Universitat
de les Illes Balears

MASTER'S THESIS

TRANSITION MAPS ON CROSS-SECTIONAL PLANES DEFINED BY LINEAR FLOWS

Francisco Alejandro Torres Juan

Master's Degree in Advanced Physics and Applied Mathematics

Centre for Postgraduate Studies

Academic Year 2021-22

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Key words:

dynamical systems, differential equations, cross sections, Krylov base

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1 Introduction

Differential equations are present nearly everywhere, since every process that evolves with time can be modelled by a differential equation. Thus, in order to fully understand the modelled process, their study becomes essential. And, regarding this study, two branches arise, each one of them with a different focus.

On the one hand, classical theory focuses on finding an explicit solution of the differential equation, for a given initial value. This solution, that is not analytically obtainable for every differential equation, completely determines the evolution of the system.

On the other hand, qualitative theory is not focused on finding an explicit solution. Instead, the focus of this branch of theory is to locate points with a specific behavior, that will determine the behavior of nearby solutions. Thus, the qualitative study also helps us determine the evolution of the system without having to compute an explicit solution. Of course, if the solution is easily obtainable, its use eases the work.

In particular, for linear differential systems, as the one given by

$$\dot{\mathbf{x}} = A\mathbf{x}, \tag{1}$$

for $A \in M_n(\mathbb{R})$ and $\mathbf{x} \in \mathbb{R}^n$, an explicit solution is always obtainable. In fact, there is a wide amount of theory and results regarding this family of systems, since linear systems are the easiest to work with. However, there are still some open questions regarding these systems.

This work is intended to cover some of these questions. In particular, the focus of this work is to fully determine the evolution of a linear system over any hyperplane $\mathcal{P}$ transversal to the flow of the system. This master's thesis is inspired by the article [7] and the work [8], both developed by J. Llibre and A. E. Teruel.

As new points with respect to the latter reference, first in this work the contact points are fully treated and classified into disjoint sets $\mathcal{L}_m$ with respect to their order of contact. Secondly, the hyperplane $\mathcal{P}$ is partitioned into two subsets $\mathcal{P}^I$ and $\mathcal{P}^O$ that gather all the points that evolve to one of the half-spaces separated by the hyperplane. On the third place, theorems about explicit expressions for the transition maps and their derivatives are stated and proven. Moreover, for dimensions $n = 2, 3$ these expressions are used to obtain explicit expressions for the transition maps in a neighborhood of the contact point.

In the last section, we present a study over a family of piecewise linear systems as an application of the theory developed throughout the work. In particular, for these systems the results developed start to shine, as it will be stated below.

1.1 Results and concepts related to the work

This subsection is devoted to locate the basic results that will be applied throughout the work, in order to preserve clarity and organization.

In the next result, and over the rest of the work, $A^0 = I_n$, the $n \times n$ identity matrix.

Theorem 1 (Cayley-Hamilton)**.** *Let $p_A(x) = \sum_{i=0}^{n} d_i x^i$ be the characteristic polynomial of the matrix A, where $d_0 = (-1)^n \det(A)$, $d_{n-1} = -\operatorname{Tr}(A)$, and $d_n = 1$. Then, $A^n = -\sum_{i=0}^{n-1} d_i A^i$.*

Corollary 1. *The matrix A verifies*

$$A^{n+m} = -\sum_{i=0}^{n-1} d_i^m A^i$$

for all $m \geq 0$.

Theorem 2 (Picard)**.** *Consider $(t_0, \mathbf{x}_0) \in D \subset \mathbb{R} \times \mathbb{R}^n$ and let $R = I_a(t_0) \times B_b(\mathbf{x}_0) \subset D$, for $I_a(t_0)$ the closed interval of radius a centered at t_0 and $B_b(\mathbf{x}_0)$ the closed $\mathbb{R}^n$ ball of radius b centered at $\mathbf{x}_0$. Let $f : D \to \mathbb{R}^n$ be a continuous function verifying the Lipschitz condition with respect to the second variable in D. Then, the initial value problem*

$$\begin{cases} \dot{\mathbf{x}} = f(t, \mathbf{x}), \\ \mathbf{x}(t_0) = \mathbf{x}_0, \end{cases}$$

has a unique solution, defined over $I_\alpha(t_0)$, where $\alpha = \min(a, \frac{b}{M})$ and $M = \max\{\|f(t, \mathbf{x})\| : (t, \mathbf{x}) \in R\}$.

Theorem 3 (Implicit Function Theorem)**.** *Let $f : (V, W) \subset \mathbb{R}^{n+m} \to \mathbb{R}^m$ be of class $C^1((V, W))$, where we consider $(\mathbf{x}, \mathbf{y})$ to be the coordinates of (V, W), and suppose there exists a point $(\mathbf{v}, \mathbf{w}) \in (V, W)$ such that $f(\mathbf{v}, \mathbf{w}) = \mathbf{0}$. If the differential matrix*

$$D_y(\mathbf{v}, \mathbf{w})$$

is regular, then there exists an open subset $U \subset V$ in a neighborhood of $\mathbf{v}$ and there exists a unique function $h : U \to \mathbb{R}^m$ of class $C^1(U)$ such that $h(\mathbf{v}) = \mathbf{w}$ and $f(\mathbf{v}, h(\mathbf{v})) = \mathbf{0}$ for all $\mathbf{v} \in U$.

2 Linear Differential Systems

In this first section, we present all the concepts and results relative to **linear differential systems**, which are given by

$$\dot{\mathbf{x}} = A(t)\mathbf{x}(t) + \mathbf{b}(t) \tag{2}$$

for $A : I \to M_n(\mathbb{R})$, $\mathbf{b} : I \to \mathbb{R}^n$ continuous functions, being $I \subset \mathbb{R}$ an interval. If $\mathbf{b}(t) \equiv \mathbf{0}$, we call the system **homogeneous**.

As it happens in any family of differential systems, there are two branches of theory for these ones: classical theory, and qualitative theory. Both branches bring useful results, so in what follows we will introduce definitions and results for each of these branches.

2.1 Classical Theory

Classical theory is focused on explicitly solving the presented differential equation, the linear system (2) in this case, in order to have a full understanding on the system. Although an explicit solution is always desired, there are some systems that are analytically unsolvable. Moreover, even for some systems where a solution is obtainable, expressions are so complex that they yield little information. For the linear systems, this is not the case, and we can always find a solution for equations of the form (2).

To obtain an expression for the solution, we present some definitions that will be needed in order to get to the mentioned expression.

A first result ensures we have a unique solution for each given initial condition.

Theorem 4. *Consider a point $(t_0, \mathbf{x}_0) \in I \times \mathbb{R}^n$. The Cauchy problem (2) with initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ has an unique solution, defined all over I.*

Proof. For closed and bounded I, the result follows straightforward from Picard's Theorem, noting that linearity of the vector field yields to Lipschitz condition over the domain $I \times \mathbb{R}^n$ with Lipschitz constant equal to $L = \max_{t \in I} \|A(t)\|$ since $A(t)$ is a continuous function.

Now, consider I left-open, right-open or open. Consider a fixed initial condition $(t_0, \mathbf{x}_0)$ and a sequence of closed and bounded intervals $\{I_n\}_{n=0}^{\infty}$ such that $t_0 \in I_n$, $I_n \subset I_{n+1}$ and $I = \bigcup_{n=0}^{\infty} I_n$. Since these intervals are closed, on all of them we have a unique solution fulfilling the initial condition. Let us label these solutions as $\{\varphi_n\}_{n=0}^{\infty}$. If we consider now the function

$$\begin{array}{rcll} \varphi : I & \to & \mathbb{R}^n & \\ t & \mapsto & \varphi(t) = \varphi_n(t) & \text{if } t \in I_n, \end{array}$$

then we have that, as I_n is closed, the function φ_n is unique. Thus, as $\varphi_{n+1}|_{I_n} = \varphi_n$, the function $\varphi(t)$ is solution of the differential equation satisfying the initial condition $(t_0, \mathbf{x}_0)$ and it is unique. □

The uniqueness of solutions for a given initial condition $(t_0, \mathbf{x}_0)$ allows us to talk about the solution $\varphi(t; t_0, \mathbf{x}_0)$, a function of t that for $t = t_0$ takes as image the initial condition $\mathbf{x}_0$. Logically, it verifies the differential equation from which it is solution, that is, $\dot{\varphi} = A(t)\varphi + \mathbf{b}(t)$, for any $t \in I$.

For the case of homogeneous systems, the next result states the algebraic structure of the set of solutions. Specifically, they form a vector space.

Theorem 5. *The set $\mathcal{A}$ of all solutions of the homogeneous system $\dot{\mathbf{x}} = A(t)\mathbf{x}$ forms an $n-$dimensional vector space.*

Proof. First of all, notice that the set $\mathcal{A}$ is a subset of the vector space $C^1(I)$. Moreover, the zero vector of this space, the function $\mathbf{0}$, is in $\mathcal{A}$. Thus, it suffices to check that this set is closed with respect to vector addition and scalar products. If we consider $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{A}$, as the differential equation is linear, it follows $\frac{d(\mathbf{x}_1+\mathbf{x}_2)}{dt} = \dot{\mathbf{x}}_1 + \dot{\mathbf{x}}_2 = A(t)(\mathbf{x}_1 + \mathbf{x}_2)$, from where $\mathbf{x}_1 + \mathbf{x}_2 \in \mathcal{A}$. Now,

consider $\mathbf{x} \in \mathcal{A}, c \in \mathbb{R}$. We have $\dot{\mathbf{x}} = A(t)\mathbf{x}$, thus multiplying by c, it follows $c\dot{\mathbf{x}} = cA(t)\mathbf{x}$, from where $c\mathbf{x} \in \mathcal{A}$.

Let us now consider the function $\varphi_s : \mathbb{R}^n \to \mathcal{A}$ given by $\varphi_s(\mathbf{x}) = \varphi(t; s, \mathbf{x})$. If we prove it is an isomorphism, we will have the dimension of $\mathcal{A}$. We have that φ_s is linear, as $\varphi_s(\mathbf{x}_1 + \mathbf{x}_2) = \varphi(t; s, \mathbf{x}_1 + \mathbf{x}_2)$. If we evaluate this expression at time $t = s$, we have $\varphi(s; s, \mathbf{x}_1 + \mathbf{x}_2) = \mathbf{x}_1 + \mathbf{x}_2$, which is the same as $\varphi(s; s, \mathbf{x}_1) + \varphi(s; s, \mathbf{x}_2)$. From uniqueness of solutions, as these both coincide in a point, they must be the same, that is, $\varphi(t; s, \mathbf{x}_1 + \mathbf{x}_2) = \varphi(t; s, \mathbf{x}_1) + \varphi(t; s, \mathbf{x}_2)$. Similar calculations show that the property holds for scalar products. In consequence, the map is linear. Consider now its kernel, $\ker \varphi_s = \{\mathbf{x} \in \mathbb{R}^n : \varphi_s(\mathbf{x}) = \mathbf{0}\} = \{\mathbf{0}\}$ by the uniqueness of solutions. Moreover, the map is surjective, as to any solution with a given initial condition can be mapped the point acting as this initial condition. Thus, it follows that φ_s is an isomorphism. □

From the result above, it follows that solutions of a linear homogeneous system form a vector space, thus in order to describe all possible solutions it suffices to know a basis of them. Moreover, as a linear combination of vectors can be written in terms of a matricial product, the following definition leads to easier calculations.

We define the **solution matrix** for a homogeneous system to be a matrix in which every column is a solution:

$$M(t) = \begin{pmatrix} \varphi(t; t_1, \mathbf{x}_1) \mid \varphi(t; t_2, \mathbf{x}_2) \mid \ldots \mid \varphi(t; t_n, \mathbf{x}_n) \end{pmatrix}. \tag{3}$$

If all columns (solutions) that form the matrix are linearly independent, we say the matrix is a **fundamental matrix**. In this case, it follows straightforward that those solutions span a basis of $\mathcal{A}$.

On the next result, there are presented some basic properties of solution matrices and fundamental ones.

Proposition 1. *Let (3) be a solution matrix for system (2) with $\mathbf{b}(t) \equiv \mathbf{0}$. Then, the next properties follow:*

a) $M(t)$ is a solution matrix if, and only if, $\dot{M}(t) = A(t)M(t)$.

b) $M(t)$ is a solution matrix if, and only if, $M(t)\mathbf{c}$ is a solution, for all $\mathbf{c} \in \mathbb{R}^n$.

c) If $M(t)$ is a solution matrix, then $M(t)C$ is a solution matrix for all matrix C.

d) $M(t)$ is a fundamental matrix if, and only if, there exists a $t_0 \in \mathbb{R}$ such that $\det(M(t_0)) \neq 0$.

e) If $M(t)$ is a fundamental matrix, then

$$\frac{d}{dt} M^{-1}(t) = -M^{-1}(t)A(t).$$

Proof. Since most of these properties are straightforward, we will just prove statement d).

For the direct implication, since $M(t)$ is fundamental, we have that the columns $\{\varphi(t; t_i, \mathbf{x}_i\}_{i=1}^n$ of the matrix are linearly independent for all time t, in particular, for some time t_0. Therefore, $\det(M(t_0)) \neq 0$.

For the converse, let us suppose there exist $\alpha_1, \ldots, \alpha_n$ not all null such that $\sum_{i=1}^n \alpha_i \varphi(t; t_i, \mathbf{x}_i) \equiv 0$, where 0 denotes the function identically zero. Now, since linear combination of solutions is a solution, it follows that $\det(M(t)) = 0$ for all time t, which is a contradiction. □

Moreover, on the next results it is shown a formula to compute the solution for both a homogeneous and a non-homogeneous system.

Proposition 2. *Let $M(t)$ be a fundamental matrix of system (2) with $\mathbf{b}(t) \equiv \mathbf{0}$ and let $\varphi(t; t_0, \mathbf{x}_0)$ be a solution of this system. Then,*

$$\varphi(t; t_0, \mathbf{x}_0) = M(t)M^{-1}(t_0)\mathbf{x}_0.$$

Proof. Let $\varphi(t; t_k, \mathbf{x}_k)$ be the columns of matrix $M(t)$. As this is a fundamental matrix, the set $\{\varphi(t; t_k, \mathbf{x}_k)\}_{k=1}^n$ is a basis of $\mathcal{A}$. Therefore, the solution $\varphi(t; t_0, \mathbf{x}_0)$ can be written in terms of the base, that is,

$$\varphi(t; t_0, \mathbf{x}_0) = \sum_{k=1}^{n} \lambda_k \varphi(t; t_k, \mathbf{x}_k) = M(t)\Lambda,$$

where $\Lambda = (\lambda_1, \ldots, \lambda_n)$. In consequence, we have $\mathbf{x}_0 = \varphi(t_0; t_0, \mathbf{x}_0) = M(t_0)\Lambda$ or, equivalently, $\Lambda = M^{-1}(t_0)\mathbf{x}_0$. Substituting this expression for Λ in the previous chain of equalities, the result follows. □

Proposition 3. *Let $M(t)$ be a fundamental matrix for the homogeneous system $\dot{\mathbf{x}} = A(t)\mathbf{x}$ and let $\varphi(t; t_0, \mathbf{x}_0)$ be a solution of the non-homogeneous system $\dot{\mathbf{x}} = A(t)\mathbf{x} + \mathbf{b}(t)$. Then,*

$$\varphi(t; t_0, \mathbf{x}_0) = M(t)\left(M^{-1}(t_0)\mathbf{x}_0 + \int_{t_0}^{t} M^{-1}(s)\mathbf{b}(s)ds\right).$$

Proof. Suppose the function $\psi(t) = M(t)C(t)$ is a solution for the non-homogeneous system. If we differentiate this expression, we have $\dot{\psi}(t) = \dot{M}(t)C(t) + M(t)\dot{C}(t) = A(t)\psi(t) + M(t)\dot{C}(t)$. Therefore, $C(t)$ must fulfill the condition $\dot{C}(t) = M^{-1}(t)\mathbf{b}(t)$. It follows then $C(t) - C(t_0) = \int_{t_0}^{t} M^{-1}(s)\mathbf{b}(s)ds$. From this, we have

$$\psi(t) = M(t)\left(C(t_0) + \int_{t_0}^{t} M^{-1}(s)\mathbf{b}(s)ds\right).$$

And, as $\psi(t_0) = M(t_0)C(t_0)$, the result follows. □

From the last result above, we almost have an explicit expression for the solution. However, computations of a fundamental matrix $M(t)$ are generally hard to perform. In some cases, the calculus gets simplified. A family in which this simplification happens are the **linear systems with constant coefficients**, which are given by

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}, \tag{4}$$

for A an $n \times n$ constant matrix, and $\mathbf{b}$ a vector of $\mathbb{R}^n$. As above, if vector $\mathbf{b}$ is null, the system will be called homogeneous.

Moreover, if $\mathbf{z}$ is a zero of equation $A\mathbf{x} + \mathbf{b}$ (we will explain later in the qualitative theory section the interest for these zeros), the change of variable $\mathbf{y} = \mathbf{x} - \mathbf{z}$ converts system (4) in a homogeneous one. Relabeling $\mathbf{y}$ as $\mathbf{x}$, it is given by

$$\dot{\mathbf{x}} = A\mathbf{x}. \tag{5}$$

For the case of A being regular, there is a unique zero, and it is given by $\mathbf{z} = A^{-1}\mathbf{b}$.

For homogeneous systems, as the one given in equation (5), we have a pretty compact expression for its fundamental matrix. It is defined as

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}, \tag{6}$$

and we will refer to this as the **exponential matrix** of the system. The following result shows us that this matrix is indeed a fundamental matrix for the system (5) and shows that the series converge.

Theorem 6. *Consider the linear homogeneous system*

$$\dot{\mathbf{x}} = A\mathbf{x},$$

for $A \in M_n(\mathbb{R})$ and $\mathbf{x} \in \mathbb{R}^n$. The fundamental matrix for this system is the exponential matrix, given by

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!},$$

and this series converges absolutely and uniformly over compact subsets of $\mathbb{R}$.

Proof. Consider the initial value problem

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x}, \\ \mathbf{x}(0) &= \mathbf{e}_k, \end{aligned}$$

where $\mathbf{e}_k$ is the $k-$th vector of the canonical basis of $\mathbb{R}^n$, and its solution $\varphi(t; 0, \mathbf{e}_k)$.

Consider, for a closed interval $I \subset \mathbb{R}$, the Picard operator

$$\begin{aligned} T : C(I, \mathbb{R}^n) &\to C(I, \mathbb{R}^n) \\ \varphi(t; 0, \mathbf{e}_k) &\mapsto \mathbf{e}_k + \textstyle\int_0^t A\varphi(t; 0, \mathbf{e}_k) ds. \end{aligned}$$

As the vector field is linear, it fulfills the Lipschitz condition with Lipschitz constant $\|A\|$; therefore, starting with any $\varphi_0 \in C(I.\mathbb{R}^n)$, the sequence of Picard iterates $\{T^i\varphi\}_{i\geq 0}$ converges uniformly in I to the unique solution of the initial value problem.

Let us consider now the sequence of iterates given by

$$\begin{array}{rcl} \varphi_0(t) & = & \mathbf{e}_k, \\ \varphi_m(t) & = & \mathbf{e}_k + \int_0^t A\varphi_{m-1}(s)ds, \quad m \geq 1. \end{array}$$

Direct computations show $\varphi_1(t) = (I_n + At)\mathbf{e}_k$, for I_n the $n \times n$ identity matrix, $\varphi_2(t) = (I_n + At + \frac{A^2t^2}{2})\mathbf{e}_k$ and, iteratively, $\varphi_m(t) = (\sum_{i=0}^m \frac{A^it^i}{i!})\mathbf{e}_k$. As this sequence converges uniformly in I to the solution$\varphi(t; 0, \mathbf{e}_k)$, we have that

$$\varphi(t; 0, \mathbf{e}_k) = \lim_{m\to\infty} \varphi_m(t) = \left(\sum_{i=0}^{\infty} \frac{A^it^i}{i!}\right)\mathbf{e}_k.$$

Let us now consider the fundamental matrix given by

$$M(t) = \left(\begin{array}{c|c|c|c} \varphi(t; 0, \mathbf{e}_1) & \varphi(t; 0, \mathbf{e}_2) & \ldots & \varphi(t; 0, \mathbf{e}_n) \end{array} \right),$$

whose columns are the limits of the sequence of iterates for each vector on the canonical basis of $\mathbb{R}^n$. From the explicit expressions for these limits of iterates, and defining the exponential matrix as

$$e^{At} = \sum_{i=0}^{\infty} \frac{A^it^i}{i!},$$

it follows

$$M(t) = (\sum_{i=0}^{\infty} \frac{A^it^i}{i!})I_n = e^{At}I_n.$$

From this expression, it follows that the exponential matrix is actually the fundamental matrix for system (5), and it is defined all over I. □

Notice that, due to its construction, the exponential matrix e^{At} commutes with the matrix A. Indeed, we have

$$\begin{array}{rcl} Ae^{At} & = & A\sum_{k=0}^{\infty} \frac{A^kt^k}{k!} = \sum_{k=0}^{\infty} \frac{A^{k+1}t^k}{k!} \\ & = & (\sum_{k=0}^{\infty} \frac{A^kt^k}{k!})A = e^{At}A. \end{array} \tag{7}$$

On the other hand, and although we have now an explicit expression for fundamental matrices, we can further simplify the computations of the exponential matrix with the aid of next result.

Proposition 4. *Let $P, P^{-1} \in M_n(\mathbb{R})$ such that $J = PAP^{-1}$, then $e^J = Pe^AP^{-1}$.*

Proof. We have

$$e^J = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(PAP^{-1})^k}{k!} = P(\lim_{n\to\infty} \sum_{k=0}^{n} \frac{A^k}{k!})P^{-1} = Pe^A P^{-1}.$$

□

As a consequence of the previous result, we just need to know the exponential matrix of the Jordan canonical form of matrix A of system (5) and the matrices of the change of variables. Thus, the problem of computing a fundamental matrix reduces, for this family of systems, to knowing the exponential matrix for the different Jordan canonical forms. And, as Jordan matrices are represented block by block, it suffices to know the exponential matrix of each of the blocks. This is summarized below:

- For blocks of the form $J_1 = \lambda I_m$, it is straightforward to show that the exponential matrix is
$$e^{J_1 t} = e^{\lambda t} I_m.$$
- For blocks of the form $J_2 = \lambda I_m + N$, where $N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$,
the exponential matrix will be given by
$$e^{J_2 t} = e^{\lambda t} \sum_{k=0}^{m} \frac{t^k N^k}{k!} = e^{\lambda t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \dots & \frac{t^{m-1}}{(m-1)!} \\ 0 & 1 & t & \dots & \frac{t^{m-2}}{(m-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & t \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$
- For blocks of the form $J_3 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, the exponential matrix will be
$$e^{J_3 t} = e^{\alpha t} \begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix}.$$
- Finally, for blocks of the form $J_4 = \text{diag}(J_3, J_3, \dots, J_3) + N^2$, for N the matrix above, the exponential matrix will be
$$e^{J_4 t} = \text{diag}(e^{J_3 t}, e^{J_3 t}, \dots, e^{J_3 t}) \sum_{k=0}^{\frac{m}{2}} \frac{t^k N^{2k}}{k!}.$$

To end up, note that this exponential matrix leads us to a solution, from Proposition 2, as we are working with homogeneous systems as given by equation (5). This solution will be then given by

$$\varphi(t; t_0, \mathbf{x}_0) = e^{A(t-t_0)}\mathbf{x}_0. \tag{8}$$

Moreover, from Proposition 3, solutions for non-homogeneous systems will be given by

$$\varphi(t; t_0, \mathbf{x}_0) = e^{A(t-t_0)}\mathbf{x}_0 + \int_{t_0}^{t} e^{A(t-s)}\mathbf{b}ds. \tag{9}$$

The previous expression can be further simplified for the cases where equation $A\mathbf{x} = \mathbf{b}$ has a solution. This is presented and proven below.

Proposition 5. *If $\mathbf{z}$ is a solution of equation $A\mathbf{x} = \mathbf{b}$, then the solution for the non-homogeneous system $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}$ starting at $\mathbf{x}_0$ can be written as*

$$\varphi(t; t_0, \mathbf{x}_0) = e^{A(t-t_0)}(\mathbf{x}_0 - \mathbf{z}) + \mathbf{z}.$$

Proof. From expression (9), by means of the change of variable $\tau = t - s$ applied to the integral, we have $\int_{t_0}^{t} e^{A(t-s)}\mathbf{b}ds = -\int_{t-t_0}^{0} e^{A\tau}\mathbf{b}d\tau = \int_{0}^{t-t_0} e^{A\tau}\mathbf{b}d\tau$. Now, if $\mathbf{z}$ is such that $A\mathbf{z} = \mathbf{b}$, the integral becomes $\int_{0}^{t-t_0} e^{A\tau}A\mathbf{z}d\tau$. Now the integral is solved straightforward, leading to $\int_{0}^{t-t_0} e^{A\tau}A\mathbf{z}d\tau = e^{A(t-t_0)}\mathbf{z} - I\mathbf{z}$. Substituting this in expression (9), we get the result. □

2.1.1 The Putzer method

In this section, it is presented an alternative method to compute the exponential matrix e^{At} of a homogeneous linear system, as the given by (5). It is due to E. G. Putzer, see e.g. [9], and it is useful since it works for any matrix A, regardless of its Jordan Canonical Form, and the calculations of this form are completely by-passed. To state and prove the main result, we just need the characteristic polynomial of matrix A, which is given by

$$|\lambda I - A| = \sum_{i=0}^{n} d_i \lambda^i,$$

for $d_n = 1$.

Then we have the following theorem, that shows a formula for the exponential matrix of matrix A and states the Putzer method of computing it.

Theorem 7. *Consider the linear differential system given by (5), and let $d_i, i = 0, 1, \ldots, n-1$ be the coefficients of the characteristic polynomial of matrix A. Let $z(t)$ be the solution of the Cauchy problem*

$$\begin{cases} \sum_{i=0}^{n} d_i z^{(i)} = 0, \\ z(0) = \dot{z}(0) = \ldots = z^{(n-2)}(0) = 0, z^{(n-1)}(0) = 1, \end{cases} \tag{10}$$

where the superscript (i) denotes the i-th derivative, being $z^{(0)} = z(t)$, $d_n = 1$ and $d_i, i = 0, \ldots, n-1$ the coefficients above. Consider the vector

$$Z(t) = \begin{pmatrix} z(t) \\ \dot{z}(t) \\ \vdots \\ z^{(n-1)}(t) \end{pmatrix},$$

and the matrix

$$C = \begin{pmatrix} d_1 & d_2 & \ldots & d_{n-1} & 1 \\ d_2 & d_3 & \ldots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{n-1} & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

Then, we have

$$e^{At} = \sum_{i=0}^{n-1} \gamma^i(t) A^i, \tag{11}$$

for $\gamma^i(t)$ the components of the vector

$$\Gamma(t) = CZ(t).$$

Proof. It suffices to show that the matrix

$$\Phi(t) = \sum_{i=0}^{n-1} \gamma^i(t) A^i$$

fulfills $\frac{d\Phi}{dt} = A\Phi$ and $\Phi(0) = I_n$, as the exponential matrix e^{At} also verifies the previous conditions and we have uniqueness of solutions.

Notice that, by construction, only $\gamma^0(t)$ involves $z^{(n-1)}(t)$, thus $\gamma^i(0) = 0$ for $i \geq 1$. In consequence, $\gamma^0(0) = 1$, from where $\Phi(0) = I_n$.

We will now prove that $\frac{d\Phi}{dt} - A\Phi = 0$. To do so, we differentiate $\Phi(t)$ with respect to t and apply the Cayley-Hamilton Theorem. We then have

$$\frac{d\Phi}{dt} - A\Phi = (\dot{\gamma}^0 + d_0\gamma^{n-1}) + \sum_{i=0}^{n-1} (\dot{\gamma}^i - \gamma^{i-1} + d_i\gamma^{n-1}) A^i,$$

from where we just have to show that

$$\begin{array}{rcl} \dot{\gamma}^0(t) & = & -d_0\gamma^{n-1}(t), \\ \dot{\gamma}^i(t) & = & \gamma^{i-1}(t) - d_i\gamma^{n-1}(t), \quad i = 1, \ldots, n-1. \end{array} \tag{12}$$

From the definition of the γ^i, we have, for each $i = 0, \ldots, n-1$,

$$\gamma^i(t) = \sum_{k=1}^{n-i-1} c_{k+i} z^{(k-1)} + z^{(n-i-1)},$$

thus

$$\dot{\gamma}^i(t) = \sum_{k=1}^{n-i-1} c_{k+i} z^{(k)} + z^{(n-i)}. \tag{13}$$

As $\gamma^{n-1} = z$, by adding $d_i z$ to both sides of the previous expression we have

$$\dot{\gamma}^i(t) + d_i \gamma^{n-1} = \sum_{k=0}^{n-i-1} c_{k+i} z^{(k)} + z^{(n-i)}. \tag{14}$$

For $i = 0$, we have

$$\dot{\gamma}^0(t) + d_0 \gamma^{n-1} = \sum_{k=0}^{n-1} c_k z^{(k)} + z^{(n)}.$$

As $z(t)$ is the solution of the Cauchy problem (10), the right-hand side of the previous expression is null. Hence, we have the first identity of expression (12).

For $i \geq 1$, by replacing i by $i-1$ in expression (13) and changing the summation index from k to $k+1$, we get to

$$\gamma^{i-1}(t) = \sum_{k=0}^{n-i-1} c_{k+i} z^{(k)} + z^{(n-i)}.$$

Now, the right-hand side of the previous expression is the same as the right-hand side of expression (14), from where it follows the second identity of expression (12) for each $i = 1, \ldots, n-1$. □

Notice that in the proof the eigenvalues make no appearance, and neither do their multiplicities. Thus, the proof is indeed valid for any matrix A, regardless of its JCF. Therefore, the method is valid for every matrix A.

2.2 Qualitative Theory

2.2.1 Phase space and orbits

Qualitative theory, in contrast of classical theory, isn't focused on finding an explicit solution. In fact, tools and results from this branch on theory help us understand the asymptotic behaviour and evolution of the solutions, without needing to compute explicit expressions for them.

We start with some definitions applied to linear systems, for the sake of formalism. Let us consider a linear differential equation

$$\dot{\mathbf{x}} = f(\mathbf{x}) := A\mathbf{x} + \mathbf{b}, \tag{15}$$

where $\mathbf{x} \in D \subset \mathbb{R}^n$, $A \in M_n(\mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^n$. As in the rest of the work we will be dealing with linear systems with constant coefficients, from here onwards definitions and results will be focused on this family of systems, although most results apply to generic systems aswell.

We will call D the **phase space**. As f in equation (15) is locally Lipschitz, for all $\mathbf{x}_0 \in D$ exists a unique solution $\varphi(t; t_0, \mathbf{x}_0)$ verifying $\mathbf{x}(t_0) = \mathbf{x}_0$, and defined all over $\mathbb{R}$. This last statement comes from an extension of Theorem 4 since A and $\mathbf{b}$ are constants, thus $I = \mathbb{R}$.

Noting that $\varphi(t + \tau; t_0, \mathbf{x}_0) = \varphi(t; t_0 - \tau, \mathbf{x}_0)$ for all $\tau \in \mathbb{R}$, without loss of generality we can consider the initial time as $t_0 = 0$, and work with $\varphi(t; \mathbf{x}_0)$, which implicitly tells us that the initial time is 0.

Remark 1. *Notice that $\varphi(t; \varphi(t_1; \mathbf{x}_0)) = \varphi(t + t_1; \mathbf{x}_0)$, as both solutions coincide for $t = 0$ and we have uniqueness of solutions. So the consideration of a solution starting at $\mathbf{x}_0$ at time $t + t_1 = 0$ leads to the same expression as the composition of a solution that at time $t = 0$ starts at $\varphi(t_1; \mathbf{x}_0)$.*

Now, given an initial point $\mathbf{x}_0 \in D$, we define an **orbit through** $\mathbf{x}_0$ to be the set

$$\gamma_{\mathbf{x}_0} = \{\varphi(t; \mathbf{x}_0) | t \in \mathbb{R}\} \subset D. \tag{16}$$

This set, for a given initial point $\mathbf{x}_0$, is the main tool to study the evolution and behaviour of the system. Moreover, the Remark 1 ensures that the set $\gamma_{\mathbf{x}_0}$ forms a group with the composition.

We have a first direct result, which ensures that different orbits do not intersect, and that the initial point for an orbit is arbitrary.

Proposition 6. *Under all previous conditions, the following properties are verified:*

a) If $\mathbf{x}_1 \in \gamma_{\mathbf{x}_0}$, then $\gamma_{\mathbf{x}_1} = \gamma_{\mathbf{x}_0}$.

b) If $\gamma_{\mathbf{x}_0} \cap \gamma_{\mathbf{x}_1} \neq \emptyset$, then $\gamma_{\mathbf{x}_0} = \gamma_{\mathbf{x}_1}$.

Proof. To prove the first statement, notice that if $\mathbf{x}_1 \in \gamma_{\mathbf{x}_0}$, then there exists $t_1 \in \mathbb{R}$ such that $\mathbf{x}_1 = \varphi(t_1; \mathbf{x}_0)$. Therefore, the orbit through $\mathbf{x}_1$ is

$$\begin{aligned} \gamma_{\mathbf{x}_1} &= \{\varphi(t; \mathbf{x}_1) | t \in \mathbb{R}\} = \{\varphi(t; \varphi(t_1; \mathbf{x}_0)) | t \in \mathbb{R}\} \\ &= \{\varphi(t + t_1; \mathbf{x}_0) | t \in \mathbb{R}\} = \{\varphi(s; \mathbf{x}_0) | s \in \mathbb{R}\} = \gamma_{\mathbf{x}_0}, \end{aligned}$$

where the third equality comes from Remark 1.

For the second statement, suppose that the intersection $\gamma_{\mathbf{x}_0} \cap \gamma_{\mathbf{x}_1}$ is not empty. Thus, consider $\mathbf{x}_2 \in \gamma_{\mathbf{x}_0} \cap \gamma_{\mathbf{x}_1}$, from the first statement of this result it follows $\gamma_{\mathbf{x}_0} = \gamma_{\mathbf{x}_2} = \gamma_{\mathbf{x}_1}$, which proves the result. □

This result also assures that the set $\{\gamma_{\mathbf{x}}\}_{\mathbf{x} \in D}$ forms a partition of the phase space. We call this partition the **phase portrait**. This portrait will allow us to visually understand the system we are dealing with.

Now we characterize possible orbits, with the aid of the next technical lemma.

Lemma 1. *If F is an additive closed proper subgroup of $\mathbb{R}$, then there exists $T > 0$ such that $F = T\mathbb{Z}$.*

Proof. Consider $T = \inf\{x \in F | x > 0\}$. As F is closed, it follows $T \in F$. Let us suppose $T = 0$, therefore for all $\epsilon > 0$ exists $x \in F$ such that $x \in (0, \epsilon)$. This implies F is dense in $\mathbb{R}$ as for any interval of length ϵ it has elements, as considering $nx \in F$ it follows $nx \in (0, n\epsilon)$. Moreover, as F is closed, it must be $F = \mathbb{R}$, which is a contradiction as F is proper. So it must be $T > 0$.

Now, to show $F = T\mathbb{Z}$, the inclusion $T\mathbb{Z} \subset F$ is direct for $T \in F$. Now, suppose we don't have the other inclusion, that is, there exists $x \in F$ such that $x \notin T\mathbb{Z}$. We can suppose $x > 0$, hence there exists n such that $nT < x < (n+1)T$, which implies $0 < x - nT < T$ with $x - nT \in F$, as both $T, x \in F$ and F is an additive group. But this last statement is a contradiction, since T is the infimum of F. Consequently, the other inclusion must be verified. □

Theorem 8. *Every orbit is either a point, homeomorphic to* $\mathbb{S}^1$ *or homeomorphic to* $\mathbb{R}$*.*

Proof. Given $\mathbf{x}_0 \in \mathbb{R}^n$, as a consequence of Remark 1 we have that $\gamma_{\mathbf{x}_0}$ forms a group with the composition of solutions. Hence, the map $\varphi(\cdot; \mathbf{x}_0) : \mathbb{R} \to \gamma_{\mathbf{x}_0}$ is a continuous group epimorphism. Therefore, $\mathbb{R}/\ker\varphi \simeq \gamma_{\mathbf{x}_0}$ where $\simeq$ denotes the isomorphism.

Now, $\ker\varphi = \{t \in \mathbb{R} | \varphi(t; \mathbf{x}_0) = \mathbf{x}_0\}$, as $\mathbf{x}_0$ is the identity element of $\gamma_{\mathbf{x}_0}$. Therefore, $\ker\varphi$ is the preimage of the identity element, from where it is a closed additive subgroup of $\mathbb{R}$, as $\{\mathbf{x}_0\}$ is a subgroup of $\gamma_{\mathbf{x}_0}$. Now, if $\ker\varphi$ is not proper, either we have $\ker\varphi = 0$, from where $\gamma_{\mathbf{x}_0} \approx \mathbb{R}$; or $\ker\varphi = \mathbb{R}$ and thus $\gamma_{\mathbf{x}_0} \approx \mathbf{x}_0$. On the other hand, if $\ker\varphi$ is proper, then $\ker\varphi = T\mathbb{Z}$. And as $\mathbb{R}/T\mathbb{Z}$ is homeomorphic to $\mathbb{S}^1$ it follows $\gamma_{\mathbf{x}_0} \approx \mathbb{S}^1$. □

From the result above, it follows that we have three different kinds of orbits. We will call a **singular point**, **critical point** or **equilibrium point** to an orbit consisting of only one point. Orbits homeomorphic to $\mathbb{S}^1$ are called **periodic orbits**.

Notice that singular points are orbits formed by just one point, hence they are defined by constant solutions starting at the specific point. From this, it follows that the vector field evaluated at this point is null. Thus, a simple way to compute singular points is to solve the equation $f(\mathbf{x}) = \mathbf{0}$.

These orbits are usually the easiest to compute, and with the aid of some basic results they characterize the local behaviour of the system in a neighborhood of them.

As for periodic orbits, they are defined from periodic solutions of the differential equation. That is, given a solution $\varphi(t; \mathbf{x}_0)$ of system (15) verifying, for a fixed $T \in \mathbb{R}^+$, $\varphi(T; \mathbf{x}_0) = \mathbf{x}_0$ and $\varphi(t; \mathbf{x}_0) \neq \mathbf{x}_0$ for all $t \in (0, T)$, then the solution defines a periodic orbit. Moreover, the period of this orbit will be T.

If a periodic orbit is isolated in the set of periodic orbits, we call this orbit a **limit cycle**.

2.2.2 Singular points and invariant subspaces

Consider a linear system given by equation (15). Its singular points will be given as the solutions of

$$A\mathbf{x} = -\mathbf{b},$$

if the previous system has solutions, i.e., the system is compatible.

Suppose we have a solution, $\mathbf{z}$, of the previous system. Then, $\mathbf{z}$ is a singular point. Moreover, if A is regular, this singular point will be the unique solution, and it will be given by $\mathbf{z} = A^{-1}\mathbf{b}$.

To simplify the study, if there exists a singular point $\mathbf{z}$ of the system, we can translate it to the origin by means of the change of variables $\mathbf{x} - \mathbf{z} \to \mathbf{x}$ and thus we get to work with homogeneous systems, whose solutions are easily computable in terms of exponential matrices.

Now, we move on to study the global behaviour of the flow. In particular, we will see that the phase space can be decomposed as a direct sum of vector subspaces such that the asymptotic behaviour of any solution contained in any of these subspaces is clear. Consequently, as the set of solutions form a vector space, as we have seen in the previous section, any solution can be written as a linear combination of solutions contained on these subspaces. Thus, the asymptotic behaviour of any solution can be deduced from these other solutions.

First of all, we must clarify what it means for a solution to be contained in a subspace. To get there, we first have to include some other concepts.

We say that a set D is **invariant by the flow** of a differential equation if for all points $\mathbf{p} \in D$ the solution $\varphi(t; \mathbf{p}) \in D$ for all $t \in \mathbb{R}$. We now introduce some examples, in order to clarify this concept.

As a first example of an invariant set, suppose $\lambda \in \mathbb{R}$ is an eigenvalue of matrix A of system (5), and consider $\mathbf{v} \in \mathbb{R}^n$ its associated eigenvector. Then, the set $R = \{\mathbf{z} + r\mathbf{v} | r \in \mathbb{R}\}$ is invariant by the flow, for $\mathbf{z}$ such that $A\mathbf{z} = \mathbf{b}$, that is, $\mathbf{z}$ is an equilibrium point of the system. To check it, take $\mathbf{x}_0 \in R$, thus $\mathbf{x}_0 = \mathbf{z} + r_0\mathbf{v}$. Now, the solution starting at $\mathbf{x}_0$ is $\varphi(t; \mathbf{x}_0) = e^{At}(\mathbf{x}_0 - \mathbf{z}) + \mathbf{z}$, as it has ben proved in Proposition 5. From this expression, we have $\varphi(t; \mathbf{x}_0) = r_0 e^{At}\mathbf{v} + \mathbf{z} = r_0 e^{\lambda t}\mathbf{v} + \mathbf{z}$. In consequence, $\varphi(t; \mathbf{x}_0) \in R$ for all values of t.

Another example: suppose $\lambda = \alpha + \mathrm{i}\beta$ is an eigenvector of A with associated generalized eigenvectors $\mathbf{v}$ and $\mathbf{w}$. Then, the plane $P = \{\mathbf{z} + r_1\mathbf{v} + r_2\mathbf{w} | r_1, r_2 \in \mathbb{R}\}$ is invariant by the flow, where $\mathbf{z}$ is an equilibrium of the system. Now, by taking a point $\mathbf{x}_0 \in P$, that is, $\mathbf{x}_0 = \mathbf{z} + r_1^0\mathbf{v} + r_2^0\mathbf{w}$, it can be checked that $\varphi(t; \mathbf{x}_0) \in P$ for all $t \in \mathbb{R}$.

The next lemma states that the previous examples are just some particular cases. It shows the invariance of generalized eigenspaces.

Lemma 2. *Let E be a generalized eigenspace associated to an eigenvalue λ of matrix A of system (5). Then, E is invariant by the flow.*

Proof. Let λ be an eigenvalue of A and $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ be the generalized eigenvectors associated to λ. Then, we have $(A - \lambda I)^{m_j}\mathbf{v}_j = \mathbf{0}$ for some minimal value $m_j \in \mathbb{Z}^+$. Let $M = \max\{m_j\}$.

We know that

$$\ker(A-\lambda I) \subset \ker(A-\lambda I)^2 \subset \ldots \subset \ker(A-\lambda I)^M \subset E.$$

Thus, as $\mathbf{v}_j \in E$, $\mathbf{w}_j = (A-\lambda I)\mathbf{v}_j \in \ker(A-\lambda I)^{m_j-1} \subset E$ and $\mathbf{w}_j = A\mathbf{v}_j - \lambda \mathbf{v}_j$ it follows $A\mathbf{v}_j \in E$. From this, it follows $A\mathbf{v} \in E$ for all $\mathbf{v} \in E$. Therefore, we have $e^{At}\mathbf{v} \in E$. □

From the previous lemma and the examples, we can consider now the subspaces generated by the union of all the eigenvectors associated to eigenvalues having either positive, negative or zero real part. These subspaces are essential to study the behaviour of solutions of linear differential equations.

Let us then consider the eigenvalues $\lambda_k = \alpha_k + i\beta_k$, with $\beta_k = 0$ for the real ones, and their associated generalised eigenvectors $\mathbf{v}_k = \mathbf{u}_k + i\mathbf{w}_k$, with $\mathbf{w}_k = \mathbf{0}$ if $\beta_k = 0$. We define the **stable subspace** $E^S = Span\{\mathbf{u}_k, \mathbf{w}_k | \alpha_k < 0\}$, the **unstable subspace** $E^U = Span\{\mathbf{u}_k, \mathbf{w}_k | \alpha_k > 0\}$ and the **center subspace** $E^C = Span\{\mathbf{u}_k, \mathbf{w}_k | \alpha_k = 0\}$, where $Span\{\mathbf{v}\}$ denotes the subspace generated by $\mathbf{v}$. These subspaces are invariant by the flow, and gather all the converging, diverging and oscillating dynamics of the system, respectively. These statements come from the following theorem.

Theorem 9. *Let E^S, E^U and E^C be the stable, unstable and central subspaces, respectively, associated to the homogeneous linear differential equation $\dot{\mathbf{x}} = A\mathbf{x}$. They verify the following two statements:*

a) They are invariant by the flow of the system.

b) $\mathbb{R}^n = E^S \oplus E^U \oplus E^C$.

Proof. The first statement is a consequence of Lemma 2. The second statement is direct. □

From the result above, it follows that solutions starting in some of these spaces will remain there for all values of t. Moreover, since the phase space is decomposed in the stable, unstable and central spaces, the linear combination of solutions is a solution, and we have uniqueness of solutions, any solution can be written as a linear combination of a solution contained in the stable space, a solution contained in the unstable space and a solution contained in the central space. Therefore, in order to describe the behaviour of solutions of a linear differential system, it is enough to describe the behaviour of solutions on the invariant spaces E^S, E^C and E^U, respectively. This leads us to the following result, that states the dynamics of solutions depending on the subspace they belong to.

Proposition 7. *Let $A \in M_n(\mathbb{R})$ and consider $\varphi(t;\mathbf{x}_0)$ the unique solution to the linear system $\dot{\mathbf{x}} = A\mathbf{x}$ with initial condition $\mathbf{x}(0) = \mathbf{x}_0$. Then:*

a) if $\mathbf{x}_0 \in E^S \backslash \{\mathbf{0}\}$, then $\lim_{t\to+\infty} \varphi(t;\mathbf{x}_0) = \mathbf{0}$ and $\lim_{t\to-\infty} \|\varphi(t;\mathbf{x}_0)\| = +\infty$.

b) if $\mathbf{x}_0 \in E^U \backslash \{\mathbf{0}\}$*, then* $\lim_{t\to-\infty} \varphi(t;\mathbf{x}_0) = \mathbf{0}$ *and* $\lim_{t\to+\infty} \|\varphi(t;\mathbf{x}_0)\| = +\infty$.

c) if $\mathbf{x}_0 \in E^C \backslash \{\mathbf{0}\}$ *and* A *diagonalizes, then there exist positive constants* m *and* M *such that* $m \leq \|\varphi(t;\mathbf{x}_0)\| \leq M$ *for all* $t \in \mathbb{R}$.

d) if A *does not diagonalize, then there exists* $\mathbf{x}_0 \in E^C \backslash \{\mathbf{0}\}$ *such that* $\lim_{t\to\pm\infty} \|\varphi(t;\mathbf{x}_0)\| = \infty$.

e) if $E^S \neq \{\mathbf{0}\}$, $E^U \neq \{\mathbf{0}\}$ *and* $\mathbf{x}_0 \in E^S \oplus E^U \backslash E^S \cup E^U$, *then* $\lim_{t\to\pm\infty} \|\varphi(t;\mathbf{x}_0)\| = \infty$.

f) if $E^U \neq \{\mathbf{0}\}$, $E^C \neq \{\mathbf{0}\}$ *and* $\mathbf{x}_0 \in E^U \oplus E^C \backslash E^U \cup E^C$, *then* $\lim_{t\to\infty} \|\varphi(t;\mathbf{x}_0)\| = \infty$ *and* $\lim_{t\to-\infty} \|\varphi(t;\mathbf{x}_0)\|$ *does not exist.*

g) if $E^S \neq \{\mathbf{0}\}$, $E^C \neq \{\mathbf{0}\}$ *and* $\mathbf{x}_0 \in E^S \oplus E^C \backslash E^S \cup E^C$, *then* $\lim_{t\to-\infty} \|\varphi(t;\mathbf{x}_0)\| = \infty$ *and* $\lim_{t\to\infty} \|\varphi(t;\mathbf{x}_0)\|$ *does not exist.*

Proof. For the first statement, consider $\mathbf{x}_0 \in E^S \backslash \{\mathbf{0}\}$. We have $\mathbf{x}_0 = \sum_{i=1}^m \alpha_i \mathbf{v}_i$ for $\mathbf{v}_i$ the generalized eigenvectors with negative real part. Now, for each of these generalized eigenvectors, we have $\|e^{At}\mathbf{v}_i\| = e^{a_i t}\|\mathbf{v}_i\|$ for a_i the real part of their associated eigenvalue. Now, taking limits, we have the result for these vectors. Since linear combination of solutions is a solution, we have the result for any $\mathbf{x}_0 \in E^S \backslash \{\mathbf{0}\}$.

The second statement is analogous to the first one.

For the third one, take $\mathbf{x}_0 \in E^C \backslash \{\mathbf{0}\}$. Under the assumption that A diagonalizes, we can write $\mathbf{x}_0 = \sum_{i=1}^k \alpha_i \mathbf{v}_i$ for $\mathbf{v}_i$ the eigenvectors of null real part. Now, for these eigenvectors, we have that $\|e^{At}\mathbf{v}_i\|$ writes as a linear combination of $\sin(\beta_i t), \cos(\beta_i t)$. Hence, the solution is bounded both above and below. The statement follows since linear combination of solutions is a solution.

For statement d), if A does not diagonalize, there exists a generalized eigenvector $\mathbf{v} \in E^S$. For this generalized eigenvector, $\|e^{At}\mathbf{v}\|$ will write as a linear combination of sines and cosines, but it will also have powers of t. Hence, the result follows.

The rest of the statements follow from the previous ones.

□

For some systems, the central subspace E^C is null. In these cases, we have $\mathbb{R}^n = E^S \oplus E^U$, and whenever this happens, we call the singular point of the system to be **hyperbolic**. These points are of special interest, since whenever there is no central subspace, the dynamics are completely determined for any solution. Moreover, some results require the singular point to be hyperbolic. It is straightforward to check that this definition coincides with the one usually presented for hyperbolic singular points: all the eigenvalues of the jacobian matrix of the system evaluated at the singular point have non-zero real part.

Moreover, from the result above, notice that, although the dynamics for both the stable and unstable subspace are characterized, dynamics for solutions on

the central subspace depend on the spectrum of matrix A. More explicitly, if A diagonalizes, solutions remain bounded; otherwise, solutions tend to ∞, spiraling if the eigenvalues are complex. Finally, in order to preserve clarity, a couple examples are presented below, for different system configurations.

Example 1. *Consider the* $3-$*dimensional system* $\dot{\mathbf{x}} = A\mathbf{x}$*, where*

$$A = \begin{pmatrix} 4 & 0 & 0 \\ -1 & 3 & 1 \\ 0 & 0 & -2 \end{pmatrix}.$$

The eigenvalues for this matrix are $\lambda_1 = 4, \lambda_2 = 3, \lambda_3 = -2$*, and the respective associated eigenvectors are* $\mathbf{v}_1 = (1, -1, 0), \mathbf{v}_2 = (0, 1, 0), \mathbf{v}_3 = (0, -1, 5)$*. Thus, the stable subspace is* $E^S = Span\{(0, -1, 5)\}$ *and the unstable subspace is* $E^U = Span\{(1, -1, 0), (0, 1, 0)\}$*. Moreover, there is no central subspace, so the singular point is hyperbolic.*

Now, if we pick a point $P \in \mathbb{R}^3$*, the position vector of this point can be decomposed as* $P = a(0, -1, 5) + b_1(1, -1, 0) + b_2(0, 1, 0)$*. Thus, the solution can be decomposed as*

$$\varphi(t; P) = a\varphi(t; (0, -1, 5)) + b_1\varphi(t; (1, -1, 0)) + b_2\varphi(t; (0, 1, 0)).$$

This is straightforward, since sum of solutions is a solution, and as for time $t = 0$ *the left-hand side and the right-hand side coincide, uniqueness of solutions yields that both expressions refer to the same solution.*

Thus, every solution that has initial condition not contained in any of the subspaces has a stable component approaching the origin and two unstable ones moving away from it, as time increases. This can also be stated from statement $e)$ *in Proposition 7.*

In Figure 1, there is a representation of the stable manifold in blue, the unstable manifold in red, and an orbit in green outside of these manifolds. Moreover, the projection of the orbit in each of these manifolds is also graphed.

Example 2. *Consider the* $4-$*dimensional system*

$$\begin{cases} \dot{x} &= ay, \\ \dot{y} &= -ax, \\ \dot{z} &= bw, \\ \dot{w} &= -bz, \end{cases}$$

which has associated matrix

$$A = \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix},$$

for $a, b > 0$.

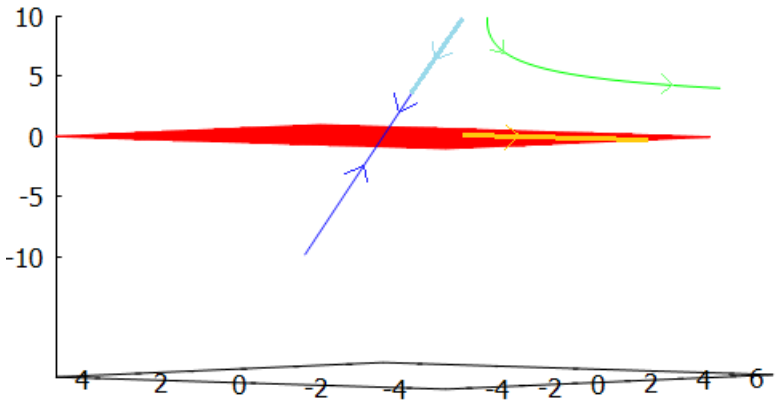


Figure 1: Unstable and stable manifolds, alongside with an orbit. In light-blue, the projection of the orbit onto the stable manifold, and in orange, the projection of the orbit onto the unstable manifold.

Its eigenvalues, which are easy to calculate, are $\pm\sqrt{a}i, \pm\sqrt{b}i$ *being each of them of multiplicity one. As all of them have zero real part, this system has no stable or unstable subspaces, having only a central one. Moreover, the central subspace has two invariant subspaces: the one corresponding to a plane* XY *generated by the eigenvectors associated to the eigenvalues* $\pm\sqrt{a}i$ *and the other generated by an analogous plane* ZW*. Orbits defined by solutions contained on each of these invariant planes rotate without a homotecy, thus they correspond to periodic orbits. The whole orbits in* $\mathbb{R}^4$ *are the product of two circumferences, hence all of them are contained in an invariant torus, filling it. However, these full solutions in* $\mathbb{R}^4$ *will define periodic orbits only in determinate conditions, that will be explained below. Therefore, only in this case the torus will be filled with periodic orbits.*

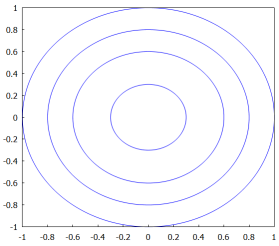
Moreover, according to statement c) in Proposition 7, as the matrix diagonalizes (over $\mathbb{C}$*), solutions remain bounded for all time* $t \in \mathbb{R}$*.*

For this family of systems, the orbits projected onto either the XY *or the* ZW *invariant planes will turn out to be periodic orbits of period* $\frac{2\pi}{a}$ *and* $\frac{2\pi}{b}$*, respectively, even if the whole solution does not define a* $4D$ *periodic orbit.*

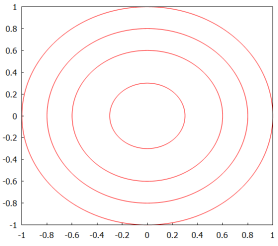
If $\frac{a}{b}$ *is rational, at some point both projected orbits will close, thus the whole solution will define a* $4D$ *periodic orbit. Moreover, the period for this orbit will be* $\frac{2\pi}{\gcd(a,b)}$*.*

On the other hand, if $\frac{a}{b}$ *is irrational, there is no number that makes both projected orbits close at the same time, thus the* $4D$ *orbit will not be periodic, even though projections onto the* XY *and* ZW *planes are periodic.*

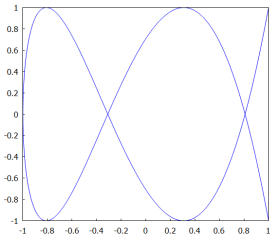
For $a = 2, b = 5$*, some solutions are represented in Figure 2. For* $a = \sqrt{2}, b = 5$*, projections on the* XZ *and* YW *plane are represented in Figure 3.*



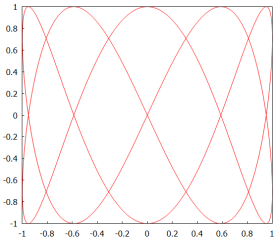
(a) Orbits on the XY plane.



(b) Orbits on the ZW plane.

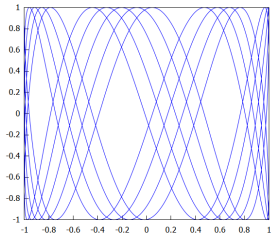


(c) Orbit on the XZ plane. Initial condition $(1, 0, 1, 0)$.

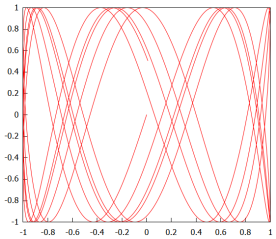


(d) Orbits on the YW plane. Initial condition $(1, 0, 1, 0)$.

Figure 2: Orbits for rational $\frac{a}{b}$ of example 2.



(a) Orbit on the XZ plane for irrational $\frac{a}{b}$.



(b) Orbit on the YW plane for irrational $\frac{a}{b}$.

Figure 3: Orbit for irrational $\frac{a}{b}$ defined by solution with initial condition $(1, 0, 1, 0)$, of example 2.

Example 3. *Consider the* $4-$*dimensional system given by*

$$\begin{cases} \dot{x} &= y + z, \\ \dot{y} &= -x + w, \\ \dot{z} &= w, \\ \dot{w} &= -z, \end{cases}$$

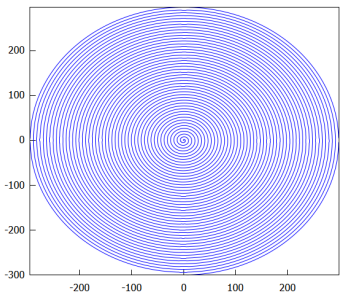
which has associated matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

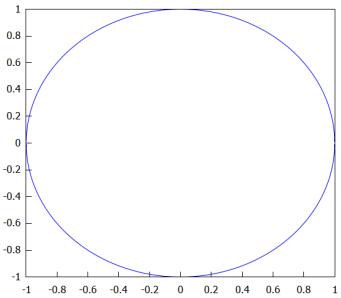
This system is very similar to the one in the previous example for $a = b = 1$*, but in this case matrix* A *does not diagonalize. Thus, from statement d) in Proposition 7, it follows that there exist unbounded solutions for this system.*

This translates as, although projections onto the $XY-$*plane or onto the* $ZW-$*plane will look like periodic orbits, the global dynamics won't be periodic, i.e. solutions will have oscillatory behavior in the* $4-$*dimensional space, but they will grow to infinity.*

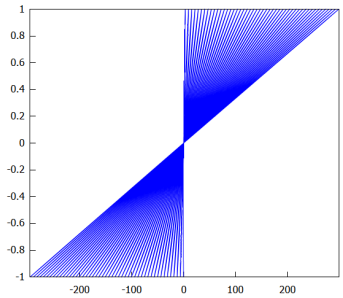
Projections of an orbit are presented in Figure 4, where it can be seen that the subspace generated by the ZW *plane is invariant, and the projected orbit is periodic there, even though with respect to toe coordinates* X *and* Y *it grows unbounded, as it can be seen in the other projections, where the* X *and* Y *bounds are near* 300*.*



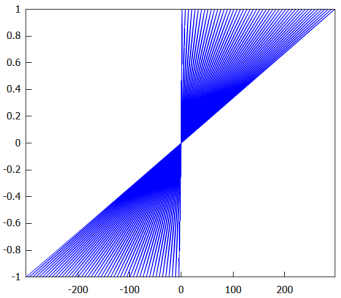
(a) Orbit on the XY plane.



(b) Orbit on the ZW plane.



(c) Orbit on the XZ plane.



(d) Orbit on the YW plane.

Figure 4: Projection of an orbit defined by a solution with initial condition $(0, 0, -1, 0)$, of example 3.

3 Transition maps: general study in $\mathbb{R}^n$

3.1 Contextualization

On the previous section there has been presented an introduction to both classical and qualitative theory applied to linear systems. From that introduction, it is easy to notice that these systems are rather simple as they form a family for which a solution can be computed, thus in most cases the qualitative work gets simplified. In this section, by taking advantage of the simplicity of these systems, we present and study a map, similar to the Poincaré map, defined over a transversal hyperplane to the flow of the linear system that sends points on this hyperplane to other points on it, following the flow of the system.

Although in this section we will be focused on linear systems to develop the theory and results of this map, it shines when dealing with a family of systems called **piecewise linear systems (PWLS for short)**, which are formed by two (or more) linear systems separated by a hyperplane, by taking this hyperplane as the cross section.

In this work we will be dealing with linear systems of constant coefficients, such as

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}, \tag{17}$$

where $A \in M_n(\mathbb{R})$ and $\mathbf{x}, \mathbf{b} \in \mathbb{R}^n$ and such that matrix A is regular, thus they have exactly one equilibrium, given by $\mathbf{z} = -A^{-1}\mathbf{b}$. Remind that, as it was seen in Section 2.2.2, all linear systems with constant coefficients having at least one equilibrium can be transformed in homogeneous systems by translating the equilibria to the origin. Thus, we will be focused on homogeneous constant coefficient systems, such as

$$\dot{\mathbf{x}} = A\mathbf{x}. \tag{18}$$

In order to be more precise, we define here the mentioned hyperplane, that is also called a cross-section for the linear system, and it is given by

$$\mathcal{P} = \{\mathbf{q} \in \mathbb{R}^n | \mathbf{k}^T\mathbf{q} = 1\}$$

for a fixed vector $\mathbf{k} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$.

Notice that, from the definition above, the hyperplane does not contain the origin, as $\mathbf{k}^T\mathbf{0} = 0$. Moreover, this hyperplane splits the whole $\mathbb{R}^n$ in two half-spaces, one of which will contain the origin. We will denote these half-paces as S_0 for the one containing the origin, and S for the other one.

As it was stated in the introduction of this section, the map we will define and study will send a point $\mathbf{q} \in \mathcal{P}$ to another point $\widehat{\mathbf{q}} \in \mathcal{P}$ through the orbit defined by the solution starting at $\mathbf{q}$. For any point $\mathbf{q} \in \mathcal{P}$, the orbit defined by the solution starting at $\mathbf{q}$ can either evolve to $\mathcal{P}$ through S_0 or through S. This latter statement allows us to define two different transition maps: Π^A, that will act on points $\mathbf{q} \in \mathcal{P}$ such that their orbit evolves through S_0, and $\widehat{\Pi}^A$, that will act on points $\mathbf{q} \in \mathcal{P}$ such that their orbit evolves through S.

However, before moving on to work with these maps, some concepts and results are presented for the sake of clarity.

3.2 Contact points, observability

First of all, let us introduce a concept and some results related to it that will be of aid to continue the study. Specifically, this concept and some of the results presented will lead to a base that will reduce the dimensionality of the problem.

We define a point $\mathbf{q} \in \mathcal{P}$ to be a **contact point of order** $m \geq 1$ of the flow defined by the system (18) with the hyperplane $\mathcal{P}$ if it satisfies

i) $\mathbf{k}^T A^l \mathbf{q} = 0, \forall l = 1, 2, \ldots, m-1$,

ii) $\mathbf{k}^T A^m \mathbf{q} \neq 0$.

If the first condition is fulfilled for every $m \geq 1$, we define the order of $\mathbf{q}$ to be infinity.

The following lemma relates the existence of a unique contact point $\mathbf{p}$ of order n to the regularity of matrix A.

Lemma 3. *If there exists a unique contact point* $\mathbf{p}$ *of order* n *of the flow of system* (18) *with the hyperplane* $\mathcal{P}$*, then* $\mathbf{k}^T A^n \mathbf{p} = (-1)^{n+1} \det(A)$ *and, therefore,* $\det(A) \neq 0$.

Proof. Let $\mathbf{p} \in \mathcal{P}$ be the unique contact point of order n of the flow defined by system $\dot{\mathbf{x}} = A\mathbf{x}$. We have $\mathbf{k}^T \mathbf{p} = 1, \mathbf{k}^T A^l \mathbf{p} = 0, \forall l = 1, 2, \ldots, n-1$ and $\mathbf{k}^T A^n \mathbf{p} \neq 0$.

From the Cayley-Hamilton Theorem, we have $A^n = -\sum_{i=0}^{n-1} d_i A^i$ for d_i the coefficients of the characteristic polynomial of matrix A, thus $\mathbf{k}^T A^n \mathbf{p} = -\sum_{i=0}^{n-1} d_i \mathbf{k}^T A^i \mathbf{p}$. Now, applying the equalities $i), ii)$ from the definition of a contact point, we have $\mathbf{k}^T A^n \mathbf{p} = (-1)^{n+1} d_0 = (-1)^{n+1} \det(A)$, and as $\mathbf{k}^T A^n \mathbf{p} \neq 0$, we have the result. □

The previous lemma states that, under the assumption of a unique contact point of order n of the flow of system (18), then the matrix A of the system will be regular. Thus, if the matrix defining the system is singular, the differential equation will not have a unique contact point of order n.

The lemma has a reciprocal under an additional condition, that we present here. We say that system (18) is **observable** relative to the hyperplane $\mathcal{P}$ if the observability matrix

$$\mathcal{O} = \left(\mathbf{k} \mid A^T \mathbf{k} \mid (A^T)^2 \mathbf{k} \mid \ldots \mid (A^T)^{n-1} \mathbf{k} \right)^T, \tag{19}$$

has rank n.

Now we present the (almost) reciprocal of Lemma 3.

Lemma 4. *If the homogeneous linear system* (18) *is observable relative to the hyperplane* $\mathcal{P}$ *and the matrix* A *is regular, then there exists a unique contact point* $\mathbf{p}$ *of order* n *of the flow of the system with the hyperplane* $\mathcal{P}$.

Proof. As the system is observable, the linear system of equations

$$\mathcal{O}\mathbf{x} = \mathbf{e}_1, \tag{20}$$

has a unique solution, where $\mathbf{e}_1$ denotes the first vector of the canonical basis. This solution, namely $\mathbf{p}$, fulfills

$$\begin{aligned} &\mathbf{k}^T\mathbf{p} = 1, \\ &\mathbf{k}^T A^i \mathbf{p} = 0, \quad i = 1, \dots, n-1, \end{aligned}$$

so this point is a contact point of order greater or equal to n.

But, as from Cayley-Hamilton theorem we have $A^n = -\sum_{i=0}^{n-1} d_i A^i$, it follows $\mathbf{k}^T A^n \mathbf{p} = (-1)^{n+1} \det(A)$ as in Lemma 3. Now, as A is regular, the last equality is not null, thus $\mathbf{p}$ is a contact point of order n. Moreover, as it is the unique solution to system (20), uniqueness follows. □

For the rest of the work, it will be assumed that system (18) is observable relative to the hyperplane $\mathcal{P}$, with matrix A regular, as it was stated in the introduction of this section. From the lemma above, under the mentioned conditions it follows the existence and uniqueness of a contact point $\mathbf{p}$ of order n.

3.3 The Krylov base

Under the assumption of existence and uniqueness of a contact point $\mathbf{p} \in \mathcal{P}$ of order n, there appears a somewhat special vector set, namely, $\mathcal{B} = \{A^k \mathbf{p}\}_{k=0}^{n-1}$. In the following result it is proved that $\mathcal{B}$ is actually a base for $\mathbb{R}^n$, which is called the **Krylov base**.

Lemma 5. *If there exists a unique contact point $\mathbf{p}$ of order n of the system (18), then the set $\mathcal{B} = \{A^k \mathbf{p}\}_{k=0}^{n-1}$ forms a basis for $\mathbb{R}^n$.*

Proof. As the set $\mathcal{B}$ consists of exactly n vectors, it suffices to show that either they are linearly independent or that they span the whole $\mathbb{R}^n$ space. We will prove the first statement. Let, then, $\sum_{i=0}^{n-1} \alpha_i A^i \mathbf{p} = \mathbf{0}$ be a null linear combination of these vectors. By multiplying both sides by $\mathbf{k}^T A$, as $\mathbf{p}$ is a contact point of order n, the equality reduces to $\alpha_{n-1} \mathbf{k}^T A^n \mathbf{p} = 0$. From this last expression, we deduce $\alpha_{n-1} = 0$. By plugging the value of $\alpha_{n-1} = 0$ in the sum and multiplying it now by $\mathbf{k}^T A^2$, we are left with $\alpha_{n-2} \mathbf{k}^T A^n \mathbf{p} = 0$, thus $\alpha_{n-2} = 0$. If we now iterate the reasoning for $\mathbf{k}^T A^j$ increasing j until $n-1$, the linear combination gets reduced to $\alpha_0 \mathbf{p} = \mathbf{0}$. And, since $\mathbf{p} \neq \mathbf{0}$ since it is a point of $\mathcal{P}$ and the origin is not contained in the hyperplane, it follows $\alpha_0 = 0$. Therefore, we have $\alpha_i = 0, \forall i = 0, 1, \dots, n-1$, thus the vectors are linearly independent and they form a basis. □

In terms of the Krylov base, $\mathcal{B}$, the cross-section is written as $\mathcal{P} = \{\mathbf{p} + \sum_{i=1}^{n-1} a_i A^i \mathbf{p} | a_i \in \mathbb{R}\}$. Thus, if we identify every point $\mathbf{q}$ in $\mathcal{P}$ with its coordinates $(a_1, a_2, \dots, a_{n-1})$, we get that $\mathcal{P}$ is isomorphic to $\mathbb{R}^{n-1}$. In the following, we will use the notation $\mathbf{q}_\mathcal{B} = (a_1, a_2, \dots, a_{n-1})$ in $\mathbb{R}^{n-1}$ to express the coordinates of a point $\mathbf{q} \in \mathcal{P}$ in the Krylov base. It follows directly that for the contact point $\mathbf{p}$ of order n, we have $\mathbf{p}_\mathcal{B} = (0, 0, \dots, 0)$.

Moreover, the representation of matrix A of system (18) on the Krylov base turns out to be rather simple. Precisely, the matrix in this base is written as

$$B = \begin{pmatrix} 0 & \dots & \dots & 0 & -d_0 \\ 1 & \ddots & & \vdots & -d_1 \\ 0 & \ddots & \ddots & \vdots & -d_2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & 1 & -d_{n-1} \end{pmatrix}, \tag{21}$$

where $d_i, i = 0, \dots, n-1$ are the coefficients of the characteristic polynomial of A. This matrix B is usually called the **controlability matrix** of system (18), and it is widely used in the study of Piecewise Linear Systems by authors like Carmona et al., see e. g. [4].

3.4 Analysis of the flow through $\mathcal{P}$

In this section, the objective is to determine and classify the points of $\mathcal{P}$ in which the flow evolves through the half-space S_0 or to the half-space S, respectively.

Let us now consider the following subsets of $\mathcal{P}$, which can be defined by an abuse of notation in terms of the Krylov base $\mathcal{B}$:

$$\mathcal{L}_m = \{(a_1, a_2, \dots, a_{n-m}, 0, \dots, 0) \in \mathbb{R}^{n-1} | a_{n-m} \neq 0\}, \text{ for } m = 1, \dots, n-1,$$
$$\mathcal{L}_n = \{(0, 0, \dots, 0) \in \mathbb{R}^{n-1}\}.$$

These sets locate and classify all the contact points of the flow of system (18) in the plane $\mathcal{P}$ with respect to their order of contact. This statement is proven in the following result.

Lemma 6. *a) The sets $\{\mathcal{L}_m\}_{m=1}^n$ form a partition of $\mathcal{P}$.*

b) For each $m = 1, 2, \dots, n$, the set $\mathcal{L}_m$ is formed by all the contact points of order m of the flow of system (18) with $\mathcal{P}$.

Proof. The first statement is direct.

For the second statement, the case $m = n$ is also direct. Now, consider $m \in \{1, 2, \dots, n-1\}$ and let $\mathbf{q}$ be a point of $\mathcal{L}_m$. As $\mathbf{q} \in \mathcal{L}_m$, we can write $\mathbf{q} = \mathbf{p} + \sum_{i=1}^{n-m} a_i A^i \mathbf{p} = \sum_{i=0}^{n-m} a_i A^i \mathbf{p}$ for $a_0 = 1, a_{n-m} \neq 0$. Then, if we multiply by $\mathbf{k}^T A^r$, it follows

$$\mathbf{k}^T A^r \mathbf{q} = \sum_{i=r}^{n-m+r} a_{i-r} \mathbf{k}^T A^i \mathbf{p}.$$

If $r \in \{1, 2, \dots, m-1\}$, then the index on the right-hand side runs from $i = r$ to $i = n - m + r \leq n - 1$. Thus, each of the summands is null as $\mathbf{p}$ is a contact point of order n, therefore $\mathbf{k}^T A^r \mathbf{q} = 0$. Otherwise, if $r = m$, the index runs from $i = m$ to $i = n$, thus it follows $\mathbf{k}^T A^m \mathbf{q} = a_{n-m} \mathbf{k}^T A^n \mathbf{p} \neq 0$. It follows that $\mathbf{q}$ is a contact point of order m.

Reciprocally, let $\mathbf{q} \in \mathcal{P}$ be a contact point of order m. From statement a), the point $\mathbf{q}$ is in exactly one of the $\mathcal{L}_k$ sets, for $k \in \{1, 2, \ldots, n\}$. Now, as we have seen that the points in $\mathcal{L}_k$ are the contact points of order k, we can deduce that $\mathbf{q} \in \mathcal{L}_m$. □

Remark 2. *From the previous result, we have that these $\mathcal{L}_m$ sets form a partition of $\mathcal{P}$. Moreover, for $m < n$, each of these sets has two connex components, having as a boundary the union $\bigcup_{i=m+1}^{n} \mathcal{L}_i$.*

Now, in order to make the previous remark a bit more clear, we present some low-dimensional examples below.

The most basic case is $n = 2$, where the hyperplane $\mathcal{P} = \mathcal{L}_1 \cup \mathcal{L}_2$ is just a straight line formed by two segments of contact points of order 1 that share a common boundary given by $\mathcal{L}_2$, that consists of $\mathbf{p}$ the contact point of order 2. In Figure 5 there is a visualization.

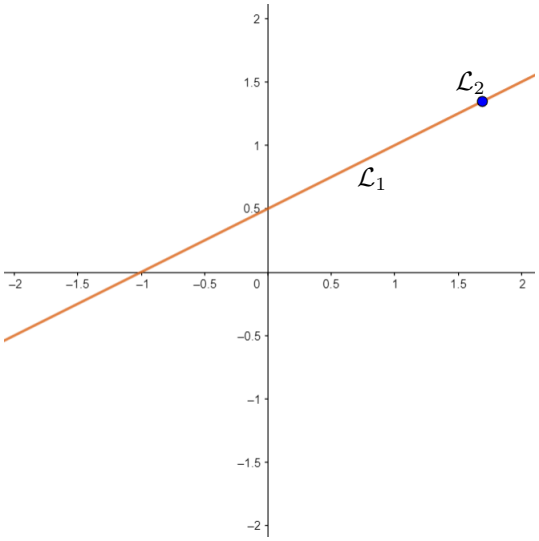


Figure 5: Set $\mathcal{L}_1$ containing the contact points of order 1, in orange, separated by the set $\mathcal{L}_2$ containing the unique contact point of order 2, in blue.

As a more illustrative example, consider the case $n = 3$. Here, we have $\mathcal{P} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$. The set $\mathcal{L}_1$ is formed by two open semi-planes that gather all the contact points of order 1, and they have as a boundary the straight line $\mathcal{L}_2 \cup \mathcal{L}_3$. The set $\mathcal{L}_2$ is formed by two segments that gather the contact points of order 2 and they have as a boundary the set $\mathcal{L}_3$, which consists in the contact point $\mathbf{p}$ of order 3. This can be visualized in Figure 6.

Once we have $\mathcal{P}$ partitioned in subsets such that each subset gathers all the points with the same contact order, now we study the behavior of the flow in terms of this order of contact. This will lead us to define a flow orientation and to classify the orbits depending on their behavior with respect to $\mathcal{P}$. Moreover,

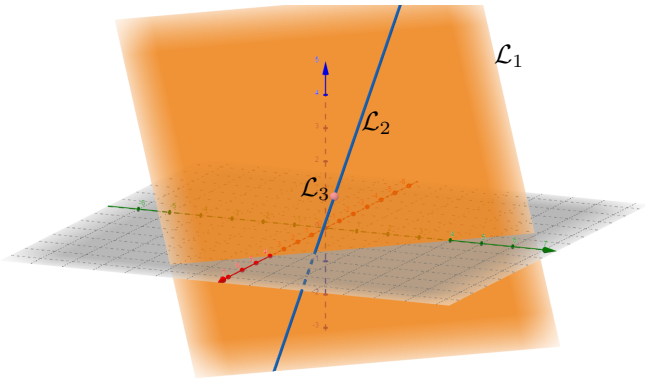


Figure 6: Set $\mathcal{L}_1$ containing the contact points of order 1, in orange, separated by the set $\mathcal{L}_2$ containing the contact points of order 2, in blue, separated as well by the set $\mathcal{L}_3$ containing the unique contact point of order 3, in pink.

it will be seen that orbits defined by solutions starting on an even order contact point will remain locally in either S_0 or S, while orbits defined by solutions starting on an odd order contact point will cross from S_0 to S or vice versa.

We start by introducing a few definitions. Let $\mathbf{q} \in \mathcal{P}$, and consider $\varphi(s;\mathbf{q})$ the solution of system (18) with initial condition $\mathbf{x}(0) = \mathbf{q}$. Let $\gamma_{\mathbf{q}}$ be the orbit through $\mathbf{q}$. If there exists $\varepsilon > 0$ such that $\mathbf{k}^T(\varphi(s;\mathbf{q}) - \mathbf{q})\mathbf{k}^T(\varphi(-s;\mathbf{q}) - \mathbf{q}) < 0$ for $s \in (0,\varepsilon)$, we say that $\gamma_{\mathbf{q}}$ is a **crossing orbit**. These orbits cross from $\mathcal{S}$ to $\mathcal{S}_0$ or vice versa, hence their name, and this allows us to define a flow orientation at $\mathbf{q}$. We say $\gamma_{\mathbf{q}}$ is **inward oriented** when it goes through $\mathbf{q}$ from $\mathcal{S}$ to $\mathcal{S}_0$, which means that $\mathbf{k}^T(\varphi(s;\mathbf{q}) - \mathbf{q}) < 0$ and $\mathbf{k}^T(\varphi(-s;\mathbf{q}) - \mathbf{q}) > 0$. If $\mathbf{k}^T(\varphi(s;\mathbf{q}) - \mathbf{q}) > 0$ and $\mathbf{k}^T(\varphi(-s;\mathbf{q}) - \mathbf{q}) < 0$, then the orbit goes through $\mathbf{q}$ from $\mathcal{S}_0$ to $\mathcal{S}$, and we say it is **outward oriented**. Finally, if there exists $\varepsilon > 0$ such that $\mathbf{k}^T(\varphi(s;\mathbf{q}) - \mathbf{q})\mathbf{k}^T(\varphi(-s;\mathbf{q}) - \mathbf{q}) > 0$ for $s \in (0,\varepsilon)$, then $\gamma_{\mathbf{q}}$ is a **non-crossing orbit**.

Notice that non-crossing orbits are contained in either $\mathcal{S}_0$ or $\mathcal{S}$. For a non-crossing orbit, we say it is **inward contained** at $\mathbf{q}$ when $\mathbf{k}^T(\varphi(s;\mathbf{q}) - \mathbf{q}) < 0$ and $\mathbf{k}^T(\varphi(-s;\mathbf{q}) - \mathbf{q}) < 0$. If $\mathbf{k}^T(\varphi(s;\mathbf{q}) - \mathbf{q}) > 0$ and $\mathbf{k}^T(\varphi(-s;\mathbf{q}) - \mathbf{q}) > 0$, then the non-crossing orbit is **outward contained** at $\mathbf{q}$.

The result presented below helps us classify the orbits $\gamma_{\mathbf{q}}$, and shows us that they depend purely on the parity of the order of contact of $\mathbf{q}$.

Lemma 7. *Let $\mathbf{q}$ be a contact point of order $1 \leq m \leq n$ of the system (18) with the hyperplane $\mathcal{P}$ and $\gamma_{\mathbf{q}}$ be the orbit through $\mathbf{q}$. Let $\mathbf{q}_{\mathcal{B}} = (a_1, \ldots, a_{n-1})$ be the coordinates of $\mathbf{q}$ in the Krylov base, and set $a_0 = 1$. Then:*

a) If m is odd, then $\gamma_{\mathbf{q}}$ is a crossing orbit to $\mathcal{P}$ at $\mathbf{q}$. Moreover:

a.1) if $(-1)^{n+1}a_{n-m}\det(A) < 0$, then $\gamma_{\mathbf{q}}$ is inward oriented,

a.2) if $(-1)^{n+1}a_{n-m}\det(A) > 0$, then $\gamma_{\mathbf{q}}$ is outward oriented.

b) If m is even, then $\gamma_{\mathbf{q}}$ is a non-crossing orbit at $\mathbf{q}$. Moreover:

b.1) if $(-1)^{n+1}a_{n-m}\det(A) < 0$, then $\gamma_{\mathbf{q}}$ is inward contained,

b.2) if $(-1)^{n+1}a_{n-m}\det(A) > 0$, then $\gamma_{\mathbf{q}}$ is outward contained.

Proof. Let $\mathbf{q} \in \mathcal{P}$ be a contact point of order m. Let $\varphi(s;\mathbf{q})$ be the solution to system (18) with initial condition $\mathbf{x}(0) = \mathbf{q}$. Applying the Taylor expansion to the solution, we have

$$\varphi(s;\mathbf{q}) = \mathbf{q} + \sum_{i=1}^{m-1} \frac{s^i}{i!} A^i\mathbf{q} + \frac{s^m}{m!}A^m\mathbf{q} + O(s^{m+1}).$$

Now, as $\mathbf{q}$ is a contact point of order m, by subtracting $\mathbf{q}$ and multiplying by $\mathbf{k}^T$ to the left, we have

$$\mathbf{k}^T(\varphi(s;\mathbf{q}) - \mathbf{q}) = \frac{s^m}{m!}\mathbf{k}^T A^m\mathbf{q} + O(s^{m+1}). \tag{22}$$

Moreover, as $\mathbf{q}_{\mathcal{B}}$ verifies $a_{n-m} \neq 0$ and $a_i = 0$ for $n-m < i \leq n-1$, then $\mathbf{q} = \sum_{i=0}^{n-m} a_i A^i\mathbf{p}$ and $\mathbf{k}^T A^m\mathbf{q} = \sum_{i=0}^{n-m} a_i\mathbf{k}^T A^{i+m}\mathbf{p}$, for $\mathbf{p} \in \mathcal{P}$ the unique contact point of order n. Now, from Lemma 3 and by the definition of contact point, the second equality reduces to

$$\mathbf{k}^T A^m\mathbf{q} = (-1)^{n+1}a_{n-m}\det(A). \tag{23}$$

Now the results from both statements are straightforward:

a) If m is odd, from equation (22) we deduce that expressions $\mathbf{k}^T(\varphi(s;\mathbf{q}) - \mathbf{q})$ and $\mathbf{k}^T(\varphi(-s;\mathbf{q}) - \mathbf{q})$ will have opposite sign. This implies that $\gamma_{\mathbf{q}}$ crosses $\mathcal{P}$ through $\mathbf{q}$. Statements $a.1), a.2)$ follow from equation (23).

b) If m is even, the reasoning is similar to the one performed in a).

□

From Lemma 7, if $m < n$ is odd, the set $\mathcal{L}_{m+1}$ splits $\mathcal{L}_m$ into two parts, one containing the points at which the orbits are inwards transversal to $\mathcal{P}$ and the other consisting of the points at which the orbits are outwards transversal to $\mathcal{P}$. These sets are called, respectively,

$$\mathcal{L}_m^I = \{(a_1,\ldots,a_{n-m},0,\ldots,0) | (-1)^{n+1}a_{n-m}\det(A) < 0\},$$
$$\mathcal{L}_m^O = \{(a_1,\ldots,a_{n-m},0,\ldots,0) | (-1)^{n+1}a_{n-m}\det(A) > 0\}.$$

Similarly, if $m < n$ is even, the set $\mathcal{L}_{m+1}$ splits $\mathcal{L}_m$ in two parts, containing each of these parts the points at which the orbits are inwards tangent to $\mathcal{P}$ and the points at which the orbits are outwards tangent to $\mathcal{P}$, respectively. Explicitly, these sets are

$$\mathcal{L}_m^I = \{(a_1,\ldots,a_{n-m},0,\ldots,0) | (-1)^{n+1}a_{n-m}\det(A) < 0\},$$
$$\mathcal{L}_m^O = \{(a_1,\ldots,a_{n-m},0,\ldots,0) | (-1)^{n+1}a_{n-m}\det(A) > 0\}.$$

Notice that this pair of sets has the same definition as the pair defined above, for odd m, even though they are different sets, since orbits defined by solutions starting at points on these sets have different behavior. However, since their behavior depends purely on the parity of m the order of contact, both these pairs of sets are well-defined and differentiated from each other.

On the other hand, for each $m < n$ each of these subsets is precisely one of the connex components of $\mathcal{L}_m$ that were stated in Remark 2.

The case $m = n$ is a bit different, because as $\mathcal{L}_n = \{\mathbf{p}\}$ either $\mathcal{L}_n^I$ is empty, or the other one will be empty. This will depend on the parity of n, the dimension of the space where the system is located, and on the sign of the determinant of matrix A.

Finally, we define $\mathcal{P}^I = \bigcup_{k=1}^n \mathcal{L}_k^I$, which is formed by all the points of $\mathcal{P}$ at which orbits evolve to $\mathcal{S}_0$, and $\mathcal{P}^O = \bigcup_{k=1}^n \mathcal{L}_k^O$, which is formed by all the points of $\mathcal{P}$ at which orbits evolve to $\mathcal{S}$.

3.5 Transition maps

Under the assumption of existence and uniqueness of a contact point $\mathbf{p}$ of order n of system (18), we can ensure the existence of a map defined over $\mathcal{P}$ that sends points on the hyperplane to points on the same hyperplane through the flow of the system. In particular, the existence of this map defined over the set $\mathcal{L}_m$ with odd m and in a neighborhood of $\mathcal{L}_{m-1}$, follows from the continuity of the flow with respect to initial conditions. We will call this map the **transition map**, and we distinguish two of them depending on the domain being a subset of $\mathcal{P}^I$ or $\mathcal{P}^O$. Precisely, we have:

$$\Pi^A : D_1 \subset \mathcal{P}^I \to \mathcal{P}^O,$$
$$\widehat{\Pi}^A : D_2 \subset \mathcal{P}^O \to \mathcal{P}^I.$$

Now, if we take a point $\mathbf{q} \in D_1$, as it belongs to $\mathcal{P}$ as well, it can be written in terms of the Krylov base, that is

$$\mathbf{q} = \mathbf{p} + \sum_{i=1}^{n-1} a_i A^i \mathbf{p}, \tag{24}$$

for $\mathbf{q}_{\mathcal{B}} = (a_1, \ldots, a_{n-1})$. The same holds for $\Pi^A(\mathbf{q}) \in \mathcal{P}^O$, leading to

$$\Pi^A(\mathbf{q}) = \mathbf{p} + \sum_{i=1}^{n-1} b_i A^i \mathbf{p}, \tag{25}$$

for $\Pi^A(\mathbf{q})_{\mathcal{B}} = (b_1, \ldots, b_{n-1})$. From expressions (24) and (25), it follows that the coordinates of $\Pi^A(\mathbf{q})$ in the Krylov base can be expressed in terms of these of $\mathbf{q}$, say $b_i = \pi_i^A(a_1, \ldots, a_{n-1})$. Thus the transition map Π^A can be determined from the set of $n-1$ functions $\{\pi_i^A\}_{i=1}^{n-1}$. A similar analysis leads us to conclude that

the other transition map, $\widehat{\Pi}^A$, can also be determined from the $n-1$ functions $\{\widehat{\pi}_i^A\}_{i=1}^{n-1}$. In following sections we will deal with the determination of the maps components. But, before moving on to that, we introduce some other expressions for these maps, that will be useful to simplify the work.

3.5.1 Expressions for the transition maps

Remind that, as the system (18) is linear, its flow is given by $\Phi(\mathbf{q},t) = e^{tA}\mathbf{q}$ for $\mathbf{q} \in \mathbb{R}^n$ a given initial condition. If we take $\mathbf{q} \in D_1$, we can deduce the existence of a function $\tau(\mathbf{q}) > 0$ such that

$$\begin{array}{l} \Pi^A(\mathbf{q}) = e^{\tau(\mathbf{q})A}\mathbf{q} \in \mathcal{P}^O, \\ \Phi(\mathbf{q},t) \notin \mathcal{P}, \forall t \in (0,\tau(\mathbf{q})). \end{array} \tag{26}$$

In a similar way, for points in $\mathbf{q} \in D_2$ there exists a function $\tau(\mathbf{q}) > 0$ verifying the analogous properties. In both cases the function $\tau(\mathbf{q})$ is called the **time of flight** of the point $\mathbf{q}$, and the procedure of obtaining this function is somewhat likely to the one of the time of flight of the Poincaré map.

Let $\mathbf{q} \in D_1$ such that $\mathbf{q}_{\mathcal{B}} = (a_1, \ldots, a_{n-1})$. From expressions (25) and (26), and applying that matrices A and $e^{\tau(\mathbf{q})A}$ commute as it was checked in equations (7), we get

$$\Pi^A(\mathbf{q}) = e^{\tau(\mathbf{q})A}\mathbf{p} + \sum_{i=1}^{n-1} a_i A^i e^{\tau(\mathbf{q})A}\mathbf{p}. \tag{27}$$

Now, from the Putzer method explained in the previous section, by means of the functions γ^i, expression (27) can be written as

$$\Pi^A(\mathbf{q}) = \sum_{i=0}^{n-1}\sum_{m=0}^{n-1} a_i \gamma^m(\tau(\mathbf{q})) A^{i+m}\mathbf{p}, \tag{28}$$

where we have considered $a_0 = 1$ for the sake of compacity of the expression. Once again an analogous expression is obtained for the other map.

The following result provides us explicit expressions for the components of the transition maps, $\{\pi_i^A\}_{i=1}^{n-1}$ and $\{\widehat{\pi}_i^A\}_{i=1}^{n-1}$. In order to prove it, we first introduce and remind some tools and concepts. The first of those will be the controlability matrix (21) of system (18) introduced above, in Section 3.3.

For each $m \geq 0$, we define the vectors $\mathbf{d}^m = (d_0^m, d_1^m, \ldots, d_{n-1}^m)$, which are given by

$$\mathbf{d}^m = B^m \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{n-1} \end{pmatrix}. \tag{29}$$

Finally, we define the functions $L, U, P : \mathbb{R}^{n-1} \to M_n(\mathbb{R})$ given by

$$L(a_1, \ldots, a_{n-1}) = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ a_1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{n-1} & \ldots & a_1 & 1 \end{pmatrix}, \tag{30}$$

$$U(a_1, \ldots, a_{n-1}) = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & a_{n-1} & \ldots & a_1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & a_{n-1} \end{pmatrix}, \tag{31}$$

$$P(a_1, \ldots, a_{n-1}) = L(a_1, \ldots, a_{n-1}) + B^{n-1} U(a_1, \ldots, a_{n-1}). \tag{32}$$

Theorem 10. *Let $\mathbf{q} \in \mathcal{P}$ with $\mathbf{q}_{\mathcal{B}} = (a_1, \ldots, a_{n-1})$, $\{\pi_i^A\}_{i=1}^{n-1}$, $\{\widehat{\pi}_i^A\}_{i=1}^{n-1}$ be the components of the transition maps Π^A, $\widehat{\Pi}^A$, respectively, and the function P defined in expression (32). Now,*

a) If $\mathbf{q} \in D_1$, then

$$\begin{pmatrix} 1 \\ \pi_1^A(a_1, \ldots, a_{n-1}) \\ \vdots \\ \pi_{n-1}^A(a_1, \ldots, a_{n-1}) \end{pmatrix} = P(a_1, \ldots, a_{n-1}) \begin{pmatrix} \gamma^0(\tau(\mathbf{q})) \\ \gamma^1(\tau(\mathbf{q})) \\ \vdots \\ \gamma^{n-1}(\tau(\mathbf{q})) \end{pmatrix}$$

b) If $\mathbf{q} \in D_2$, then

$$\begin{pmatrix} 1 \\ \widehat{\pi}_1^A(a_1, \ldots, a_{n-1}) \\ \vdots \\ \widehat{\pi}_{n-1}^A(a_1, \ldots, a_{n-1}) \end{pmatrix} = P(a_1, \ldots, a_{n-1}) \begin{pmatrix} \gamma^0(\tau(\mathbf{q})) \\ \gamma^1(\tau(\mathbf{q})) \\ \vdots \\ \gamma^{n-1}(\tau(\mathbf{q})) \end{pmatrix}$$

Proof. We will prove the first statement, as the second is analogous.

Suppose then $\mathbf{q} \in D_1$. From expression (28), if we take $r = m + i$, it follows

$$\begin{aligned} \Pi^A(\mathbf{q}) &= \sum_{r=0}^{n-1} \left(\sum_{m=0}^{r} a_{r-m} \gamma^m(\tau(\mathbf{q})) \right) A^r \mathbf{p} + \sum_{r=n}^{2n-2} \left(\sum_{m=r-(n-1)}^{n-1} a_{r-m} \gamma^m(\tau(\mathbf{q})) \right) A^r \mathbf{p} \\ &= \sum_{r=0}^{n-1} \left(\sum_{m=0}^{r} a_{r-m} \gamma^m(\tau(\mathbf{q})) \right) A^r \mathbf{p} + \sum_{r=0}^{n-2} \left(\sum_{m=r+1}^{n-1} a_{n+r-m} \gamma^m(\tau(\mathbf{q})) \right) A^{n+r} \mathbf{p}, \end{aligned}$$

for $a_0 = 1$. From the corollary 1, we have

$$\begin{aligned}\Pi^A(\mathbf{q}) &= \sum_{r=0}^{n-1}\left(\sum_{m=0}^{r} a_{r-m}\gamma^m(\tau(\mathbf{q}))\right)A^r\mathbf{p} - \sum_{r=0}^{n-2}\left(\sum_{m=r+1}^{n-1} a_{n+r-m}\gamma^m(\tau(\mathbf{q}))\right)\sum_{i=0}^{n-1} d_i^r A^i\mathbf{p} \\ &= \sum_{r=0}^{n-1}\left(\sum_{m=0}^{r} a_{r-m}\gamma^m(\tau(\mathbf{q}))\right)A^r\mathbf{p} - \sum_{i=0}^{n-1}\left(\sum_{r=0}^{n-2}\left(\sum_{m=r+1}^{n-1} a_{n+r-m}\gamma^m(\tau(\mathbf{q}))\right)d_i^r\right)A^i\mathbf{p}.\end{aligned}$$

If we rearrange the indices in the summations of the second term, we have

$$\begin{aligned}\Pi^A(\mathbf{q}) &= \sum_{r=0}^{n-1}\left(\sum_{m=0}^{r} a_{r-m}\gamma^m(\tau(\mathbf{q}))\right)A^r\mathbf{p} - \sum_{i=0}^{n-1}\left(\sum_{j=1}^{n-1}\left(\sum_{k=1}^{j} a_{n-k}d_i^{j-k}\right)\gamma^j(\tau(\mathbf{q}))\right)A^i\mathbf{p} \\ &= \sum_{r=0}^{n-1}\left(\sum_{m=0}^{r} a_{r-m}\gamma^m(\tau(\mathbf{q})) - \sum_{j=1}^{n-1}\left(\sum_{k=1}^{j} a_{n-k}d_r^{j-k}\right)\gamma^j(\tau(\mathbf{q}))\right)A^r\mathbf{p} \\ &= \sum_{r=0}^{n-1}\left(a_r\gamma^0(\tau(\mathbf{q})) + \sum_{m=1}^{r}\left(a_{r-m} - \sum_{k=1}^{m} a_{n-k}d_r^{m-k}\right)\gamma^m(\tau(\mathbf{q}))\right. \\ &\quad \left. - \sum_{m=r+1}^{n-1}\left(\sum_{k=1}^{m} a_{n-k}d_r^{m-k}\right)\gamma^m(\tau(\mathbf{q}))\right)A^r\mathbf{p}.\end{aligned}$$

Now, remind that $\Pi^A(\mathbf{q}) = \mathbf{p} + \sum_{i=1}^{n-1}\pi_i^A(\mathbf{q}_\mathcal{B})A^i\mathbf{p}$. Therefore, both the previous expression and this one are written in terms of the Krylov basis. Hence, the coefficients of the terms $A^i\mathbf{p}, i = 0, 1, \ldots, n-1$ will coincide for both expressions. Explicitly, the term $r = 0$ in the equation above will coincide with the coordinate of $\mathbf{p}$ in the second expression, that is,

$$1 = a_0\gamma^0(\tau(\mathbf{q})) - \sum_{m=1}^{n-1}\left(\sum_{k=1}^{m} a_{n-k}d_0^{m-k}\right)\gamma^m(\tau(\mathbf{q})).$$

On the other hand, for the rest of the coordinates $A^r\mathbf{p}$, where $r = 1, \ldots, n-1$, we obtain the following equalities for the coefficients:

$$\begin{aligned}\pi_r^A(a_1, \ldots, a_{n-1}) = a_r\gamma^0(\tau(\mathbf{q})) &+ \sum_{m=1}^{r}\left(a_{r-m} - \sum_{k=1}^{m} a_{n-k}d_r^{m-k}\right)\gamma^m(\tau(\mathbf{q})) \\ &- \sum_{m=r+1}^{n-1}\left(\sum_{k=1}^{m} a_{n-k}d_r^{m-k}\right)\gamma^m(\tau(\mathbf{q})).\end{aligned}$$

Finally, rearranging the second equality for each $r = 1, \ldots, n-1$ to determine the part that comes from matrix L and the part that comes from matrix $B^{n-1}U$, we have

$$\pi_r^A(a_1, \ldots, a_{n-1}) = \sum_{m=0}^{r} a_{r-m}\gamma^m(\tau(\mathbf{q})) - \sum_{m=1}^{n-1}\left(\sum_{k=1}^{m} a_{n-k}d_r^{m-k}\right)\gamma^m(\tau(\mathbf{q})).$$

Now, it is straightforward to check that this expression for each $r = 1, \ldots, n-1$ is exactly the $r+1$-th row of the matricial product

$$P(a_1, \ldots, a_{n-1}) \begin{pmatrix} \gamma^0(\tau(\mathbf{q})) \\ \gamma^1(\tau(\mathbf{q})) \\ \vdots \\ \gamma^{n-1}(\tau(\mathbf{q})) \end{pmatrix},$$

and the equality for the first coordinate corresponds to the first row of the previous product. Hence, the result follows. □

In theorem 10 we have obtained explicit expressions of the components of the transition maps in terms of the coordinates of the point $\mathbf{q}$ and its time of flight $\tau(\mathbf{q})$. Moreover, the time of flight is implicitly defined in the first equation. Now, whenever we are able to solve the equation involving the time of flight, we will be able to fully determine the transition map. In particular, if we know the time of flight τ_0 of a point $\mathbf{q}$, we can try to apply the Implicit Function Theorem in order to obtain an explicit expression for the time of flight in a neighborhood of τ_0, and thus determining the transition map in a neighborhood of $\mathbf{q}$. As an example, for the contact point of order n $\mathbf{p}$, we know that its time of flight is $\tau = 0$, so we can try to apply the previous reasoning near this point. This will be done for dimensions $n = 2$ and $n = 3$ in the following section.

On the other hand, notice also that matrix $P(a_1, \ldots, a_{n-1})$ only depends on the coordinates of $\mathbf{q}$ and the coefficients of the characteristic polynomial of matrix A. Moreover, information on the eigenvalues is organized within the functions $\gamma^i(\tau)$.

3.5.2 Expressions for the derivatives of transition maps

Now we address the problem of computing the partial derivatives of the transition maps. In order to prove the main result we introduce the next technical lemma.

Lemma 8. *Matrices B, P defined above commute, i.e.,*

$$BP = PB$$

Proof. The proof of the lemma is a direct calculation of both matricial products, since all the elements in each matrix are known. □

On the other hand, the controlability matrix defined in (21) allows us to find a system of differential equations for the functions γ^i introduced in Section 2.1.1. Even more, the result can be stated as a corollary from Theorem 7, since there it is shown that the functions satisfy the corresponding equations presented below.

Lemma 9. *Let $\Gamma(t) = (\gamma^i(t))_{i=0}^{n-1}$ be the vectorial function which components are the functions defined in Theorem 7. Then, $\Gamma(t)$ is the solution of the linear differential system $\dot{\Gamma}(t) = B\Gamma(t)$ satisfying the initial condition $\Gamma(0) = \mathbf{e}_1$, where $\mathbf{e}_1 = (1, 0, \ldots, 0)$ is the first element of the canonical base of $\mathbb{R}^n$.*

Proof. Notice that in the proof of Theorem 7 expressions (12) were found to be true. Moreover, in earlier stages of the theorem it was also shown that the components verify the initial conditions mentioned. Thus, this result is direct from the proof of that theorem. □

The previous result will be useful for the main result of this subsection, and in following sections as well, as it allows us to find series expansions for functions γ^i in a neighborhood of $t = 0$, by means of the system of differential equations they fulfill. Matricially, the expansions are written as

$$\Gamma(t) = \sum_{k=0}^{m} B^k \Gamma(0) \frac{t^k}{k!} + O(t^{m+1}), \tag{33}$$

and they give us an easier way of obtaining explicit expressions for these functions γ^i.

Now we present and prove the main result.

Theorem 11. *Let $\mathbf{q} \in \mathcal{P}$ and $\{\pi_i^A\}_{i=1}^{n-1}, \{\widehat{\pi}_i^A\}_{i=1}^{n-1}, P, B, \gamma^i(t)$ defined as above. Now,*

a) if $\mathbf{q} \in D_1$ with $\mathbf{q}_{\mathcal{B}} = (a_1, \ldots, a_{n-1})$, then

$$\begin{pmatrix} 0 \\ \frac{\partial \pi_1^A}{\partial x_j}(a_1, \ldots, a_{n-1}) \\ \vdots \\ \frac{\partial \pi_{n-1}^A}{\partial x_j}(a_1, \ldots, a_{n-1}) \end{pmatrix} = \frac{\partial P}{\partial x_j} \begin{pmatrix} \gamma^0(\tau(\mathbf{q})) \\ \gamma^1(\tau(\mathbf{q})) \\ \vdots \\ \gamma^{n-1}(\tau(\mathbf{q})) \end{pmatrix} + \frac{\partial \tau}{\partial x_j} B \begin{pmatrix} 1 \\ \pi_1^A \\ \vdots \\ \pi_{n-1}^A \end{pmatrix}.$$

b) if $\mathbf{q} \in D_2$ with $\mathbf{q}_{\mathcal{B}} = (a_1, \ldots, a_{n-1})$, then

$$\begin{pmatrix} 0 \\ \frac{\partial \widehat{\pi}_1^A}{\partial x_j}(a_1, \ldots, a_{n-1}) \\ \vdots \\ \frac{\partial \widehat{\pi}_{n-1}^A}{\partial x_j}(a_1, \ldots, a_{n-1}) \end{pmatrix} = \frac{\partial P}{\partial x_j} \begin{pmatrix} \gamma^0(\tau(\mathbf{q})) \\ \gamma^1(\tau(\mathbf{q})) \\ \vdots \\ \gamma^{n-1}(\tau(\mathbf{q})) \end{pmatrix} + \frac{\partial \tau}{\partial x_j} B \begin{pmatrix} 1 \\ \widehat{\pi}_1^A \\ \vdots \\ \widehat{\pi}_{n-1}^A \end{pmatrix}.$$

Proof. We will prove the first statement, as the second is analogous.

From Theorem 10, the partials of the transition map verify $\frac{\partial \pi_i^A}{\partial x_j} = \frac{\partial P}{\partial x_j}\gamma(\tau(\mathbf{q})) + \frac{\partial \tau}{\partial x_j} P \dot{\Gamma}(\tau(\mathbf{q}))$. The result now follows by applying Lemmas 8 and 9. □

Note that expressions of $\frac{\partial \tau}{\partial x_j}$ can be explicitly obtained from the first equation in the above Theorem.

3.6 Geometrical aspects

We work out now some geometrical aspects related to the hyperplane $\mathcal{P}$ under the assumption of a unique contact point $\mathbf{p}$ of order n.

We have the following as a first result:

Proposition 8. *Under the assumption of existence and uniqueness of a contact point* $\mathbf{p}$ *of order* n*, the hyperplane* $\mathcal{P}$ *can not be parallel to any invariant subspace.*

Proof. As we are dealing with linear systems of the form $\dot{\mathbf{x}} = A\mathbf{x}$, invariant subspaces are either eigenspaces or generalized eigenspaces. Thus, it suffices to show that $\mathcal{P}$ can not be parallel to neither of these.

Suppose then that $\mathcal{P}$ is parallel to an eigenspace

$$E_\lambda = \{\mathbf{v} \in \mathbb{R}^n | A\mathbf{v} = \lambda\mathbf{v}\}.$$

Hence, there exists a vector $\mathbf{v} \in E_\lambda$ such that $\mathbf{p} + \mathbf{v} \in \mathcal{P}$. Hence, it follows $\mathbf{k}^T\mathbf{v} = 0$. Moreover, since $\mathbf{v}$ is an eigenvector of eigenvalue λ, we have $\mathbf{k}^T A\mathbf{v} = \lambda\mathbf{k}^T\mathbf{v} = 0$. On the other hand, in terms of the Krylov base we have

$$\mathbf{v} = \sum_{i=1}^{n-1} \alpha_i A^i \mathbf{p}, \tag{34}$$

hence $A\mathbf{v} = \sum_{i=1}^{n-1} \alpha_i A^{i+1}\mathbf{p}$. Thus, from this expression and the previous equation it follows $\alpha_{n-1}(-1)^{n+1}\det(A) = 0$. From this, either $\det(A) = 0$, which leads to a contradiction, or $\alpha_{n-1} = 0$. In this latter case, we can iterate the reasoning with $A^k\mathbf{v}, k \geq 2$, and at some point we will get to $\det(A) = 0$ since as $\mathbf{v}$ is an eigenvector, its non-null by definition, thus $\alpha_j \neq 0$ for some $j = 1, 2, \ldots, n-1$. Hence, the hyperplane $\mathcal{P}$ can not be parallel to an eigenspace.

Let us suppose now that the hyperplane $\mathcal{P}$ is parallel to the generalized eigenspace

$$V_\lambda = \ker(A - \lambda I)^m.$$

Hence, there exists a vector $\mathbf{v} \in V_\lambda$ such that $\mathbf{p} + \mathbf{v} \in \mathcal{P}$. Thus, in terms of the Krylov base, we have once again expression (34) for the coordinates of $\mathbf{v}$. Now, since the generalized eigenspaces verify the chain of inclusions

$$\ker(A - \lambda I) \subset \ker(A - \lambda I)^2 \subset \ldots \subset \ker(A - \lambda I)^{m-1} \subset \ker(A - \lambda I)^m = V_\lambda,$$

we have that there exists a vector $\mathbf{v} \in \ker(A - \lambda I)^m$ such that $(A - \lambda I)\mathbf{v} = \mathbf{w}$ for some vector $\mathbf{w} \in \ker(A - \lambda I)^{m-1}$. Since V_λ is parallel to $\mathcal{P}$, its subsets will also be parallel to the hyperplane. Thus, we have $\mathbf{k}^T\mathbf{v} = 0$, $\mathbf{k}^T\mathbf{w} = 0$. Hence, it follows $\mathbf{k}^T A\mathbf{v} = 0$. And this leads to a contradiction analogous to that of the eigenspaces shown above.

Therefore, we conclude that the hyperplane $\mathcal{P}$ can not be parallel to any invariant subspace. □

Another question of interest is the location of the points of D_1 (resp. D_2) that share time of flight. For $\tau > 0$, let $\mathcal{P}_\tau$ be the set of all points in D_1 with time of flight τ, and $\Pi^A(\mathcal{P}_\tau)$ be its image set by the corresponding transition map. The next lemma shows that both these sets are contained into $(n-2)$-dimensional affine manifolds of $\mathcal{P}$.

Lemma 10. *Consider a fixed value $\tau > 0$. Then, the set $\mathcal{P}_\tau$ is contained in the affine manifold $\{\mathbf{q} \in \mathbb{R}^n | \mathbf{k}^T\mathbf{q} = 1, \mathbf{k}^T e^{\tau A}\mathbf{q} = 1\}$ and the set $\Pi^A(\mathcal{P}_\tau)$ is contained in the affine manifold $\{\mathbf{q} \in \mathbb{R}^n | \mathbf{k}^T\mathbf{q} = 1, \mathbf{k}^T e^{-\tau A}\mathbf{q} = 1\}$. Moreover, both these manifolds are $(n-2)$-dimensional.*

Proof. Consider $\mathbf{q} \in \mathcal{P}_\tau$. Then, $\mathbf{q} \in \mathcal{P}$ and $e^{\tau A}\mathbf{q} \in \mathcal{P}$, that is, $\mathbf{k}^T\mathbf{q} = 1$ and $\mathbf{k}^T e^{\tau A}\mathbf{q} = 1$. If we consider now the image of $\mathbf{q}$ by Π^A, $\Pi^A(\mathbf{q}) = \widehat{\mathbf{q}} = e^{\tau A}\mathbf{q}$, we have $\mathbf{k}^T\widehat{\mathbf{q}} = \mathbf{k}^T e^{\tau A}\mathbf{q} = 1$ and $\mathbf{k}^T e^{-\tau A}\widehat{\mathbf{q}} = \mathbf{k}^T\mathbf{q} = 1$, which proves the first part of the result.

Now, suppose that the manifold $\{\mathbf{q} \in \mathbb{R}^n | \mathbf{k}^T\mathbf{q} = 1, \mathbf{k}^T e^{\tau A}\mathbf{q} = 1\}$ is $(n-1)$-dimensional, which implies that the system formed by equations

$$\begin{cases} \mathbf{k}^T\mathbf{q} = 1, \\ \mathbf{k}^T e^{A\tau}\mathbf{q} = 1, \end{cases}$$

has rank 1 and it is indeterminate. That is, the manifold $\{\mathbf{q} \in \mathbb{R}^n | \mathbf{k}^T\mathbf{q} = 1, \mathbf{k}^T e^{\tau A}\mathbf{q} = 1\}$ fills the whole hyperplane $\mathcal{P}$. This implies that any point in $\mathcal{P}$ has flight time τ. Therefore, the functions $\gamma^i(\tau)$ are constant. Under this assumption, from Lemma 9 it follows $\gamma^i(\tau) = 0, \forall i = 0, 1, \ldots, n-1$ since the controllability matrix B is regular, as it is a change of basis from matrix A which is also regular. But, if all the functions γ^i are null, from Theorem 10 it follows

$$\begin{pmatrix} 1 \\ \pi_1^A(a_1, \ldots, a_{n-1}) \\ \vdots \\ \pi_{n-1}^A(a_1, \ldots, a_{n-1}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

which in particular leads to $1 = 0$, a contradiction. Hence, it follows that the manifold needs to be $(n-2)$-dimensional. An analogous reasoning holds for the other manifold. □

As it was expected, the same result holds for the other map, $\widehat{\Pi}^A$.

Now, for a fixed time of flight τ, the previous lemma allows us to simplify the study of the problem for all the points sharing the time of flight τ, as we can reduce the dimension of $\mathcal{P}$ and work with the mentioned $(n-2)$-dimensional manifolds. For instance, for dimension $n = 3$ the result tells us that the transition maps map line segments contained in $\mathcal{P}$ into line segments contained in $\mathcal{P}$.

In the next result, it is shown that for solutions starting at points $\mathbf{q} \in \mathcal{P}$ written in terms of the Krylov base

$$\{A^i\mathbf{p}\}_{i=0}^{n-1} = \left\{ \left. \frac{d^i\varphi(t;\mathbf{p})}{dt^i} \right|_{(0;\mathbf{p})} \right\}_{i=0}^{n},$$

their coordinates remain the same for all values of time t.

Lemma 11. *Let* $\mathbf{p}$ *be the unique contact point of order* n *of system* (18)*. Then, for any point* $\mathbf{q} \in \mathcal{P}$ *such that* $\mathbf{q}_{\mathcal{B}} = (a_1, a_2, \ldots, a_{n-1})$*, we have*

$$\varphi(t;\mathbf{q}) = \sum_{i=0}^{n-1} a_i A^i e^{tA}\mathbf{p} = \sum_{i=0}^{n-1} a_i \frac{d^i \varphi(t;\mathbf{p})}{dt^i},$$

for all time t*, where* $a_0 = 1$*.*

Proof. On the one hand, we know that the solution of system (18) writes as $\varphi(t;\mathbf{x}) = e^{tA}\mathbf{x}$, for any point $\mathbf{x} \in \mathbb{R}^n$. In particular, if we take a point $\mathbf{q} \in \mathcal{P}$, it follows $\varphi(t;\mathbf{q}) = e^{At}\mathbf{q}$.

On the other hand, in terms of the Krylov base we have

$$\mathbf{q} \quad = \textstyle\sum_{i=0}^{n-1} a_i A^i \mathbf{p},$$

where $a_0 = 1$. Thus, it follows

$$e^{At}\mathbf{q} = e^{At} \sum_{i=0}^{n-1} a_i A^i \mathbf{p}.$$

The result follows reminding that matrices A and e^{At} commute. □

4 Transition map in the neighbourhood of the contact point

In this section, we focus on a local study of the transition map for dimensions $n = 2$ and $n = 3$, by means of trying to solve the first equation of Theorem 10, in order to get an explicit expression for the time of flight. As it was stated right after the mentioned theorem, we will apply the Implicit Function Theorem onto the first equation, as for the contact point $\mathbf{p}$ of order n we know that its time of flight will be $\tau = 0$. Thus, the expansions will be local, precisely in a neighborhood of $\tau = 0$.

Remind that the study could also be performed near any other point $\mathbf{q} \in \mathcal{P}$ for which its time of flight is known.

4.1 Study on $\mathbb{R}^2$

Let us now consider a two-dimensional linear differential system given by

$$\dot{\mathbf{x}} = A\mathbf{x}, \tag{35}$$

with $D = \det(A) > 0$. The study for the case $D < 0$ is analogous, since the only differences are the orientation of the subsets $\mathcal{L}_1^I$ and $\mathcal{L}_1^O$, that will have a different sign on their last coordinate, and the contact point $\mathbf{p}$ of order 2 belonging to the subset $\mathcal{L}_2^O$.

The hyperplane in this case will be simply a straight line, given by

$$\mathcal{P} = \{\mathbf{q} \in \mathbb{R}^2 : \mathbf{k}^T\mathbf{q} = 1\} \tag{36}$$

for a fixed non-zero vector $\mathbf{k} \in \mathbb{R}^2$.

Suppose there exists a unique contact point $\mathbf{p}$ of order 2 with the straight line $\mathcal{P}$, that will verify $\mathbf{p}_{\mathcal{B}} = 0 \in \mathbb{R}$. Then, the set $\mathcal{L}_2 = \{0\}$ splits $\mathcal{L}_1$ into two parts: $\mathcal{L}_1^I = \{a_1 : a_1 > 0\}$ and $\mathcal{L}_1^O = \{a_1 : a_1 < 0\}$, by the characterization defined at the end of Section 3.4. The domain of the transition map Π^A defined over $\mathcal{P}$ is

$$D_1 \subset \mathcal{P}^I = \bigcup_{i=1}^{2} \mathcal{L}_i^I,$$

where $\mathcal{L}_2^I = \{\mathbf{p}\}$ since $n = 2$ is even and $D > 0$. Thus, the contact point will be in D_1, and it will have time of flight $\tau(\mathbf{p}) = 0$.

In a similar way, the domain of the transition map $\widehat{\Pi}^A$ defined over $\mathcal{P}$ is

$$D_2 \subset \mathcal{P}^O = \bigcup_{i=1}^{2} \mathcal{L}_i^O,$$

where the set $\mathcal{L}_2^O$ is empty. However, we can extend the domain D_2 to $\mathbf{p}$ by extending the time of flight function assigning $\tau(\mathbf{p}) = 0$. It can be checked that this extension is continuous by arguments about continuous dependence of solutions of differential equations on initial conditions.

Now, for two-dimensional systems as the one given by (35), and using the tools and concepts defined for the generic n-dimensional ones, here we can obtain approximations $\widehat{\pi}_1^A, \tau$ in terms of the coordinate a_1 of a point $\mathbf{q} \in \mathcal{P}$, expressed in terms of the Krylov base. This claim is proven in the lemma below.

Lemma 12. *Consider a two-dimensional linear system (35) with $D = \det(A) > 0$, $T = \mathrm{Tr}(A)$, a straight line (36) such that the flow of the system has exactly one contact point $\mathbf{p}$ of order two with $\mathcal{P}$. Then, for $\mathbf{q} \in \mathcal{P}$, the component of the transition map $\widehat{\Pi}^A$ and the time of flight τ defined over $\mathcal{P}$ verify the following series expansion, in terms of $\mathbf{q}_{\mathcal{B}} = (a_1)$,*

$$\begin{aligned}
\widehat{\pi}_1^A(a_1) &= -a_1 + \frac{2T}{3}a_1^2 - \frac{4T^2}{9}a_1^3 + \frac{2}{135}T\left(22T^2 - 9D\right)a_1^4 \\
&+ \frac{4}{405}\left(27DT^2 - 26T^4\right)a_1^5 \\
&+ \frac{2}{945}\left(27D^2T - 176DT^3 + 100T^5\right)a_1^6 + \mathcal{O}(a_1^7), \\
\tau(a_1) &= -2a_1 + \frac{2T}{3}a_1^2 + \frac{2}{9}\left(3D - 2T^2\right)a_1^3 - \frac{4}{135}T\left(27D - 11T^2\right)a_1^4 \\
&- \frac{2}{405}\left(81D^2 - 180DT^2 + 52T^4\right)a_1^5 \\
&+ \frac{2}{567}T\left(243D^2 - 268DT^2 + 60T^4\right)a_1^6 + \mathcal{O}(a_1^7).
\end{aligned}$$

Proof. Let $\mathbf{q} \in D_2$, where $D_2 = \mathrm{Dom}(\widehat{\Pi}^A)$. From Theorem (10) and expression (32), we have

$$\begin{pmatrix} 1 \\ \widehat{\pi}_1^A(a_1) \end{pmatrix} = P(a_1) \begin{pmatrix} \gamma^0(\tau(\mathbf{q})) \\ \gamma^1(\tau(\mathbf{q})) \end{pmatrix}, \tag{37}$$

where $P(a_1)$ is given by equation (32), that is, $P(a_1) = L(a_1) + BU(a_1)$, considering the matrices B, L, U defined as in expressions (21), (30) and (31) for $n = 2$. From the explicit expressions of each of these, we get that P is given by

$$P(a_1) = \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -D \\ 1 & T \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & a_1 \end{pmatrix} = \begin{pmatrix} 1 & -a_1 D \\ a_1 & 1 + a_1 T \end{pmatrix}.$$

Thus, the time of flight of $\mathbf{q}$, which can be expressed in terms of $\mathbf{q}_{\mathcal{B}}$, that is, $\tau(a_1)$, and the image of that point by the map $\widehat{\Pi}^A$, which will have one component $\widehat{\pi}_1^A(a_1)$, are determined implicitly by the equations

$$1 = \gamma^0(\tau(a_1)) - a_1 D\gamma^1(\tau(a_1)), \tag{38}$$

$$\widehat{\pi}_1^A(a_1) = a_1\gamma^0(\tau(a_1)) + (1 + a_1 T)\gamma^1(\tau(a_1)), \tag{39}$$

where functions γ^0, γ^1 are the ones defined in Putzer's method, thus they are the coefficients of the solution $\Gamma(t)$ of the linear system $\dot{\Gamma} = B\Gamma$ with initial condition $\Gamma(0) = \mathbf{e}_1$, as it was shown in Lemma (9).

Now, by applying Lemma (9) iteratively, one can easily obtain Taylor series expansions for both $\gamma^0(\tau), \gamma^1(\tau)$ in a neighborhood of $\tau = 0$ by means of the differential system these equations form and relating the successive derivatives with their respective equations. Matricially, this corresponds to what was shown in expression (33). Explicitly, these expansions are

$$\begin{aligned}
\gamma^0(\tau) &= 1 - \frac{D}{2}\tau^2 - \frac{DT}{6}\tau^3 + \frac{1}{24}D\left(D - T^2\right)\tau^4 \\
&+ \frac{1}{120}DT\left(2D - T^2\right)\tau^5 - \frac{1}{720}D\left(D^2 - 3DT^2 + T^4\right)\tau^6 \\
&- \frac{DT\left(D - T^2\right)\left(3D - T^2\right)}{5040}\tau^7 + \mathcal{O}(\tau^8), \\
\gamma^1(\tau) &= \tau + \frac{T}{2}\tau^2 + \frac{1}{6}\left(T^2 - D\right)\tau^3 - \frac{1}{24}T\left(2D - T^2\right)\tau^4 \\
&+ \tfrac{1}{120}\left(D^2 - 3DT^2 + T^4\right)\tau^5 + \frac{1}{720}T\left(D - T^2\right)\left(3D - T^2\right)\tau^6 \\
&+ \frac{-D^3 + 6D^2T^2 - 5DT^4 + T^6}{5040}\tau^7 + \mathcal{O}(\tau^8).
\end{aligned}$$

Now, by a procedure of indeterminate coefficients, we obtain the following series expansion of τ in terms of a_1,

$$\begin{aligned}
\tau(a_1) &= -2a_1 + \frac{2T}{3}a_1^2 + \frac{2}{9}\left(3D - 2T^2\right)a_1^3 - \frac{4}{135}T\left(27D - 11T^2\right)a_1^4 \\
&- \frac{2}{405}\left(81D^2 - 180DT^2 + 52T^4\right)a_1^5 \\
&+ \frac{2}{567}T\left(243D^2 - 268DT^2 + 60T^4\right)a_1^6 \\
&+ \mathcal{O}(a_1^7).
\end{aligned} \tag{40}$$

By substituting expression (40) in the expansions for γ^0, γ^1, we get both these functions in terms of a_1:

$$\begin{aligned}
\gamma^0(a_1) &= 1 - 2Da_1^2 + \frac{8}{3}DTa_1^3 + \frac{2}{9}D\left(9D - 14T^2\right)a_1^4 \\
&- \frac{8}{135}DT\left(81D - 58T^2\right)a_1^5 \\
&- \frac{2}{405}D\left(405D^2 - 1656DT^2 + 748T^4\right)a_1^6 + \mathcal{O}(a_1^7), \\
\gamma^1(a_1) &= -2a_1 + \frac{8T}{3}a_1^2 + \frac{2}{9}\left(9D - 14T^2\right)a_1^3 - \frac{8}{135}T\left(81D - 58T^2\right)a_1^4 \\
&- \frac{2}{405}\left(405D^2 - 1656DT^2 + 748T^4\right)a_1^5 \\
&+ \frac{16T\left(1215D^2 - 2124DT^2 + 692T^4\right)}{2835}a_1^6 + \mathcal{O}(a_1^7).
\end{aligned} \tag{41}$$

Finally, by substituting the expansions for γ^0, γ^1, τ in terms of a_1 into equation (39), we get

$$\begin{aligned}
\widehat{\pi}_1^A(a_1) &= -a_1 + \frac{2T}{3}a_1^2 - \frac{4T^2}{9}a_1^3 + \frac{2}{135}T\left(22T^2 - 9D\right)a_1^4 \\
&+ \frac{4}{405}\left(27DT^2 - 26T^4\right)a_1^5 + \frac{2}{945}\left(27D^2T - 176DT^3 + 100T^5\right)a_1^6 \\
&+ \mathcal{O}(a_1^7),
\end{aligned}$$

as we wanted to show.

□

Notice that we do not talk about approximations of π_1^A. This comes from the fact that points belonging to D_1 evolve to the half-space that contains the origin, thus these points will belong to solutions defining orbits that surrounds the equilibrium. Hence, their time of flight will not be in a neighborhood of 0.

For the case of $D < 0$, Lemma 12 will instead give a valid approximation for the component π_1^A of the transition map Π_1^A, since the singular point will be a saddle, thus orbits belonging to S_0 will not surround the origin.

On the other hand, Lemma 12 allows us to completely determine the position and time of flight for any point $\mathbf{q} \in \mathcal{P}$ in a neighborhood of the contact point $\mathbf{p}$. Moreover, if we take a look at the first terms in the series expansion for $\widehat{\pi}_1^A(a_1)$, we have

$$\begin{aligned}
\tfrac{d\widehat{\pi}_1^A(a_1)}{da_1} &= -1 + \ldots, \\
\tfrac{d^2\widehat{\pi}_1^A(a_1)}{da_1^2} &= \tfrac{4T}{3} + \ldots.
\end{aligned}$$

The previous expressions allow us to know the behaviour of $\widehat{\Pi}^A(a_1)$ near $a_1 = 0$. Precisely, the expressions obtained assure that the function is decreasing in a neighborhood of $a_1 = 0$, while local convexity will depend on the sign of T. As an application, these statements can be used to describe the local behaviour

of a pair of transition maps on a Piecewise Linear System, defined over the separation hyperplane taken as a cross section, as it will be shown in the next section. With the analysis of its growth and convexity, we can check if there exists a point in which both maps coincide, that will correspond to a limit cycle, in a neighborhood of the contact point $\mathbf{p}$.

4.2 Study on $\mathbb{R}^3$

In this section, we consider a three-dimensional system analogous to the one defined in expression (35), verifying also $D = \det(A) > 0$. This latter consideration is not significant, and results for the case $D < 0$ are analogous to the presented.

Let us consider the existence of a unique contact point $\mathbf{p}$ of order 3 with a plane

$$\mathcal{P} = \{\mathbf{q} \in \mathbb{R}^3 : \mathbf{k}^T\mathbf{q} = 1\} \tag{42}$$

in the phase space, where $\mathbf{k} \in \mathbb{R}^3$ is a fixed vector. In this case, we have $\mathbf{p}_{\mathcal{B}} = (0,0)$, so $\mathcal{L}_3 = \{(0,0)\}$ splits $\mathcal{L}_2$ into $\mathcal{L}_2^I = \{(a_1,0) : a_1 < 0\}$ and $\mathcal{L}_2^O = \{(a_1,0) : a_1 > 0\}$. Similarly, the set $\mathcal{L}_2$ splits $\mathcal{L}_1$ into $\mathcal{L}_1^I = \{(a_1,a_2) : a_2 < 0\}$ and $\mathcal{L}_1^O = \{(a_1,a_2) : a_2 > 0\}$. Now, the domain of the transition map Π^A defined over $\mathcal{P}$ is

$$D_1 \subset \mathcal{P}^I = \bigcup_{i=1}^{3} \mathcal{L}_i^I,$$

where $\mathcal{L}_3^I = \emptyset$, and therefore the point $\mathbf{p} \notin D_1$. However, one can extend D_1 to $\mathbf{p}$ by extending the time of flight function τ assigning $\tau(\mathbf{p}) = 0$. It can be shown that this extension is continuous by arguments about continuous dependence of solutions of differential equations on initial conditions.

Let $p_A(x) = x^3 - Tx^2 + Mx - D$ be the characteristic polynomial of the matrix A, where T denotes the trace of A and M the sum of its principal minors. Let $\mathbf{q} \in \mathcal{P}$ such that $\mathbf{q}_{\mathcal{B}} = (a_1,a_2) \in D_1$. By applying Theorem 10, its flying time $\tau(\mathbf{q})$ can be implicitly determined from the first equation, that is,

$$1 = \gamma^0(\tau(a_1,a_2)) + Da_2\gamma^1(\tau(a_1,a_2)) + (Da_1 + DTa_2)\gamma^2(\tau(a_1,a_2)). \tag{43}$$

Moreover, the image of $\mathbf{q}$ by the transition map Π^A is explicitly determined by equations

$$\begin{array}{rcl} \pi_1^A(a_1,a_2) & = & a_1\gamma^0(\tau(a_1,a_2)) + (1 - Ma_2)\gamma^1(\tau(a_1,a_2)) \\ & + & ((D - MT)a_2 - Ma_1)\gamma^2(\tau(a_1,a_2)), \\ \pi_2^A(a_1,a_2) & = & a_2\gamma^0(\tau(a_1,a_2)) + (a_1 + Ta_2)\gamma^1(\tau(a_1,a_2)) \\ & + & (1 + Ta_1 + (T^2 - M)a_2)\gamma^2(\tau(a_1,a_2)). \end{array} \tag{44}$$

Now, we want to get explicit expressions for the time of flight τ in terms of the coordinates of a point $\mathbf{q}_{\mathcal{B}} = (a_1,a_2)$ and explicit expressions for the functions γ^i, in order to fully determine the image of $\mathbf{q}$ by the transition map Π^A depending only on its position.

As a first approach, let us consider a fixed time of flight $\tau = \tau_0$. From Lemma 10, we know that the points from $\mathcal{P}$ having time of flight τ_0 will be contained in a straight line, and so will be their images. Moreover, we can obtain explicit expressions for these straight lines, as in expression (43), for a fixed time of flight, functions $\gamma^i, i = 0, 1, 2$ are constants. Thus, we are able to compute an explicit expression of a_1 in terms of a_2 only with linear terms, that is, the equation of a straight line. Explicitly, the expression we obtain is

$$a_1 = \frac{1-\gamma^0}{\gamma^2 D} - a_2 \frac{\gamma^1 + T\gamma^2}{\gamma^2}. \tag{45}$$

Now, as we have said, from Lemma 10, the image of the points sharing time of flight τ_0 will be another straight line, and we are able to compute an explicit expression for this other straight line, by combining equations (44) and equation (45). By substituting the expression for a_1 in the previous equation and after some straightforward calculations, we get to the following expressions for the transition map components:

$$\begin{array}{rcl} \pi_1(a_1, a_2) & = & \gamma^1 + \frac{1-\gamma^0}{\gamma^2 D}(\gamma^0 - M\gamma^2) \\ & - & a_2 \left((\frac{\gamma^1}{\gamma^2} + T)(\gamma^0 - M\gamma^2) + M\gamma^1 - (D - MT)\gamma^2 \right), \\ \pi_2(a_1, a_2) & = & \gamma^2 + \frac{1-\gamma^0}{\gamma^2 D}(\gamma^1 + T\gamma^2) \\ & - & a_2 \left((\frac{\gamma^1}{\gamma^2} + T)(\gamma^1 + T\gamma^2) - \gamma^0 - T\gamma^1 - (T^2 - M)\gamma^2 \right). \end{array} \tag{46}$$

If we label the terms of the previous equations as A_1, A_2, B_1, B_2 such that

$$\begin{array}{rcl} \pi_1(a_1, a_2) & = & A_1 - a_2 A_2, \\ \pi_2(a_1, a_2) & = & B_1 - a_2 B_2, \end{array}$$

it is straightforward that π_1, π_2 fulfill the equation of a straight line, namely,

$$B_2\pi_1 - A_2\pi_2 = B_2 A_1 - A_2 B_1.$$

Therefore, for a constant time τ_0 we not only know that points sharing that time of flight will be contained in line segments that will be mapped to other line segments, but we can also determine explicit expressions for both these line segments. Moreover, the equations we have derived are exact, since no series approximations have been used. However, in order to keep things simple, functions $\gamma^i, i = 0, 1, 2$ are derived by series approximations, as it is done below, in expression (47).

Now, for τ variable, thus we work in a more generic setting, we start by finding an approximate expression of τ in a neighborhood of 0 using Taylor expansions. For functions γ^i, series expansions in a neighborhood of $\tau = 0$ can be obtained by successive applications of Lemma 9, since the successive derivatives can be substituted by either the expression on the differential equation or its derivatives, which by successive substitution will lead to some coefficient. This,

once again, corresponds matricially to expression (33). The explicit expressions are below,

$$\begin{aligned}\gamma^0(\tau) &= 1 + D\tfrac{\tau^3}{3!} + TD\tfrac{\tau^4}{4!} + \dots,\\ \gamma^1(\tau) &= \tau - M\tfrac{\tau^3}{3!} + (D - TM)\tfrac{\tau^4}{4!} + \dots,\\ \gamma^2(\tau) &= \tfrac{\tau^2}{2} + T\tfrac{\tau^3}{3!} + (T^2 - M)\tfrac{\tau^4}{4!} + \dots.\end{aligned} \tag{47}$$

Now, substituting equations (47) in equation (43), we obtain an expansion of (43) in power series of τ, which can be rewritten as $1 = 1 + D\tau(eq(\tau) + O(\tau^4))$, where

$$\begin{aligned}eq(\tau) &= a_2 + (a_1 + Ta_2)\tfrac{\tau}{2} + (1 - Ma_2 + (a_1 + Ta_2)T)\tfrac{\tau^2}{3!}\\ &+ (T + a_2(D - TM) + (a_1 + Ta_2)(T^2 - M))\tfrac{\tau^3}{4!}\end{aligned} \tag{48}$$

Suppose $\widehat{\tau}$ is a zero of $eq(\tau)$. Then, $\widehat{\tau}$ is an approximation to the time of flight τ, satisfying $|eq(\tau) - eq(\widehat{\tau})| = O(\tau^4)$.

One way of approaching the problem of obtaining expressions of τ, π_1, π_2 in terms of the coordinates a_1, a_2 of a point $\mathbf{q} \in \mathcal{P}$ written in terms of the Krylov base is to consider a_1 fixed. In this setting, the procedure of indeterminate coefficients applied to equation (48) to obtain a series of $\tau(a_1, a_2)$ gets reduced to a series of $\tau(a_2)$ with coefficients depending of a_1. By substituting the obtained series in functions γ^i and thus substituting γ^i in equations (44), we obtain expressions of π_1 and π_2 in terms of a_2 for each fixed a_1. This is summarized and proved in the following result.

Proposition 9. *Consider $\mathbf{q} \in \mathcal{P}$ such that $\mathbf{q}_{\mathcal{B}} = (a_1, a_2)$, with $a_1 \neq 0$ and fixed. Then, the time of flight and the components of the transition map Π^A verify the following series expansion in a neighborhood of $a_2 = 0$:*

$$\begin{aligned}\tau(a_2) &= \tfrac{-2a_2}{a_1} + \tfrac{(2Ta_1-4)a_2^2}{3a_1^3} - \tfrac{((4T^2-6M)a_1^2-10Ta_1+16)a_2^3}{9a_1^5}\\ &+ \tfrac{((44T^3-108MT+54D)a_1^3+(216M-156T^2)a_1^2+300Ta_1-400)a_2^4}{135a_1^7}\\ &+ \mathcal{O}(a_2^5),\\ \pi_1(a_2) &= a_1 - \tfrac{2a_2}{a_1} + \tfrac{(2Ta_1-4)a_2^2}{3a_1^3} + \tfrac{(6Da_1^3+(-4T^2+6M)a_1^2+10Ta_1-16)a_2^3}{9a_1^5}\\ &- \tfrac{(90DTa_1^4+(-44T^3+108MT-144D)a_1^3+(156T^2-216M)a_1^2-300Ta_1+400)a_2^4}{135a_1^7}\\ &+ \mathcal{O}(a_2^5),\\ \pi_2(a_2) &= -a_2 + \tfrac{(2Ta_1+2)a_2^2}{3a_1^2} - \tfrac{(4T^2a_1^2+2Ta_1-8)a_2^3}{9a_1^4}\\ &+ \tfrac{((44T^3-18MT-36D)a_1^3+(-6T^2-54M)a_1^2-120Ta_1+200)a_2^4}{135a_1^6}\\ &+ \mathcal{O}(a_2^5).\end{aligned} \tag{49}$$

Proof. For a fixed $a_1 \neq 0$, by means of an indeterminate coefficients procedure applied to equation (48) we obtain the series of $\tau(a_2)$ presented. The series for $\pi_1(a_2)$ and $\pi_2(a_2)$ follow by substituting the approximations given in expression (47) for γ^i and the series of $\tau(a_2)$ into equations (46). □

Notice that the previous result holds for $a_1 \neq 0$. In fact, the points of $\mathcal{P}$ that, expressed in the Krylov base, fulfill $a_1 = 0$ are precisely the points of the set $\mathcal{L}_2$,

for $a_2 \neq 0$, or the contact point $\mathbf{p}$ of order 3. Both these points have time of flight $\tau = 0$, which is a branch that was eliminated in order to obtain equation (48). Consequently, the previous result will give an approximation of the time of flight and the coordinates of the image of contact points of order 1.

Moreover, as the series expansion obtained for τ has as its first terms the odd powers of a_1, it looks like it will be valid for $a_1 \to 0$ whenever $a_2 = \mathcal{O}(a_1^2)$. This last statement would imply that the domain D_1 of the transition map would have as boundary $\{(a_1, a_2) | a_2 = \mathcal{O}(a_1^2)\}$. This is still to be proven, and it is one topic where to continue this work, as it will be stated in the last section.

5 Applications to Piecewise Linear Systems

The theory and results we have developed so far is interesting and useful by its own, since it allows us to know and fully classify the behavior of solutions on any hyperplane transversal to the flow, under suitable conditions. However, there exists a family of systems in which these results become even more useful. This is the family of **piecewise linear systems** (PWLS, for short), that has been mentioned throughout the text. PWLS consist on a set of linear systems defined on a partition of the phase space, in such a way that any linear system acts on a set of the partition. Even when different partitions of $\mathbb{R}^n$ can be considered, it is usual to consider half-spaces separated by parallel hyperplanes. As an example, a PWLS with two zones is formed by equations of the form

$$\dot{\mathbf{x}} = \begin{cases} A_L\mathbf{x} + \mathbf{b}_L, & \mathbf{k}^T\mathbf{x} \leq 1, \\ A_R\mathbf{x} + \mathbf{b}_R, & \mathbf{k}^T\mathbf{x} \geq 1, \end{cases} \tag{50}$$

where $A_i \in M_n(\mathbb{R})$ and $\mathbf{x}, \mathbf{k}, \mathbf{b}_i \in \mathbb{R}^n$, for $i = L, R$. Here, the hyperplane $\mathbf{k}^T\mathbf{x} = 1$ splits the phase space $\mathbb{R}^n$ into two half-spaces $\mathbf{k}^T\mathbf{x} < 1$ and $\mathbf{k}^T\mathbf{x} > 1$, where over each of these regions the vector field is defined by the respective linear system.

Notice that depending on the values of the components of matrices A_i and vectors $\mathbf{b}_i$, these systems can either be continuous or discontinuous. Focusing on PWLS with two zones, we have that for continuous PWLS, the contact point and the Krylov base is shared between both zones. On the other hand, discontinuous PWLS can be of different kinds, depending on the contact points, the orientation of the Krylov base for each zone and the location of the sets $\mathcal{L}_m^I$ and $\mathcal{L}_m^O$ for $m = 1, 2, \ldots, n$. If both linear zones share the contact point, and the sets $\mathcal{L}_m^I$ of one linear system are located where the sets $\mathcal{L}_m^O$ of the other linear system are located, then we still have uniqueness of solutions despite the discontinuity of the whole PWLS, for a configuration where each linear system has a singular point in its region of definition. For the case of singular points belonging to the same region of definition, uniqueness of orbits will hold if the sets $\mathcal{L}_m^I$ for each linear system share location and the sets $\mathcal{L}_m^O$ for each linear term share location as well. If either the contact point is not shared or the respective sets $\mathcal{L}_m^I, \mathcal{L}_m^O$ for each zone aren't located as mentioned above, there will be regions of **sliding** for the orbits, leading to complex schemes that will not be considerate here.

The family of PWLS, both continuous and discontinuous, is of great interest, as it allows the qualitative behaviour of some non-linear systems to be obtained through linear ones, allowing a more simplified study of the dynamics, but maintaining the qualitative behaviour of the flow. There is a wide amount of theory and results devoted to this family of systems, see e.g. articles [2], [6], [1], [4] and books [5], [8].

The theory we have developed in this work allows us to analyze the flow in a neighborhood of the contact point, which is an issue which has not been considered previously anywhere, at least in a phase spaces of dimension greater than 2.

A comprehensive study of these systems would fill a whole work. Instead, in this section we will devote ourselves to study a family of 2-dimensional discontinuous PWLS sharing the contact point of order 2, in which the results we have obtained throughout the work are applied. This study will serve as an application of the theory developed in the previous sections, and in particular we will state some conditions on the PWLS for it to exhibit a bifurcation similar to the one of Hopf, but located around the contact point instead of the equilibrium. As it has been stated above, the theory developed in this work allows us to perform a local study, in a neighborhood of the contact point.

Now, since the system we will work with is discontinuous, we first study each linear part in order to determine the behavior of the system near its boundary. Thus, we will be able to properly define the PWLS domain for each linear part, including the definition over the separation straight line and on the contact point of order 2. In this example, we will consider the separation straight line to be

$$\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^2 | \mathbf{k}^T \mathbf{x} = 1\}, \quad \mathbf{k}^T = (1, 1). \tag{51}$$

Now we study one of the linear parts. In particular, for given D_0, T_0, let us consider the linear system

$$\dot{\mathbf{x}} = A_0 \mathbf{x}, \quad \mathbf{k}^T \mathbf{x} \leq 1, \tag{52}$$

where $A_0 = \begin{pmatrix} T_0 & -1 \\ D_0 & 0 \end{pmatrix}$. This system has the origin as a singular point, trace given by T_0 and determinant given by D_0. Under the assumption of A_0 being regular, this equilibrium is unique, and we have ensured the existence and uniqueness of a contact point of order 2. Moreover, the behavior of the equilibrium will depend on the signs of these two parameters. On the other hand, the contact point of order 2 for this system is given by the following system of equations for the coordinates of $\mathbf{p}_0$:

$$\begin{cases} p_1 + p_2 = 1, \\ T_0 p_1 - p_2 + D_0 p_1 = 0. \end{cases}$$

These equations have as solution the point

$$\mathbf{p}_0 = \left(\frac{1}{1 + T_0 + D_0}, \frac{T_0 + D_0}{1 + T_0 + D_0} \right).$$

Moreover, the Krylov base associated to this system will be given by $\{\mathbf{p}_0, A_0\mathbf{p}_0\} = \{(\frac{1}{1+T_0+D_0}, \frac{T_0+D_0}{1+T_0+D_0}), (\frac{-D_0}{1+T_0+D_0}, \frac{D_0}{1+T_0+D_0})\}$.

Now, since the system is defined over the half-space given by $\{\mathbf{x} \in \mathbb{R}^2 | \mathbf{k}^T\mathbf{x} \leq 1\}$, the orbits will only be defined in this domain. In particular, over the boundary given by the straight line defined in (51), the set $\mathcal{L}_1^O$ will consist of end points of orbits, and the set $\mathcal{L}_1^I$ will consist of starting points of orbits. From this statement, the transition map actually defined for this system will be Π^{A_0}, following the notation introduced in the work. In what follows, we will label this transition map as Π_0, and its component will be labelled as π_0.

Remind that, as the dimension is $n = 2$ one of the subsets $\mathcal{L}_2^I, \mathcal{L}_2^O$ will be empty, and the other one will contain the contact point $\mathbf{p}_0$. This last statement will depend on the sign of D_0. In particular, if $D_0 < 0$, then the contact point $\mathbf{p}_0 \in \mathcal{L}_2^O$, and it will be the only point of the orbit that will actually belong to the domain. On the other hand, if $D_0 > 0$, then the contact point belongs to $\mathcal{L}_2^I$ and the orbit through it is a non-crossing orbit inward contained, see Lemma 7(b.1). Therefore, it will be defined before and after the contact point.

For the second linear system let us consider, for given T, D, the linear system given by

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \mathbf{k}^T\mathbf{x} \geq 1, \tag{53}$$

where $A = \begin{pmatrix} T & -1 \\ D & 0 \end{pmatrix}$. This second linear system has also the equilibrium at the origin, and it has as trace the parameter T and as determinant the parameter D. Once again, we assume matrix A to be regular, thus the equilibrium is unique and there exists a unique contact point of order 2. Notice, however, that in this case the singular point is located outside the domain of definition of the linear system. From the point of view of PWLS, these singular points are called **virtual singular points**, and even though they are not actual equilibria of the system, their study helps to determine the behavior of their associated linear part. In particular, its behavior will depend on the signs of the parameters T and D. Like before, the contact point of order 2 for this system is given by the following system of equations for the coordinates of $\mathbf{p}_1$:

$$\begin{cases} p_1 + p_2 = 1, \\ Tp_1 - p_2 + Dp_1 = 0. \end{cases}$$

These equations have as solution the point

$$\mathbf{p}_1 = \left(\frac{1}{1+T+D}, \frac{T+D}{1+T+D}\right).$$

Moreover, the Krylov base associated to this system will be given by $\{\mathbf{p}_1, A\mathbf{p}_1\} = \{(\frac{1}{1+T+D}, \frac{T+D}{1+T+D}), (\frac{-D}{1+T+D}, \frac{D}{1+T+D})\}$.

As this linear system is defined over the half-space given by $\{\mathbf{x} \in \mathbb{R} | \mathbf{k}^T\mathbf{x} \geq 1\}$, orbits will only be defined over this domain. Following a reasoning similar to the one performed for the previous system, in this case we arrive at the conclusion that the set $\mathcal{L}_1^I$ will consist of end points of orbits, and the set $\mathcal{L}_1^O$ will consist of

start points of orbits. For this system, the transition map actually defined will be $\widehat{\Pi}^A$. We will label it as Π_1, and its component as π_1.

Once again, as $n = 2$ is the dimension, one of the subsets $\mathcal{L}_2^I, \mathcal{L}_2^O$ will be empty, and the other one will contain the contact point $\mathbf{p}_1$, depending on the sign of the determinant D. In particular, if $D > 0$, then the contact point will belong to $\mathcal{L}_2^I$ and since the system is defined above the straight line, the contact point will be the only point of the orbit actually belonging to the domain. If $D < 0$, then the contact point will be in $\mathcal{L}_2^O$ and the orbit will be defined before and after the contact, see Lemma 7(b.1).

Now, in order to define properly the region of definition of the whole PWLS, we introduce some conditions for each linear part. First of all, we want the system to share the contact point in order to avoid regions of sliding and thus keeping the study as simple as possible. This condition leads to $T + D = T_0 + D_0$. Secondly, since we want monodromic behavior in a neighborhood of the contact point, it must be fulfilled that the set $\mathcal{L}_1^I$ for the system defined by matrix A_0 must be located exactly where the set $\mathcal{L}_1^I$ is for the system defined by matrix A, and we want an analogous property for the sets $\mathcal{L}_1^O$. This implies that the determinants of both matrices must have different signs, that is, $DD_0 < 0$. Moreover, since we will work in a neighborhood of the contact point, in order to have small amplitude limit cycles, we must have $D_0 < 0$ and $D > 0$, see Figure 7a and Figure 7b. Moreover, in this setting, the shared contact point is the only point of its associated orbit actually in the domain of each linear system. Thus, on this point, we can define the PWLS to be $\mathbf{0}$ and it will act as a pseudo-equilibria.

Let us then consider, for given $D > 0, D_0 < 0, T, T_0$ such that $T+D = T_0+D_0$, the PWLS given by

$$\begin{cases} \dot{\mathbf{x}} &= A_0\mathbf{x}, & \{\mathbf{k}^T\mathbf{x} < 1\} \cup \{\mathbf{k}^T\mathbf{x} = 1, \mathbf{x} \in \mathcal{L}_1^I\}, \\ \dot{\mathbf{x}} &= \mathbf{0}, & \mathbf{x} = \mathbf{p}, \\ \dot{\mathbf{x}} &= A\mathbf{x}, & \{\mathbf{k}^T\mathbf{x} > 1\} \cup \{\mathbf{k}^T\mathbf{x} = 1, \mathbf{x} \in \mathcal{L}_1^O\}, \end{cases} \tag{54}$$

where $\mathbf{k}^T = (1, 1)$, $A_0 = \begin{pmatrix} T_0 & -1 \\ D_0 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} T & -1 \\ D & 0 \end{pmatrix}$, just as above.

This PWLS is written in what is called the **Liénard form**, a generic form for most PWLS. Thus, the study of this system will be of aid in the study of most PWLS with two zones. Moreover, this Liénard form is just a change of basis from the Krylov base in which the vectors of the base have been reordered. Remind that the Krylov base matrix representation is given by (21), the controlability matrix.

As it has been stated above, if $T + D = T_0 + D_0$, both linear parts share the contact point of order 2. We will label the contact point as $\mathbf{p}$. However, the Krylov base differs for each system, being the vectors $A_0\mathbf{p}$ and $A\mathbf{p}$ proportional with negative constant, that is, $A_0\mathbf{p} = -\gamma^2 A\mathbf{p}$, for some $\gamma \in \mathbb{R}\backslash\{0\}$. Moreover, the proportionality constant can be easily computed, since both vectors are known. By taking norms on both sides of the previous equality, we have that γ^2

is then given by

$$\gamma^2 = \frac{\|A_0\mathbf{p}\|}{\|A\mathbf{p}\|} = \frac{\sqrt{\frac{D_0^2}{(1+D_0+T_0)^2}}\sqrt{2}}{\sqrt{\frac{D^2}{(1+D+T)^2}}\sqrt{2}} = \left|\frac{D_0}{D}\right| = -\frac{D_0}{D}, \tag{55}$$

since $D_0 < 0$, $D > 0$.

On the other hand, under the assumptions $D_0 < 0$, $D > 0$ we have that the sets $\mathcal{L}_1^I$, $\mathcal{L}_1^O$ coincide for each linear part. Therefore, even when the system is discontinuous, we still have uniqueness of solutions and also monodromic behavior near the contact point $\mathbf{p}$.

Regarding the contact point, notice that we have defined the vector field at $\mathbf{p}$ to behave like a singular point. This comes from the fact that, for the setting $D_0 < 0$, $D > 0$, the contact point will be the only point of its associated orbit to actually belong to the domain of definition for each linear system. Thus, from the PWLS point of view, an orbit starting at $\mathbf{p}$ will not move from it. Hence, this point will be a pseudo-equilibria.

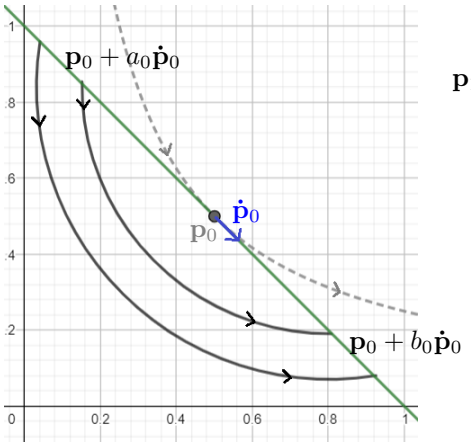
From the definition of the PWLS, the transversal hyperplane given by (51) acts as the the boundary between both linear zones. In particular, we have Π_0 as the transition map through the system defined by matrix A_0 and Π_1 as the transition map through the system defined by matrix A. Since $D_0 < 0$, $D > 0$ and thus the respective sets $\mathcal{L}_1^I$, $\mathcal{L}_1^O$ for each system share location, the composition of both transition maps will be well-defined. We will use this composition later on.

Now, since we will work with the series expansions obtained in Lemma 12, the study will be local, in a neighborhood of the contact point $\mathbf{p}$. Under the setting $D_0 < 0$, $D > 0$, small amplitude limit cycles, in terms of the Krylov base, can appear for this PWLS in a neighborhood of $\mathbf{p}$. In what follows, we will study conditions for a limit cycle to appear.

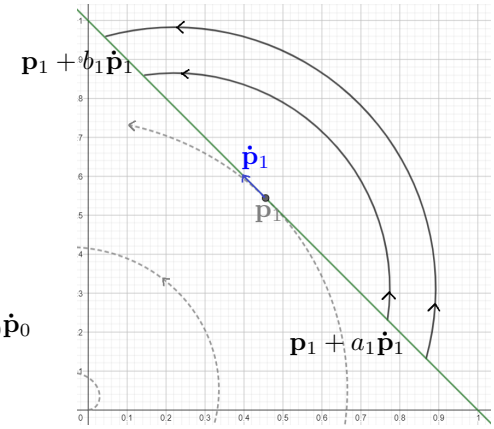
We have that, if the Krylov base for the linear part on the left of $\mathcal{P}$ is labelled as $B_0 = \{\mathbf{p}, A_0\mathbf{p}\}$ and the Krylov base for the linear part on the right is labelled as $B_1 = \{\mathbf{p}, A\mathbf{p}\}$, then a point $\mathbf{q} \in \mathcal{P}$ writes as $\mathbf{q}_{B_0} = a$ with respect to B_0 and it writes as $\mathbf{q}_{B_1} = b$ with respect to B_1. Remind that, under the assumption $D_0 < 0$, $D > 0$ the proportionality constant for the second vector of each Krylov basis writes as $\gamma^2 = -\frac{D_0}{D}$. That is, $\mathbf{A}_0\mathbf{p} = \frac{D_0}{D}A\mathbf{p}$. Therefore, since $\mathbf{q} = \mathbf{p} + aA_0\mathbf{p}$ and $\mathbf{q} = \mathbf{p} + bA\mathbf{p}$, it follows that $aA_0\mathbf{p} = bA\mathbf{p}$. Now, from the proportionality condition, it follows $-a\gamma^2 = b$, that is, $a\frac{D_0}{D} = b$.

Now, the transition map for the linear system defined by matrix A_0 takes a point $a_0 \in \mathcal{L}_1^I = \{a : (-1)^3 aD_0 < 0\} = \{a : a < 0\}$ and takes it to a point $b_0 \in \mathcal{L}_1^O = \{a : (-1)^3 aD_0 > 0\} = \{a : a > 0\}$, where the coordinate is written with respect to B_1. On the other hand, the transition map for the linear system defined by matrix A maps a point $a_1 \in \mathcal{L}_1^O = \{a : (-1)^3 aD > 0\} = \{a : a < 0\}$, written with respect to B_2, to a point $b_1 \in \mathcal{L}_1^I = \{a : (-1)^3 aD < 0\} = \{a : a > 0\}$.

In order to study the stability of the contact point, we will study the difference between the starting point and the end point of the composition of the transition



(a) Transition map for $D < 0$. Contact point $\mathbf{p}_0$ in gray, vector $\dot{\mathbf{p}}_0$ in blue, orbits starting at points $\mathbf{p}_0 + a_0\dot{\mathbf{p}}_0, a_0 < 0$ and ending at points $\mathbf{p}_0 + b_0\dot{\mathbf{p}}_0, b_0 > 0$, in black.



(b) Transition map for $D > 0$. Contact point $\mathbf{p}_1$ in gray, vector $\dot{\mathbf{p}}_1$ in blue, orbits starting at points $\mathbf{p}_1 + a_1\dot{\mathbf{p}}_1, a_1 < 0$ and ending at points $\mathbf{p}_1 + b_1\dot{\mathbf{p}}_1, b_1 > 0$, in black.

Figure 7: Configurations for different sign of D.

maps. Since the coordinate in one Krylov base writes in terms of the other Krylov base, we can consider the inverse transition map, that is, the transition map for matrix $-A_0$, and subtract both transition maps expressed on the same base. This is performed in what follows, and leads us to an expression that will be labelled as the **displacement function**.

By means of the series expansion obtained in Lemma 12, the previous statement yields the following expressions

$$\begin{aligned} b_0 = \pi_0(a_0) &= -a_0 + \tfrac{2T_0}{3}a_0^2 - \tfrac{4T_0^2}{9}a_0^3 + \tfrac{2}{135}T_0(22T_0^2 - 9D_0)a_0^4 + \dots, \\ b_1 = \pi_1(a_1) &= -a_1 + \tfrac{2T}{3}a_1^2 - \tfrac{4T^2}{9}a_1^3 + \tfrac{2}{135}T(22T^2 - 9D)a_1^4 + \dots. \end{aligned} \tag{56}$$

If we invert time on the first equation, as the system is linear, it is converted in system $\dot{\mathbf{x}} = -A_0\mathbf{x}$, thus the second coordinate in the Krylov base B_1 has its sign inverted. To add up, the trace is changed to $-T_0$, and the determinant remains constant as the dimension is $n = 2$, an even number. Moreover, the transition map for this new system maps the point $-b_0$ to the point $-a_0$. All these statements yield the equation

$$-a_0 = b_0 - \frac{2T_0}{3}b_0^2 + \frac{4T_0^2}{9}b_0^3 - \frac{2}{135}T_0(22T_0^2 - 9D_0)b_0^4 + \dots. \tag{57}$$

On the other hand, the coordinates a_1, b_1 can be written in terms of B_1 as $a_1 = \frac{D_0}{D}b_0$ and $b_1 = \frac{D_0}{D}a_0$. Thus, the second equation of expression (56) writes,

simplifying a factor of $\frac{D_0}{D}$ on both sides and multiplying the equation by -1, as

$$-a_0 = b_0 - \frac{2T}{3}\frac{D_0}{D}b_0^2 + \frac{4T^2}{9}\left(\frac{D_0}{D}\right)^2 b_0^3 - \frac{2}{135}T(22T^2-9D)\left(\frac{D_0}{D}\right)^3 b_0^4 + \dots . \quad (58)$$

By last, if we subtract equation (57) from equation (58), we get to the displacement function that was mentioned above. This map takes the form $\delta(b_0) = \tilde{\pi}(b_0) - \pi^{-1}(b_0)$ and it is given by

$$\begin{aligned} 0 \quad &= \tfrac{2}{3}b_0^2\Big((-T\tfrac{D_0}{D} + T_0) + \tfrac{2}{3}(T^2\left(\tfrac{D_0}{D}\right)^2 - T_0^2)b_0 \\ &\quad + \tfrac{1}{45}(-T(22T^2-9D)\left(\tfrac{D_0}{D}\right)^3 + T_0(22T_0^2 - 9D_0))b_0^2 + \dots\Big). \end{aligned}$$

Now, we define

$$\begin{aligned} f(b_0) \quad &= \left(-T\tfrac{D_0}{D} + T_0\right) + \tfrac{2}{3}\left(T^2\left(\tfrac{D_0}{D}\right)^2 - T_0^2\right)b_0 \\ &\quad + \tfrac{1}{45}\left(-T(22T^2-9D)\left(\tfrac{D_0}{D}\right)^3 + T_0(22T_0^2 - 9D_0)\right)b_0^2 + \dots, \end{aligned} \quad (59)$$

hence the displacement function can be written as $\delta(b_0) = \frac{2}{3}b_0^2 f(b_0)$.

The trivial solution of the previous equation, $b_0 = 0$, corresponds to the coordinate of the contact point $\mathbf{p}$. Now, since we have $b_0 > 0$ due to the PWLS configuration, we are interested in the non-trivial positive solutions of the term $f(b_0)$. If there exists any of such positive solutions, it will correspond to a non-zero coordinate b_0, that will translate as a limit cycle. On the other hand, if $f(b_0)$ is not null for some b_0^*, its sign will determine the stability of the contact point or the limit cycle near this value b_0^*.

From this last statement, for small values of b_0, the sign of the independent term $-T\frac{D_0}{D} + T_0$ of $f(b_0)$ determines the behavior of the flow in a neighborhood of the contact point $\mathbf{p}$. This follows from the fact that, locally, if this term is negative, the end point will be closer to $\mathbf{p}$ than the start point, and similarly, if this term is positive, the end point will be further away from $\mathbf{p}$ than the start point. Therefore, we can state a first result about the local behavior of the contact point $\mathbf{p}$, in which the inequality gets simplified.

Proposition 10. *Let $\mathbf{p}$ be the contact point of the PWLS (54) fulfilling $T + D = T_0 + D_0$, and $\mathbf{q} \in \mathcal{P}$ such that $\mathbf{q}_{B_0} = a_0$ and $\pi_0(a_0) = b_0$. Then:*

a) if $D + T < 0$, the contact point $\mathbf{p}$ is stable,

b) if $D + T > 0$, the contact point $\mathbf{p}$ is unstable.

Proof. Since the function $f(b_0)$ in equation (59) acts as a displacement function for the transition maps composition, the sign of its first term will determine the local behavior of the contact point $\mathbf{p}$. Now, if we substitute $D_0 = T + D - T_0$ in the independent term of $f(b_0)$, we have

$$-T\frac{D_0}{D} + T_0 = -T(1 + \frac{T - T_0}{D}) + T_0 = (T_0 - T)(1 + \frac{T}{D}) = \frac{T_0 - T}{D}(D + T).$$

Now, since $D_0 < 0$, it follows $D < T_0 - T$ and, as $D > 0$, we deduce $T_0 - T > 0$. Hence, the previous expression has a positive first factor. In consequence, the sign of the whole term will coincide with the sign of $D + T$. Then, the result follows directly. □

The previous result is rather interesting. On the one hand, it yields that the contact point has behavior similar to that of an equilibrium, without being a proper singular point. On the other hand, it states a change of stability for a specific parameter combination, particularly in $-T\frac{D_0}{D} + T_0 = 0$. Therefore, the system might suffer a bifurcation similar to the Hopf bifurcation. This bifurcation will lead to the birth of a limit cycle in a neighborhood of the PWLS. Moreover, for $-T\frac{D_0}{D} + T_0 = 0$ in equation (59), then $b_0 = 0$ is a double root of $f(b_0)$ since both the independent and linear parts get cancelled. This condition implies that $TD_0 = DT_0$, and as D and D_0 have opposite signs, it follows that T and T_0 will also have opposite signs.

Now, at the value $T = \frac{D}{D_0}T_0$, applying the condition $T + D = T_0 + D_0$ we arrive at the values

$$\begin{array}{l} T_0 = -D_0, \\ T = -D, \end{array} \tag{60}$$

that fulfill both conditions. From this, in order to simplify the study, we will consider the following family of PWLS:

$$\left\{ \begin{array}{rcll} \dot{\mathbf{x}} & = & \begin{pmatrix} T_0 + \mu & -1 \\ -T_0 & 0 \end{pmatrix} \mathbf{x}, & \{\mathbf{k}^T\mathbf{x} < 1\} \cup \{\mathbf{k}^T\mathbf{x} = 1, \mathbf{x} \in \mathcal{L}_1^I\}, \\ \dot{\mathbf{x}} & = & \mathbf{0}, & \mathbf{x} = \mathbf{p}, \\ \dot{\mathbf{x}} & = & \begin{pmatrix} T + \mu & -1 \\ -T & 0 \end{pmatrix} \mathbf{x}, & \{\mathbf{k}^T\mathbf{x} > 1\} \cup \{\mathbf{k}^T\mathbf{x} = 1, \mathbf{x} \in \mathcal{L}_1^O\}, \end{array} \right. \tag{61}$$

for $\mu \in \mathbb{R}$ a parameter. Consider $f_\mu(b_0)$ to be the function $f(b_0)$ defined in expression (59) where now we have the trace $T + \mu$ and the determinant $-T$ of the matrix of the linear system in $\mathbf{k}^T\mathbf{x} \geq 1$, instead of T and D; and the trace $T_0 + \mu$ and the determinant $-T_0$ of the matrix of the linear system in $\mathbf{k}^T\mathbf{x} \leq 1$, instead of T_0 and D_0. After some simplifications, the displacement function will now be given explicitly by

$$\begin{array}{rl} f_\mu(b_0) & = \mu\left(\frac{T-T_0}{T}\right) + \frac{2}{3}\mu\left(\frac{T_0-T}{T}\right)\left(\mu\left(\frac{T_0+T}{T} + 2T_0\right)\right) b_0 \\ & + \frac{1}{45}\left(9\frac{T_0^2}{T}(T - T_0) + \mu\left(9 + 22T_0 - 9\frac{T_0^2}{T^2}\right.\right. \\ & \left.\left. -22\mu^2\frac{T_0^2}{T^3} - 66\mu\frac{T_0^2}{T^2} - 66\frac{T_0^2}{T}\right)\right) b_0^2 \\ & + \dots . \end{array} \tag{62}$$

For $\mu = 0$, this PWLS is exactly the original one for the case $T = \frac{D}{D_0}T_0$. Thus, $f_0(b_0)$ has null independent and linear terms.

For $\mu \neq 0$, since the independent term is given by $\mu(\frac{T-T_0}{T})$, and T_0, T have opposite signs, then the term $(\frac{T-T_0}{T})$ is positive. Thus, the sign of μ will determine the stability of the contact point for this new system. Precisely:

- if $\mu < 0$, then $\mathbf{p}$ will be stable,
- if $\mu > 0$, then $\mathbf{p}$ will be unstable.

Moreover, the contact point will still be the same for both linear parts, and it will be given by $\mathbf{p} = (\frac{1}{1+\mu}, \frac{\mu}{1+\mu})$. Since we want to study the bifurcation, we will consider μ in a neighborhood of 0, thus the value $\mu = -1$ will not be reached. Now, for $-1 << \mu < 0$, the contact point will be located on the fourth quadrant, and for $0 < \mu << 1$ the contact point will lie on the first quadrant. Thus, the bifurcation, if it exists, will happen precisely when the contact point crosses the X-axis.

We now state the main result of the study, in which it is proven that the family of PWLS (61) suffers a bifurcation in which a limit cycle is born whenever the contact point changes its stability.

Theorem 12. *Consider the PWLS given by (61), and let $f_\mu(b_0)$ be the function defined in (59) for this PWLS. We have:*

a) if $\mu = 0$, then $f_0(b_0)$ has a double zero at $b_0 = 0$,

b) if $-1 << \mu < 0$, then $f_\mu(b_0)$ has a simple positive zero $b_0 = b_0(\mu)$ such that $\lim_{\mu \to 0} b_0(\mu) = 0$. Therefore, there exists a limit cycle that is born at the contact point and surrounds it. Moreover, the limit cycle is unstable.

Proof. The first statement is straightforward to check. To prove it, we will rewrite the function $f_\mu(b_0)$ as $f_\mu(b_0) = t_0 + t_0 t_1 b_0 + t_2 b_0^2 + \mathcal{O}(b_0^3)$, where $t_0 = \mu \frac{T - T_0}{T}$ and so on, following expression (62). Hence, for $\mu = 0$, the term t_0 vanishes, thus the function $f_0(b_0)$ starts with, at least, quadratic terms. It now suffices to show that $t_2 \neq 0$. We have

$$t_2|_{\mu=0} \quad = \frac{9T_0^2}{T}(T - T_0).$$

Now, since $T < 0, T_0 > 0$, it follows that $t_2|_{\mu=0}$ is always positive. In particular, it is not null. Therefore, the root is double.

For the second statement, we will label b_0 as x, and we will apply the Inverse Function Theorem (IFT, for short) to the function $g(\mu, x) := f_\mu(x)$. On the one hand, form the previous statement we have that $g(0, 0) = 0$. Hence, the point $(0, 0)$ is a root of the function g. On the other hand, we have $\frac{\partial g}{\partial \mu}\Big|_{(0,0)} = \frac{T - T_0}{T} > 0$, since $t_0 = \mu \frac{T - T_0}{T}$. Therefore, by the IFT there exists a function $\mu(x)$ defined in a neighborhood $(-x_0, x_0)$ of $x = 0$ such that $\mu(0) = 0$ and $g(\mu(x), x) = 0, \forall x \in (-x_0, x_0)$.

Now, this tells us that in a neighborhood of $(0, 0)$ we have, for each value of x, a value of the parameter $\mu(x)$ such that it yields a root of the whole function $f_\mu(x)$, that is, a limit cycle. Since the coordinate b_0 was assumed to be positive, we are only interested in values of $x \in (0, x_0)$. Moreover, since $\mu(0) = 0$, the function $\mu(x)$ written as a polynomial, starts, at least, with linear terms. Hence, its first value will be given by $\mu'(0)$. And this value can be computed by implicit

derivation. Indeed, we have that, since $g(\mu(x), x) = 0$, then $\frac{d}{dx}(g(\mu(x), x)) = 0$. If we compute the derivative explicitly, we have

$$0 = \frac{d}{dx}(g(\mu(x), x)) \mid_{x=0} = \left(\frac{\partial g}{\partial \mu}\mu' + \frac{\partial g}{\partial x}\right)\Big|_{(\mu,x)=(0,0)}. \tag{63}$$

Now, since $\frac{\partial g}{\partial x} = t_0 t_1 = \mu\frac{T-T_0}{T}t_1$, we have $\frac{\partial g}{\partial x}\Big|_{(\mu,x)=(0,0)} = 0$. Thus, from the previous equation it follows $\mu'(0) = 0$ since $\frac{\partial g}{\partial \mu}\Big|_{x=0} > 0$. Therefore, the sign of $\mu(x)$ will depend on the sign of $\mu''(0)$ since both the independent and linear terms are null. And this value can be obtained once again from implicit differentiation. From equation (63), if we differentiate again with respect to x, we have

$$0 = \left(\frac{d}{dx}\left(\frac{\partial g}{\partial \mu}\right)\mu' + \frac{\partial g}{\partial \mu}\mu'' + \frac{\partial^2 g}{\partial x \partial \mu}\mu' + \frac{\partial^2 g}{\partial x^2}\right)\Big|_{x=0} = \left(\frac{\partial g}{\partial \mu}\mu'' + \frac{\partial^2 g}{\partial x^2}\right)\Big|_{(\mu,x)=(0,0)},$$

since $\mu'(0) = 0$. Thus, we have $\mu''(0) = -\frac{2T}{T-T_0}t_2$, since $\frac{\partial^2 g}{\partial x^2}\Big|_{(\mu,x)=(0,0)} = 2t_2$. Hence, $\mu''(0) < 0$, thus we have the existence of the limit cycle for each $x \in (0, x_0)$.

Finally, in order to state its stability, notice that as $\mu(x) < 0$ in a neighborhood of $x = 0$, the term t_0 is negative, while the term $t_2 > 0$. Therefore, the function $f_{\mu(x)}(x)$ is negative for $0 < x << 1$ and becomes positive for bigger values of x. Hence, the displacement function has negative sign for orbits inside the region delimited by the limit cycle, and it has positive sign for orbits outside of that region. Therefore, the limit cycle is unstable. □

This result ensures that whenever the parameter μ decreases from $\mu = 0$, an unstable limit cycle appears for the PWLS (61) surrounding the contact point, that will be stable since $\mu < 0$. This kind of bifurcation is similar to the Hopf one, but here the limit cycle is born near the contact point instead of a singular point, whenever the contact point changes its stability. Moreover, notice that, in the proof of the previous result, we have seen that the dependence of the parameter μ with respect to x is of order x^2. In particular, we have obtained $\mu = -\frac{2t_2 T}{T-T_0}x^2 + \ldots$. Therefore, if we invert this first term in the series we have

$$x = \sqrt{\frac{T_0 - T}{2Tt_2}}|\mu|^{\frac{1}{2}} + \ldots.$$

From this expression, we have that $x(\mu)$ is not an analytic function and therefore it is not possible to apply the IFT to obtain this function instead of $\mu(x)$ as we have done in the proof of the previous result. Moreover, this expression states that the dependence of the amplitude with respect to the parameter is of order $\frac{1}{2}$, which coincides with the dependence of the amplitude with respect to the parameter in the smooth case. This is curious, since this dependence differs from the dependence of a Hopf-like bifurcation near a singular point for PWLS, where the dependence is linear, see e.g. [3]. In Figure 8 it is shown the bifurcation diagram, where it can be checked that indeed the dependence is of order $\frac{1}{2}$.

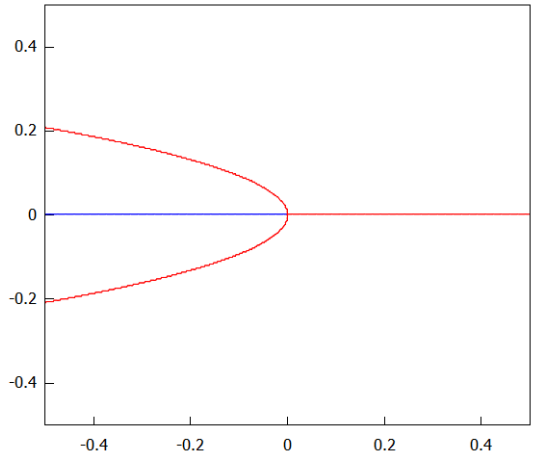
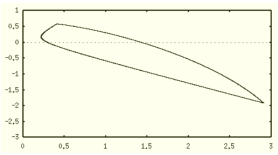
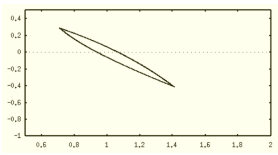


Figure 8: Bifurcation diagram for the bifurcation. On the $X-$axis, the values of μ. On the Y axis, the coordinate b_0. In red, values for which the limit cycle and the contact point are unstable. In blue, values for which the contact point is stable.

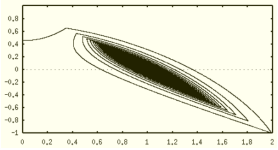
To sum up, a numerical simulation is shown in Figure 9, where for negative μ the limit cycle is located, and for positive μ a diverging solution is plotted, with no limit cycle.



(a) Limit cycle for $\mu = -0.2$.



(b) Limit cycle for $\mu = -0.02$.



(c) Orbit for $\mu = 0.02$, diverging from the contact point.

Figure 9: Orbits for the PWLS for $T = -0.8$, $T_0 = 1$.

6 Future Work

Even though this work has introduced a wide amount of concepts and results, hence it has been extensive, there are still some points to be studied or amplified. On the one hand, this study has opened some questions that are still unanswered, while other points have not even been considered due to different factors.

Hence, in this section we make a summary of these open questions and topics, in order to gather all possible continuations to this work.

On the first place, regarding Section 4, we worked out series expansions for dimensions 2 and 3, where the latter expressions were a bit trickier to obtain. A procedure similar to the one performed could give expressions for higher dimensions, but it was not considered here. Thus, this is another point in which to continue this work.

On the same section, in particular for the case $n = 3$, Proposition 9 seemed to imply that the set D_1 would have as a boundary the set $\{(a_1, a_2)|a_2 = \mathcal{O}(a_1^2)\}$, and points in D_1 would be points with $a_2 = a_1^k$, for $k \in \mathbb{R}^+, k < 2$. However, this conjecture has not been proved, and it is another question that remains open. Moreover, since we have dealt with local approximations, the limit of this domain is also local, and nothing can be stated about the whole domain D_1, only in a neighborhood of the contact point $\mathbf{p}$, for points $\mathbf{q}$ with time of flight near to 0.

On the third place, in Section 5, where we studied a general family of PWLS, we defined a particularization of that family that always yielded a bifurcation similar to the subcritical Hopf one, where an unstable limit cycle was born for a parameter decreasing from 0. Thus, for this PWLS family, we could study conditions for the system to exhibit a wide variety of kinds of bifurcations, e.g. the birth of a pseudo-homoclinic loop connecting the invariant axis of a saddle on one linear system with a monodromic orbit for the other linear system.

By last, in Section 5 we used the theory and results developed throughout the work and applied them to a PWLS. For $\mathbb{R}^2$ there exist alternatives to our methods that yield good results as well. However, the advantage of our work to some of these alternatives is the fact that it is easily generalizable to higher dimensions. Hence, it allows us to study linear systems and PWLS in dimensions $n \geq 3$.

References

[1] Claudio Buzzi, Claudio Pessoa, and Joan Torregrosa.
"Piecewise linear perturbations of a linear center." In: *Discrete and Continuous Dynamical Systems* 33.9 (2013), pp. 3915–3936.
DOI: https://doi.org/10.3934/dcds.2013.33.3915.

[2] Victoriano Carmona, Fernando Fernández-Sánchez, and Elisabeth García-Medina.
"Including homoclinic connections and T-point heteroclinic cycles in the same global problem for a reversible family of piecewise linear systems."
In: *Applied Mathematics and Computation* 296 (2017), pp. 33–41.
DOI: https://doi.org/10.1016/j.amc.2016.10.008.

[3] Victoriano Carmona et al.
"Bifurcation of invariant cones in piecewise linear homogeneous systems". In: *International Journal of Bifurcation and Chaos* 15.1 (2005), pp. 2469–2484.
DOI: https://doi.org/10.1142/S0218127405013423.

[4] Victoriano Carmona et al.
"On Simplifying and Classifying Piecewise-Linear Systems".
In: *IEEE Transactions on Circuits and Systems Part I: Fundamental Theory and Applications* 49.5 (2002), pp. 609–620.
DOI: https://doi.org/10.1109/TCSI.2002.1001950.

[5] Aleksei Fedorovich Filippov.
Differential equations with discontinuous righthand sides.
Springer Science & Business Media, 2013.

[6] Emilio Freire, Enrique Ponce, and Francisco Torres. "A general mechanism to generate three limit cycles in planar Filippov systems with two zones".
In: *Nonlinear Dynamics* 78.1 (2014), pp. 251–263.
DOI: https://doi.org/10.1007/s11071-014-1437-7.

[7] Jaume Llibre and Antonio E. Teruel.
"Existence of Poincaré maps in piecewise linear differential systems in $\mathbb{R}^n$".
In: *International Journal of Bifurcation and Chaos* 14.8 (2004), pp. 2843–2851. DOI: https://doi.org/10.1142/S0218127404010874.

[8] Jaume Llibre and Antonio E. Teruel.
Introduction to the Qualitative Theory of Differential Systems: Planar, Symmetric and Continuous Piecewise Linear Systems.
Springer Basel, Birkhäuser Advanced Texts Basler Lehrbücher, 2014.

[9] E. J. Putzer. "Avoiding the Jordan Canonical Form in the Discussion of Linear Systems with Constant Coefficients".
In: *The American Mathematical Monthly* 73.1 (1966), pp. 2–7.
DOI: https://doi.org/10.2307/2313914.