzy metric space and let* $A \subseteq X$. *A point* $x \in X$ *is a p-acc point of A, for* t_0, *if and only if there exists a sequence* $\{a_n\}$ *in* $A - \{x\}$ *such that* $\lim_n M(x, a_n, t_0) = 1$ *i.e.,* $\{a_n\}$ *is p-convergent to* x *for* t_0.

Proof. Suppose x is a p-acc point of A. For $n = 2, 3, \dots$ we have that $(B(x, \frac{1}{n}, t_0) - \{x\}) \cap A \neq \emptyset$. Then, we can construct a sequence $\{a_n\}$, taking

$a_n \in A$ with $a_n \neq x$, such that $M(x, a_n, t_0) > 1 - \frac{1}{n}$ for each $n \geq 2$. Then, $\lim_n M(x, a_n, t_0) = 1$.

Conversely, suppose that $\{a_n\}$ is a sequence in $A - \{x\}$ such that $\lim_n M(x, a_n, t_0) = 1$. Then, for $\varepsilon \in]0, 1[$ we can find n_ε such that $M(x, a_n, t_0) > 1 - \varepsilon$ for all $n \geq n_\varepsilon$, i.e. $a_n \in B(x, \varepsilon, t_0)$ with $a_n \neq x$. Then $(B(x, \varepsilon, t_0) - \{x\}) \cap A \neq \emptyset$.

□

Definition 4.3. The p-closure of a set A of X for $t_0 > 0$, denoted $\overset{\sim t_0}{A}$, is the set $A \cup \{x \in X : x$ is a p-acc point of A for $t_0\}$. The p-closure of A, denoted $\tilde{A}$, will be $\tilde{A} = \bigcup_{t>0} \overset{\sim t}{A}$.

Under this notation, the following proposition is immediate.

Proposition 4.4. *Let $(X, M, *)$ be a fuzzy metric space and let $A \subseteq X$. Then,*

(i) $x \in \overset{\sim t_0}{A}$ *if and only if for each* $\varepsilon \in]0, 1[$ *we have that* $B(x, \varepsilon, t_0) \cap A \neq \emptyset$.

(ii) $x \in \overset{\sim t_0}{A}$ *if and only if there exists a sequence* $\{a_n\}$ *in* A *such that* $\lim_n M(x, a_n, t_0) = 1$, *i.e.* $\{a_n\}$ *is p-convergent to* x, *for* t_0.

(iii) $\overline{A} \subset \overset{\sim t}{A}$, *for all* $t > 0$, *where* $\overline{A}$ *denotes the closure of* A *in* τ_M.

(iv) If $t_1 \geq t_0$ *then* $\overset{\sim t_1}{A} \supset \overset{\sim t_0}{A}$.

(v) If X *is principal then* $\overset{\sim t}{A} = \overline{A}$, *for all* $t > 0$, *and then* $\tilde{A} = \overline{A}$.

Definition 4.5. Let $\{x_n\}$ be a sequence in a fuzzy metric space X. A point x of X is called a p-cluster point of $\{x_n\}$ for $t_0 > 0$ if $\{x_n\}$ is frequently in $B(x, r, t_0)$ for each $r \in]0, 1[$, i.e. for each $r \in]0, 1[$ we have that given $n \in \mathbb{N}$ we can find $m \geq n$ such that $x_m \in B(x, r, t_0)$.

Remark 4.6. *If x is a p-cluster point of $\{x_n\}$ for t_0, then, obviously, x is a p-cluster point of $\{x_n\}$ for each $t \geq t_0$. Also, if $\{x_n\}$ is eventually constant, i.e. there exists $n_0 \in \mathbb{N}$ such that $x_n = x$ for all $n \geq n_0$, then x is the unique cluster point of $\{x_n\}$ for $t > 0$.*

Proposition 4.7. *Let $\{x_n\}$ be a sequence in a fuzzy metric space X. A point $x \in X$ is a p-cluster point of $\{x_n\}$ for t_0 if and only if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which is p-convergent to x for t_0.*

Proof. Let $x \in X$ be a p-cluster point of $\{x_n\}$ for t_0. Since $\{x_n\}$ is frequently in $B(x, r, t_0)$ for each $r \in]0, 1[$, then for $m = 2$ we can take $x_{n_2} \in B(x, \frac{1}{2}, t_0)$. By induction on m, we can construct the subsequence $\{x_{n_m}\}$ of $\{x_n\}$ where $n_m > n_{m-1}$ and $x_{n_m} \in B(x, \frac{1}{m}, t_0)$. Then $M(x, x_{n_m}, t_0) > 1 - \frac{1}{m}$ for each $m \geq 2$ and thus $\lim_n M(x, x_{n_m}, t_0) = 1$.

Conversely, suppose x is not a p-cluster point of $\{x_n\}$ for $t_0 > 0$. Then, we can find $r_0 \in]0, 1[$ such that $\{x_n\}$ is not frequently in $B(x, r_0, t_0)$. Therefore $\{x_n\}$ is eventually in $X - B(x, r_0, t_0)$. Thus, every subsequence of $\{x_n\}$ is eventually in $X - B(x, r_0, t_0)$, and so it cannot be p-convergent to x for t_0. □

Next we show a characterization of p-cluster points by means of p-closure.

Theorem 4.8. *Let $\{x_n\}$ be a sequence in a fuzzy metric space X. Then $\bigcap \overset{\sim t_0}{A_n}$ is the set of p-cluster points of $\{x_n\}$ for t_0, where $A_n = \{x_m : m \geq n\}$ for $n \in \mathbb{N}$.*

Proof. Suppose x is a p-cluster point of $\{x_n\}$ for t_0. Then $\{x_n\}$ is frequently in $B(x, \varepsilon, t_0)$ for all $\varepsilon \in]0, 1[$ and thus, for each $r \in]0, 1[$ we have that $A_n \cap B(x, r, t_0) \neq \emptyset$ for all $n \in \mathbb{N}$. Then, $x \in \overset{\sim t_0}{A_n}$ for all $n \in \mathbb{N}$ i.e., $x \in \bigcap \overset{\sim t_0}{A_n}$.

Conversely, if x is not a p-cluster point of $\{x_n\}$ for t_0, then there exists $r_0 \in]0, 1[$ such that $\{x_n\}$ is not frequently in $B(x, r_0, t_0)$, i.e., for some $n_0 \in \mathbb{N}$ we have that $x_n \notin B(x, r_0, t_0)$ for all $n \geq n_0$. Then $B(x, r_0, t_0) \cap A_n = \emptyset$ for $n \geq n_0$, and therefore x is not in $\overset{\sim t_0}{A_n}$. □

Example 4.9. Consider the fuzzy metric space $(X, M, \wedge)$ of Example 3.8.

(a) Let $\{y_n\}$ be a non-eventually constant sequence in X. We claim that $\{y_n\}$ is p-convergent if and only if $\{y_n\}$ is convergent to 1, in the usual topology of $\mathbb{R}$. Further, in that case $\{y_n\}$ is p-convergent to 1, only for $t \geq 1$.

Indeed, suppose that $\{y_n\}$ is p-convergent to $x < 1$. Then $M(x, y_n, t) \leq \max\{x, t\} < 1$ whenever $y_n \neq x$ and $t < 1$, and $M(x, y_n, t) \leq x$ whenever $y_n \neq x$ and $t \geq 1$. Hence $\{y_n\}$ is not p-convergent to x, for any $t > 0$.

Now, suppose $\{y_n\}$ is p-convergent to 1 for some $t > 0$. We claim that $t \geq 1$. Indeed, in other case if $t < 1$, $M(1, y_n, t) \leq t < 1$ whenever $y_n \neq 1$ and hence $\{y_n\}$ is not p-convergent for $t < 1$.

Finally, $\{y_n\}$ is p-convergent to 1 for $t \geq 1$ if and only if $\lim_n M(1, y_n, t) = \lim_n y_n = 1$, i.e., if and only if $\{y_n\}$ is a convergent sequence to 1, in the usual topology of $\mathbb{R}$.

(b) Let $A \subset X$. If x is a p-acc point of A then necessarily $x = 1$.

Indeed, by Proposition 4.2 we can find a sequence $\{y_n\}$ in $A - \{x\}$ which is p-convergent to x. Now, $\{y_n\}$ is not eventually constant, since in that case $\{y_n\}$ converges in A. So, by (a), $\{y_n\}$ is p-convergent to 1 for $t \geq 1$, and further $\{y_n\}$ is convergent to 1, in the usual topology of $\mathbb{R}$.

(c) Let A be a nonempty subset of X, and suppose $1 \notin A$. Since there are not non-eventually p-convergent sequences for $t < 1$, then by (a), we have that $\overset{\sim t}{A} = A$ for each $0 < t < 1$, and $\overset{\sim t}{A} = A \cup \{1\}$ if and only if A contains a sequence $\{y_n\}$ that converges to 1, in the usual topology of $\mathbb{R}$, and $t \geq 1$.

We continue approaching the following question: given a nested sequence of sets $\{A_n\}$ of a fuzzy metric space X with p-fuzzy diameter zero, can we find $t > 0$ such that $\{\overset{\sim t}{A}\}$ has p-fuzzy diameter zero? In the next we answer affirmatively to such a question. First, we prove the following useful lemma.

Lemma 4.10. *Let A be a subset of the fuzzy metric space $(X, M, *)$. Then $\phi_{\overset{\sim t}{A}}(3t) \geq \phi_A(t)$ for all $t > 0$.*

Proof. Fix $t_0 > 0$. Let $x, y \in \overset{\sim t_0}{A}$. By (ii) of Proposition 4.4 we can find two sequences $\{x_n\}$ and $\{y_n\}$ in A, which are p-convergent for t_0, to x and y, respectively. Then

$$\begin{aligned} M(x, y, 3t_0) &\geq M(x, x_n, t_0) * M(x_n, y_n, t_0) * M(y_n, y, t_0) \geq \\ &\geq M(x, x_n, t_0) * \phi_A(t_0) * M(y_n, y, t_0) \end{aligned}$$

and when n tends to ∞ we have that $M(x, y, 3t_0) \geq \phi_A(t_0)$ and hence $\phi_{\overset{\sim t_0}{A}}(3t_0) \geq \phi_A(t_0)$.

□

The announced answer to the aforesaid question is provided below.

Proposition 4.11. *Let $\{A_n\}$ be a nested sequence of sets of a fuzzy metric space X. If $\{A_n\}$ has p-fuzzy diameter zero for some $t_0 > 0$, then $\{\overset{\sim t_0}{A_n}\}$ has p-fuzzy diameter zero, for some $t_1 \geq t_0$.*

Proof. Suppose $\{A_n\}$ has p-fuzzy diameter zero for $t_0 > 0$. By the previous lemma we have that $\lim_n \phi_{\overset{\sim t_0}{A_n}}(3t_0) \geq \lim_n \phi_{A_n}(t_0) = 1$ and hence, by Proposition 3.7, $\{\overset{\sim t_0}{A_n}\}$ has p-fuzzy diameter zero for $t_1 = 3t_0 \geq t_0$. □

The converse of the preceding proposition is not true, in general. Indeed, if for each $n \in \mathbb{N}$ we consider $A_n = \{x_m : m \geq n\}$ in the fuzzy metric space of Example 3.8, then by (c) in Example 4.9 we know that $\overset{\sim t_0}{A_n} = A_n$ for each $n \in \mathbb{N}$, when we consider $0 < t_0 < 1$. Moreover, $\overset{\sim t_0}{A_n}$ has p-fuzzy diameter zero for $t_1 = 1 \geq t_0$ because of $\lim_n \phi_{\overset{\sim t_0}{A_n}}(1) = \lim_n \phi_{A_n}(1) = 1$. Nevertheless, $\{A_n\}$ has not p-fuzzy diameter zero for t_0 since $\lim_n \phi_{A_n}(t_0) = t_0$. However, we can prove the following version related to the reciprocal of Proposition 4.11.

Proposition 4.12. *Let $\{A_n\}$ be a nested sequence of sets of the fuzzy metric space X. If there exists $t_0 > 0$ such that $\{\overset{\sim t_0}{A_n}\}$ has p-fuzzy diameter zero for some $t_1 > 0$, then $\{A_n\}$ has p-fuzzy diameter zero for t_1.*

Proof. Suppose there exist $t_0 > 0$ such that $\{\overset{\sim t_0}{A_n}\}$ has p-fuzzy diameter zero for some $t_1 > 0$. Since $A_n \subset \overset{\sim t}{A_n}$ (for all $t > 0$) then $\lim_n \phi_{A_n}(t_1) \geq \lim_n \phi_{\overset{\sim t_0}{A_n}}(t_1) = 1$, and hence $\{A_n\}$ has p-fuzzy diameter zero for t_1. □

On account of Propositions 4.11 and 4.12 we obtain the following two corollaries.

Corollary 4.13. *Let $\{A_n\}$ be a nested sequence of sets of a fuzzy metric space X. They are equivalent:*

(i) *$\{A_n\}$ has p-fuzzy diameter zero.*

(ii) *There exists $t_0 >$ such that $\{\overset{\sim t_0}{A_n}\}$ has p-fuzzy diameter zero.*

Corollary 4.14. *Let $\{A_n\}$ be a nested sequence of sets of the fuzzy metric space X. If there exists $t_0 > 0$ such that $\{\overset{\sim t_0}{A_n}\}$ has p-fuzzy diameter zero, then we can find $t_1 > 0$ such that for each $t > t_1$ we have that $\{\overset{\sim t}{A_n}\}$ has p-fuzzy diameter zero.*

5. *p*-completeness

This section is devoted to characterize w-p-completeness by means of nested sequences. With this aim, we start recalling the following weaker notion than Cauchy sequence introduced by Gregori et al. in [5].

Definition 5.1. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called p-Cauchy for $t_0 > 0$ if given $\varepsilon \in]0,1[$ we can find $n_\varepsilon \in \mathbb{N}$ such that $M(x_m, x_n, t_0) > 1 - \varepsilon$ for all $m, n \geq n_\varepsilon$, or equivalently $\lim_{m,n} M(x_m, x_n, t_0) = 1$.

Obviously, $\{x_n\}$ is Cauchy if and only if it is p-Cauchy for all $t > 0$. Under this notation we have the following proposition.

Proposition 5.2. *Every p-convergent sequence is p-Cauchy.*

Proof. Suppose $\{x_n\}$ is a p-convergent sequence to x for t_0. Let $\varepsilon \in]0,1[$. We can choose $\delta \in]0,1[$ such that $(1-\delta)*(1-\delta) > 1-\varepsilon$. Then, there exists $n_\varepsilon \in \mathbb{N}$ such that $M(x_n, x_m, 2t_0) \geq M(x_n, x, t_0) * M(x, x_m, t_0) > (1-\delta)*(1-\delta) > 1 - \varepsilon$, for all $m, n \geq n_\varepsilon$, and hence $\{x_n\}$ is p-Cauchy for $2t_0$.

□

Observe that in the previous demonstration that it is actually showed that a p-convergent sequence for $t_0 > 0$ is p-Cauchy for $2t_0$. So, it arises the following open question.

Question 5.3. *Is every p-convergent sequence for $t_0 > 0$ a p-Cauchy sequence for t_0?*

Obviously, the converse of Proposition 5.2 is not true, in general. Indeed, if we consider the fuzzy metric space $(X, M, \wedge)$ of Example 3.8 and take $Y = X - \{1\}$. Then, $\{x_n\}$ is a p-Cauchy sequence in Y which is not p-convergence. Nevertheless, such a reciprocal becomes true when a p-Cauchy sequence in addition has a cluster point, as shows the following result.

Proposition 5.4. *Every p-Cauchy sequence with a p-cluster point is p-convergent.*

Proof. Let $\{x_n\}$ be a p-Cauchy sequence for $t_1 > 0$ and suppose that x is a p-cluster point of $\{x_n\}$ for $t_2 > 0$.

Let $\varepsilon \in]0,1[$ and consider $\delta \in]0,1[$ such that $(1-\delta) * (1-\delta) > 1 - \varepsilon$. Then, for such a $\delta \in]0,1[$ we can find $n_\delta \in \mathbb{N}$ such that it satisfies (simultaneously) $M(x_m, x_n, t_1) > 1-\delta$ and $M(x, x_n, t_2) > 1-\delta$ for all $n \geq n_\delta$. Then $M(x, x_n, t_1+t_2) \geq M(x, x_{n_\varepsilon}, t_2) * M(x_{n_\varepsilon}, x_n, t_1) \geq (1-\delta)*(1-\delta) > 1-\varepsilon$ for all $n \geq n_\delta$ and so $\lim_n M(x, x_n, t_1+t_2) = 1$, and hence $\{x_n\}$ is p-convergent to x (for $t_1 + t_2 > 0$). $\square$

An immediate corollary of the previous result is the following one.

Corollary 5.5. *If $\{x_n\}$ is a p-Cauchy sequence in a fuzzy metric space X, then it can have at most one p-cluster point.*

Definition 5.6. A fuzzy metric space $(X, M, *)$ is called w-p-complete (respectively p-complete) if every p-Cauchy sequence is p-convergent (respectively, convergent). (Compare with definition of p-complete in [5]). It is also said that M or X is complete.

The relationship among completeness, p-completeness and w-p-completeness is shown in the following diagram of implications.

$$\begin{array}{ccc} & & w\text{-}p\text{-complete} \\ & \nearrow & \\ p\text{-complete} & & \\ & \searrow & \\ & & \text{complete} \end{array}$$

If X is principal then, obviously, every w-p-complete space is p-complete (in Example 19 of [5], there is a complete principal fuzzy metric space which is not w-p-complete).

From the above definitions and the last paragraph, we obtain the following corollary.

Corollary 5.7. *X is p-complete if and only if X is principal and w-p-complete.*

Next, we characterize w-p-complete fuzzy metric spaces by means of a nested sequence of sets of X.

Theorem 5.8. *Let $(X, M, *)$ be a fuzzy metric space. Then X is w-p-complete if and only if for every nested sequence $\{A_n\}$ which has p-fuzzy diameter zero there exists $t > 0$ such that $\bigcap \overset{\sim}{A_n}^t = \{x\}$, for some $x \in X$.*

Proof. Suppose X is w-p-complete. Let $\{A_n\}$ be a nested sequence which has p-fuzzy diameter zero for $t_0 > 0$. We construct a sequence $\{a_n\}$ taking $a_n \in A_n$ for each $n \in \mathbb{N}$. Since $\{A_n\}$ has p-fuzzy diameter zero, given $r \in]0, 1[$ there exists $n_r \in \mathbb{N}$ such that $M(x, y, t_0) > 1 - r$ for all $x, y \in A_n$ with $n \geq n_r$. In particular, $M(a_m, a_n, t_0) > 1 - r$ for all $m, n \geq n_r$, i.e., $\{a_n\}$ is p-Cauchy, and therefore, by hypothesis $\{a_n\}$ is p-convergent to (some) $x \in X$, for (some) $t \geq t_0$. In addition, $\{A_n\}$ has also p-fuzzy diameter zero for that $t > 0$ attending to Remark 3.4.

Now, $a_m \in A_n$ for all $m \geq n$ and then, by (ii) of Proposition 4.4 $x \in \overset{\sim}{A_n}^t$ for all $n \in \mathbb{N}$. Now, by Proposition 4.11, $\{\overset{\sim}{A_n}^t\}$ has p-fuzzy diameter zero for some $t_1 \geq t$. Therefore, by Proposition 3.5, $\bigcap \overset{\sim}{A_n}^t = \{x\}$.

Conversely, let $\{x_n\}$ be a p-Cauchy sequence in X for $t_0 > 0$. Define $A_n = \{x_n, x_{n+1}, \dots\}$ for all $n \in \mathbb{N}$. For a given $r \in]0, 1[$ we can find $n_r \in \mathbb{N}$ such that $M(x_m, x_n, t_0) > 1 - r$ for all $m, n \geq n_r$. Then $\{A_n\}$ is a nested sequence that has p-fuzzy diameter zero for $t_0 > 0$. By hypothesis, there exists $t > 0$ such that $\bigcap \overset{\sim}{A_n}^t = \{x\}$. Now, by Corollary 4.14, there exists $t_1 > \max\{t_0, t\}$ such that $\{\overset{\sim}{A_n}^{t_1}\}$ has p-fuzzy diameter zero. Moreover, by (iv) of Proposition 4.4, $x \in \overset{\sim}{A_n}^{t_1}$ for all $n \in \mathbb{N}$, and by Proposition 3.5, $\bigcap \overset{\sim}{A_n}^{t_1} = \{x\}$. Then, for $\varepsilon \in]0, 1[$ we can find $n_\varepsilon \in \mathbb{N}$ such that $M(y, z, t_1) > 1 - \varepsilon$ for all $y, z \in \overset{\sim}{A_n}^{t_1}$ and $n \geq n_\varepsilon$. In particular, $M(x, x_n, t_1) > 1 - \varepsilon$ for all $n \geq n_\varepsilon$, i.e., $\{x_n\}$ is p-convergent to x (for t_1), and hence X is w-p-complete. □

Example 5.9. Consider the fuzzy metric space $(X, M, *)$ of Example 3.8 and 4.9. We will prove that X satisfies Theorem 5.8 and thus X is w-p-complete.

Let $\{A_n\}$ be a nested sequence of sets of X which has p-fuzzy diameter zero. If $\{A_n\}$ is eventually constant, i.e., there exists $n_0 \in \mathbb{N}$ such that $A_n = A$ for $n \geq n_0$, then by Proposition 3.5, $A = \{x\}$ for some $x \in X$, and by (ii) of Proposition 4.4, for each $n \geq n_0$, $\overset{\sim}{A_n}^t = \{x\}$ for all $t > 0$. Suppose now that $\{A_n\}$ is not eventually constant, and without loss of generality, that it has p-fuzzy diameter zero for some $t_1 \geq 1$.

For each $n \in \mathbb{N}$ take $y_n \in A_n$ and consider the sequence $\{y_n\}$. Take $\varepsilon \in]0,1[$. There exists $n_\varepsilon \in \mathbb{N}$ such that, for each $n \geq n_\varepsilon$ we have that $M(x,y,t_0) = min\{x,y\} > 1-\varepsilon$ for all $x,y \in A_n$ with $x \neq y$. Obviously, $1-\varepsilon < y_n \leq 1$ for all $n \geq n_\varepsilon$ and then $\{y_n\}$ converges to 1, in the usual topology of $\mathbb{R}$. Then by (a) of Example 4.9, $\{y_n\}$ is p-convergent to 1 for $t_0 = 1$. Now, $y_m \in A_n$ for all $m \geq n$ and then 1 is a p-acc point of A_n for $t_0 = 1$, $n \in \mathbb{N}$. So $1 \in \bigcap \overset{\sim t_0}{A_n}$. Now, by Proposition 4.11 we have that $\{\overset{\sim t_1}{A_n}\}$ has p-fuzzy diameter zero and $1 \in \bigcap \overset{\sim t_1}{A_n}$, and by Proposition 3.5, $\bigcap \overset{\sim t_1}{A_n} = \{1\}$.

Finally, X is not p-complete since it is not principal.

6. Only for the standard fuzzy metric

Let (X,d) be a metric space and M_d the standard fuzzy metric deduced from d. If $\{x_n\}$ is a sequence in X, it is well known [3] that $\{x_n\}$ is d-Cauchy if and only if it is M_d-Cauchy, and also $\{x_n\}$ is d-convergent if and only if it is M_d-convergent, since $\tau(d) = \tau_{M_d}$. Further, (X,d) is complete if and only if (X,M_d) is complete.

Proposition 6.1. *Let A be a non-empty subset of (X,M_d). Then $\phi_A(t) = \dfrac{t}{t+diam(A)}$ for $t>0$.*

Proof. Let $t > 0$. Then

$$\begin{aligned}\phi_A(t) &= \inf\{M_d(x,y,t) : x,y \in A\} = \inf\left\{\frac{t}{t+d(x,y)} : x,y \in A\right\} = \\ &= \frac{t}{t+\sup\{d(x,y) : x,y\in A\}} = \frac{t}{t+\mathrm{diam}(A)}.\end{aligned}$$

□

Proposition 6.2. *Let $\{A_n\}$ be a nested sequence of sets of X. They are equivalent:*

(i) $\{A_n\}$ has p-fuzzy diameter zero in (X,M_d).

(ii) $\lim_n diam(A_n) = 0$.

(iii) $\{A_n\}$ has fuzzy diameter zero in (X,M_d).

Proof. By the last proposition, $\lim_n \phi_{A_n}(t_0) = 1$ for some $t_0 > 0$ is equivalent to $\lim_n \operatorname{diam}(A_n) = 0$ and it is equivalent to $\lim_n \phi_{A_n}(t) = 1$ for all $t > 0$. □

In a similar way, the following proposition can be obtained.

Proposition 6.3. *Let $\{x_n\}$ be a sequence in the standard fuzzy metric space (X, M_d). Then*

(i) $\{x_n\}$ is p-Cauchy if and only if $\{x_n\}$ is Cauchy.

(ii) $\{x_n\}$ is p-convergent if and only if $\{x_n\}$ is convergent.

Further,

(iii) (X, M_d) is w-p-complete if and only if (X, M_d) is p-complete if and only if (X, M_d) is complete.

Now, as a corollary of our Theorem 5.8 we obtain the well-known characterization of the completeness of a metric space by means of a nested sequence of closed sets.

Corollary 6.4. *Let (X, d) be a metric space. They are equivalent:*

(i) (X, d) is complete.

(ii) Every nested sequence of closed sets $\{F_n\}$ with $\lim_n$ diam$(F_n) = 0$ has a singleton intersection.

Proof. Suppose (X, d) is complete. Then (X, M_d) is complete and consequently it is w-p-complete. Let $\{F_n\}$ be a sequence of closed sets with $\lim_n \operatorname{diam}(F_n) = 0$. Then, by Proposition 6.2, $\{F_n\}$ has p-fuzzy diameter zero in (X, M_d), and hence, by Theorem 5.8, there exists $t_0 > 0$ such that $\bigcap \overset{\sim\, t_0}{F_n} = \{x\}$. Now, $\overset{\sim\, t}{F_n} = \overline{F_n}$ $(= F_n)$ for all $t > 0$ since M_d is principal and then $\bigcap F_n = \{x\}$.

Conversely, let $\{A_n\}$ be a nested sequence of sets of X which has p-fuzzy diameter zero in (X, M_d). Then, by Proposition 6.2, $\{A_n\}$ has fuzzy diameter zero, and following the arguments in the proof of by Lemma 1 of [8], we conclude that $\{\overline{A_n}\}$ has fuzzy diameter zero in (X, M_d). Now, by Proposition 6.2, $\lim_n \operatorname{diam}(\overline{A}_n) = 0$. Then, by hypothesis, $\bigcap \overline{A}_n = \{x\}$ and by (v) of Proposition 4.4, $\bigcap \overset{\sim\, t}{A_n} = \{x\}$, for all $t > 0$, since M_d is principal. Hence, by Theorem 5.8 we have that (X, M_d) is w-p-complete, and by (iii) of Proposition 6.3 (X, M_d) is complete. Consequently (X, d) is complete. □

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